Hyperbolic isometries versus symmetries of links

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Knots, Groups and 3-Manifolds in Marseille

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Symmetries of links

- $L \subset S^3$ a link
- A finite group $G < \mathrm{Diff}^+(S^3, L)$ is called a group of symmetries.
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- Then $G \lhd \mathrm{Isom}^+(S^3 \setminus L)$.
- Let $\mathrm{Sym}^+(S^3, L) \lhd \mathrm{Isom}^+(S^3 \setminus L)$ be the maximal subgp. of symmetries.
- For a knot $L = K$, $\mathrm{Sym}^+(S^3, K) = \mathrm{Isom}^+(S^3 \setminus K)$ (Gordon-Luecke)

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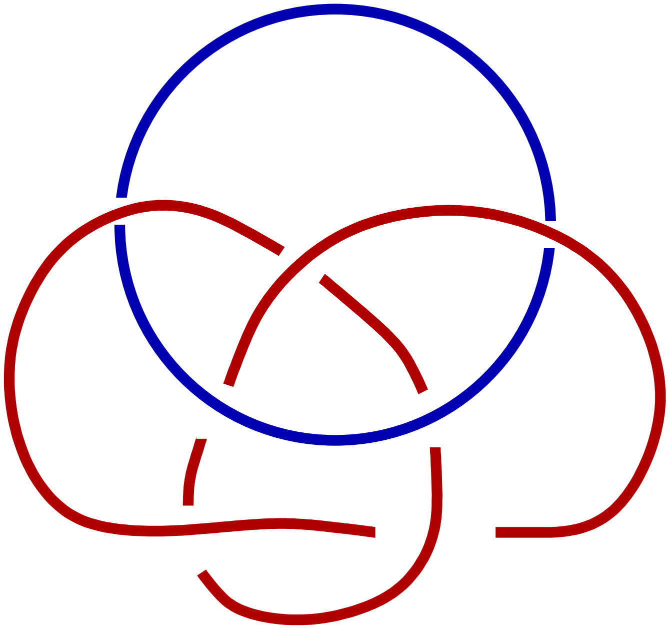
Question: How different $\mathrm{Sym}^+(S^3, L)$ and $\mathrm{Isom}^+(S^3 \setminus L)$ can be?

Example 7^2_8

- $\mathrm{Sym}(S^3, 7^2_8) = D_2$ The components cannot be switched: trefoil and trivial.

- $\mathrm{Isom}(S^3 \setminus 7^2_8) = D_4$ The cusps may be switched S.R. Henry & J.R. Weeks, using SnapPea



Theorem (Paoluzzi & P.):

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- L has large number of components $\sim 10|G|$
- Kojima: there exist M^3 closed and hyperbolic such that
  $G = \mathrm{Isom}^+(M^3)$ acts freely.
- Cooper-Long: there exists M^3 closed hyperbolic $\mathbf{Q}$-homology sphere
  such that $G < \mathrm{Isom}^+(M^3)$ acts freely.

About the proof

- It is easy to construct $L \subset S^3$ such that:

$$G < \mathrm{Isom}^+(S^3 \setminus L)$$

- The difficult thing is to rigidify the action (i.e. to modify $L \subset S^3$ so that

$$G = \mathrm{Isom}^+(S^3 \setminus L))$$

- Can modify Cooper-Long's so that $G = \mathrm{Isom}^+(M^3)$ for some hyperbolic rational homolgy sphere M^3.

Finding $\mathcal{L}$ ***s.t.*** $G < \mathrm{Isom}^+(S^3 \setminus \mathcal{L})$

Proposition

Every finite G *acts effectively and freely by isometries on some hyperbolic link exterior* $S^3 \setminus \mathcal{L}$

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1 Start with M^3 closed such that G acts freely on M^3

Take a presentation $G = F_n/H$ for F_n= free rank n.

$$\pi_1(S^2 \times S^1 \# \overset{(n)}{\cdots} \# S^2 \times S^1) = F_n$$

$M^3 \to S^2 \times S^1 \# \cdots \# S^2 \times S^1$ covering with $\pi_1(M^3) = H$.

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1 Start with M^3 closed such that G acts freely on M^3

2 Remove $\mathbb{L}_0 \subset M^3$ s.t. $M^3 \setminus \mathbb{L}_0 = S^3 \setminus \mathcal{L}_0$. (Lickorish-Wallace)

Consider the orbits $\mathbb{L}_1 = \bigcup_{g \in G} g\mathbb{L}_0$ and apply general position.

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Consider the orbits $\mathbb{L}_1 = \bigcup_{g \in G} g\mathbb{L}_0$ and apply general position.

3 By Myers' Theorem, there is a hyperbolic knot $\mathbb{K} \subset (M^3 \setminus \mathbb{L}_1)/G$ such that $(M^3 \setminus \mathbb{L}_1)/G \setminus \mathbb{K}$ is hyperbolic.

Lift $\mathbb{L}_1/G \cup \mathbb{K}$ to $\mathbb{L} \subset M^3$

Then $M^3 \setminus \mathbb{L} = S^3 \setminus \mathcal{L}$

Myer's theorem and hyperbolization

Myer's thm.

On every compact orientable M^3 and every homotopy class there exists a knot $K \subset M^3$ in this homotopy class such that $M^3 \setminus \mathcal{N}(K)$ is irreducible, atoroidal, acylindrical and not Seifert fibered.

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- Can apply Thurston's hyperbolization to the exterior.
- Acylindrical:
  - allows to choose the curves on the boundary corresponding to rank one cusps (pared manifolds).
  - The boundary $(\subset \partial M^3)$ is totally geodesic.

- We have $G < \mathrm{Isom}^+(S^3 \setminus \mathcal{L})$.
- Want to modify $\mathcal{L}$ so that $G = \mathrm{Isom}^+(S^3 \setminus \mathcal{L})$ by means of a rigid object:

Rigidifying the action

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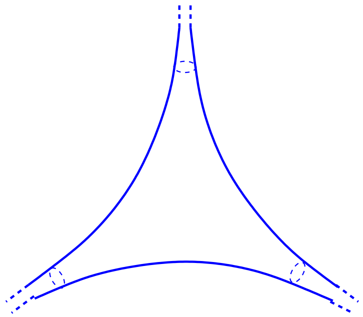
- Want to modify $\mathcal{L}$ so that $G = \mathrm{Isom}^+(S^3 \setminus \mathcal{L})$ by means of a rigid object:

  - Shortest geodesic: have an arbitrarily large stabilizer, difficult to control.

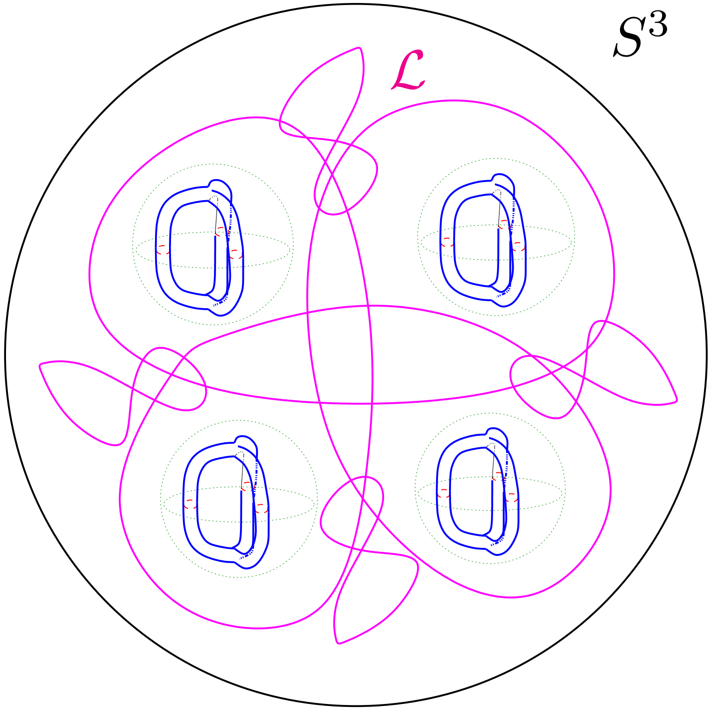
  - Pants:
    - Its orientation preserving symmetry group is Σ_3.
    - Its hyperbolic structure is unique.
    - It is small: simple loops are ∂-parallel or contractible.



Rigidify: pairs of pants

Start with $G < \mathrm{Isom}^+(S^3 \setminus \mathcal{L})$. Take H genus-2 handelbody G-equiv.

$g_i\, H \cap g_j\, H = \emptyset$ for $g_i \neq g_j \in G$ $\qquad$ $H \subset B^3 \subset S^3 \setminus \mathcal{L}$



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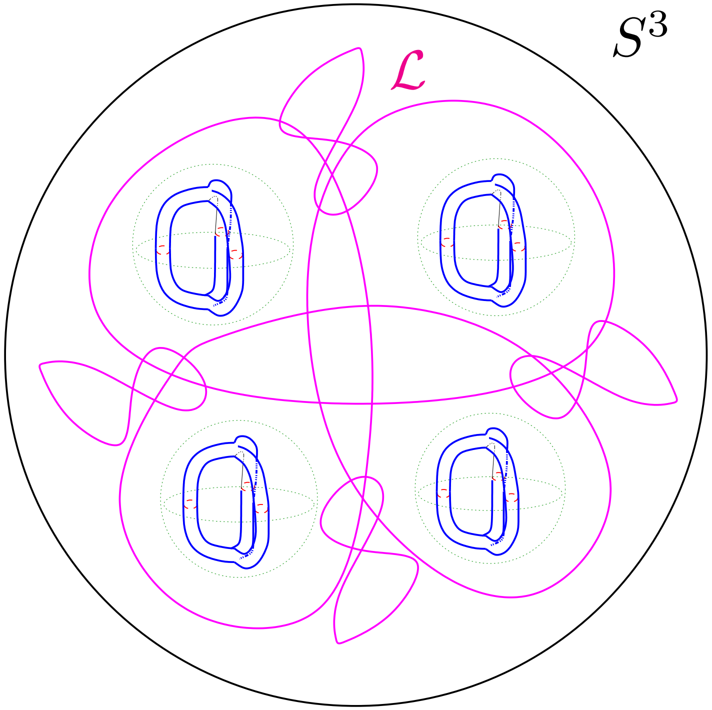
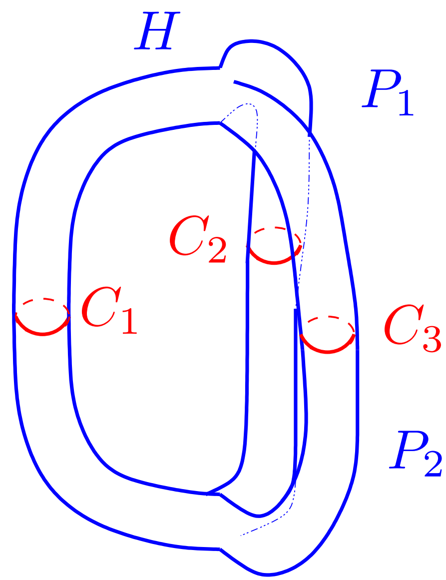




$H \subset B^3 \subset S^3 \setminus \mathcal{L}$ $\qquad$ $\partial H \setminus (C_1 \cup C_2 \cup C_3) = P_1 \cup P_2$

Remove:
- The C_i and their orbits
- curves in $int(H)$ and $S^3 \setminus \mathcal{L} \setminus \bigcup_g gH$ to be hyperbolic

Links in the handelbody

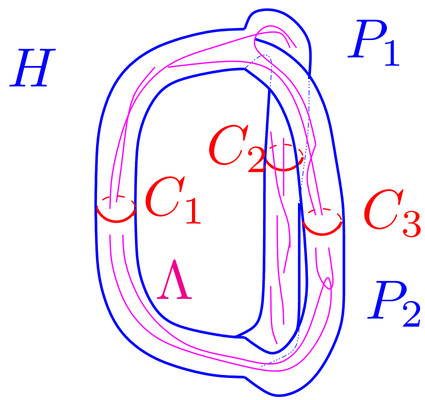
Find a link $\Lambda \subset H$ such that $\mathcal{H} = H \setminus (\Lambda \cup C_1 \cup C_2 \cup C_3)$:

1: *$\mathcal{H}$ is hyperbolic with totally geodeodesic boundary $P_1 \cup P_2$.*

2: *$\mathcal{H}$ has a unique shortest geodesic (arbitrarily short).*

3: $\mathrm{Isom}^+(\mathcal{H})$ *is trivial.*

4: *Every pair of pants properly embedded in $\mathcal{H}$ is ∂-parallel.*



$C_i \rightarrow$ Rank one cusps, $\Lambda \rightarrow$ rank two cusps.

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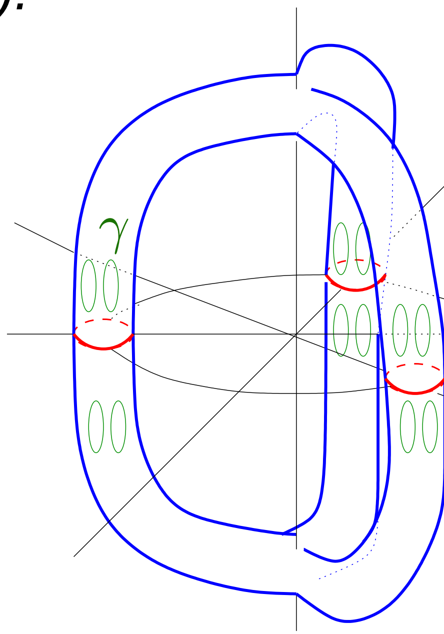
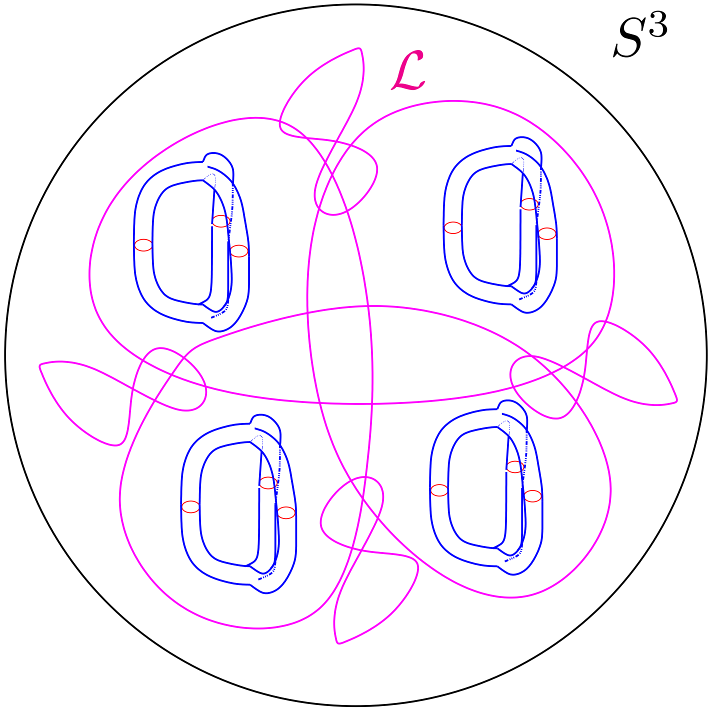


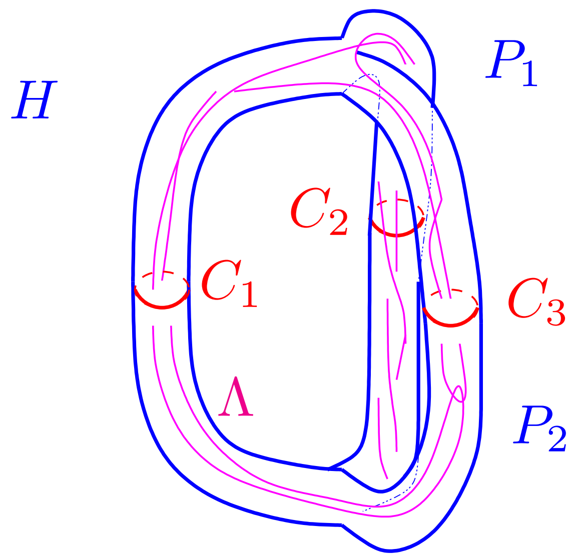
$\mathbf{D}_3 \times \mathbf{Z}/2$ acts on $(H, C_1 \cup C_2 \cup C_3)$

- Remove a trivial curve γ and its orbits
- Remove another equivarian curve

  to become hypebolic (Myer's, equiv. version)
- $1/n_i$ surgery on γ and translates, $n_1 > n_2 > \cdots > n_{12} \gg 1$.

- Remove a G-equivariant hyperbolic link in $S^3 \setminus (\mathcal{L} \cup \bigcup_g (int(H) \cup C_1 \cup C_2 \cup C_3))$ (Myer's).
- Glue it to $|G|$ copies of $\mathcal{H} = H \setminus (C_1 \cup C_2 \cup C_3 \cup \Lambda)$ along the ∂-pants. The pants are totally geodesic.





$S^3 \setminus (\mathcal{L} \cup \bigcup_g (int(H) \cup C_1 \cup C_2 \cup C_3))$ $\mathcal{H} \setminus (C_1 \cup C_2 \cup C_3 \cup \Lambda)$

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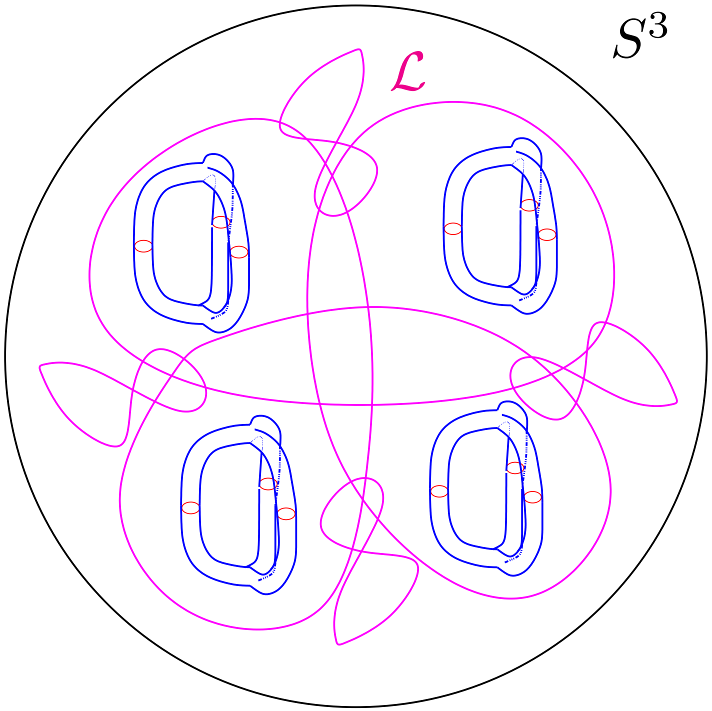

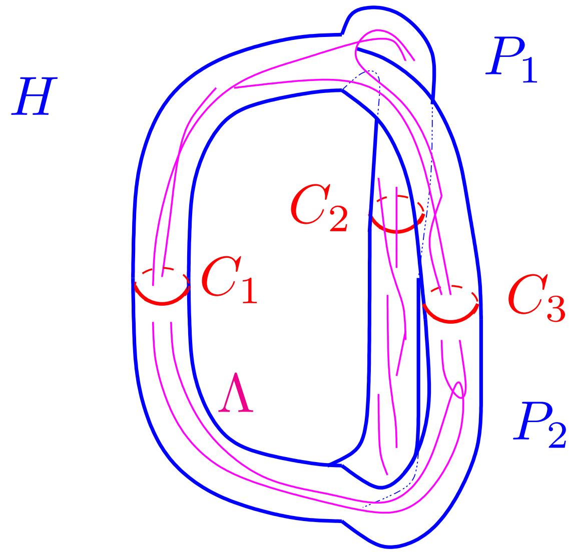

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The result is a link exterior $S^3 \setminus L$.

$\mathrm{Isom}^+(S^3 \setminus L) \subseteq G$

$\mathcal{H} = H \setminus (C_1 \cup C_2 \cup C_3 \cup \Lambda)$. $P_1 \cup P_2 = \partial\mathcal{H}$

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- Let $h \in \mathrm{Isom}^+(S^3 \setminus L)$.
- Let $\eta \subset \mathcal{H}$ be the shortest geodesic (together with $g\,\eta$, $g \in G$).
- Can assume $h\eta = \eta \Rightarrow h\,\mathcal{H} \cap \mathcal{H} \neq \emptyset$.

Claim: $\boxed{h\,\mathcal{H} = \mathcal{H}}$

(since $\mathcal{H}$ is rigid $\Rightarrow h = Id$).

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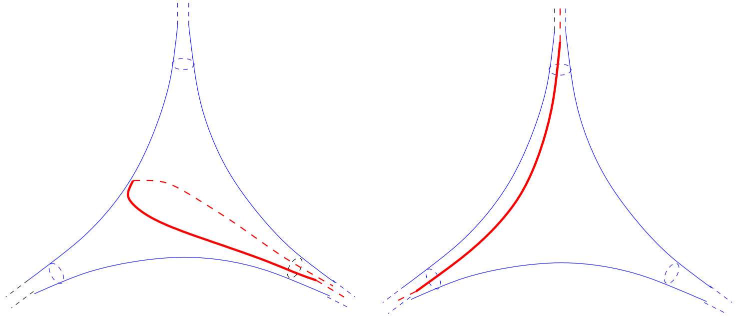
- $P_i \cap h\,P_j \neq \emptyset$ because $\begin{cases} \mathcal{H} \text{ does not contain proper pants.} \\ h\,\mathcal{H} \cap \mathcal{H} \neq \emptyset \end{cases}$
- Want to show: $P_i = h\,P_j$ by looking at intersections $P_i \cap h\,P_j \neq \emptyset$

Proof that $P_i = h\,P_j$

If $P_i \neq h\,P_j$, then $P_i \cap h\,P_j \neq \emptyset$ is a union of geodesics.

2 kinds of geodesics in the pants:



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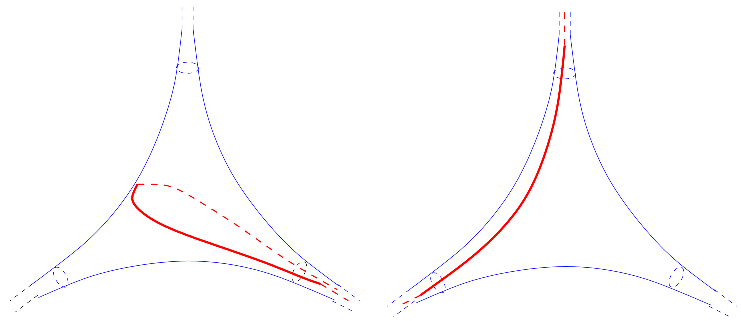
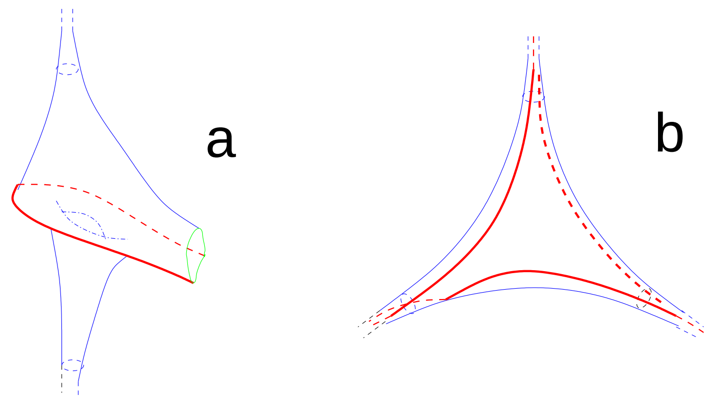


Since $P_1 \cup P_2 = \partial\mathcal{H}$ separates, $h\,P_j \cap \partial\mathcal{H}$ is one of:

P_j



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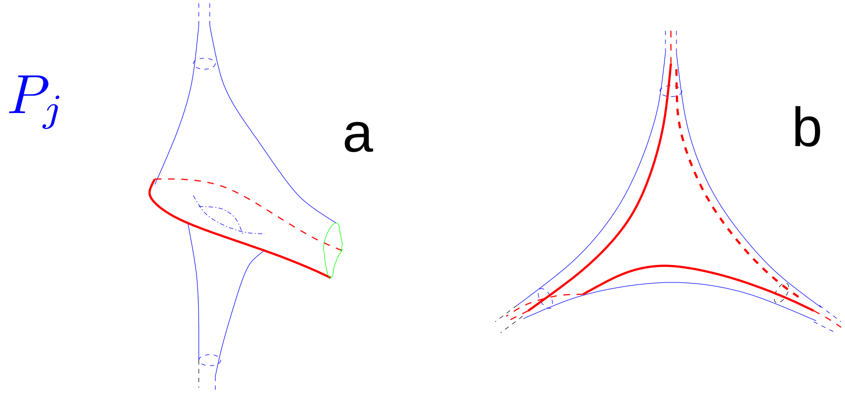
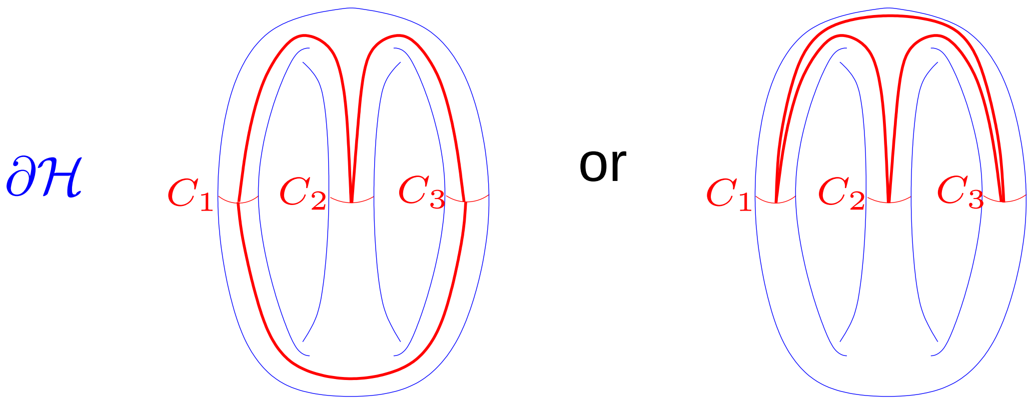


Case a: annulus in $\overline{\mathcal{H}}$

Case b: disc in $\overline{\mathcal{H}}$. The intersection of the disc with $\partial\mathcal{H}$ is:



Rational homology spheres

Theorem (D. Cooper and D. Long):

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The same argument, removing curves $\mathbf{Q}$-homologous to 0 and doing $1/n_i$ surgery with $n_i \gg 1$, gives:

$$G = \mathrm{Isom}^+(M^3)$$

because an isometry of M^3 induces an isometry of M^3 minus the shortest geodesics.