Three-Dimensional Orbifolds and their Geometric Structures

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This volume is dedicated to
Professor Laurent Siebenmann
for his 65th birthday

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Introduction

In this book, we present important recent results on the geometry and topology of 3-dimensional manifolds and orbifolds. Orbifolds are natural generalizations of manifolds, and can be roughly described as spaces which locally look like quotients of manifolds by finite group actions. They were introduced by I. Satake, under the name V-manifold, and their importance in dimension 3 emerged from the work of W. Thurston, who used them as tools for geometrizing 3-manifolds. Orbifolds occur in many contexts, for instance as orbit spaces of group actions on manifolds, or as leaf spaces of certain foliations.

A basic idea behind geometrization is the concept of *uniformization,* which for us means studying a manifold M by putting a structure on its universal cover $\tilde{M}$ that is preserved by the action of the fundamental group $\pi_1 M$. If the structure is rigid enough, this gives information about M. More specifically, we shall call *geometry* a homogeneous, simply-connected, unimodular Riemannian manifold, and say that a manifold is *geometric* if it is diffeomorphic to the quotient of a geometry by a discrete subgroup of its isometry group.

It has been known since the beginning of the twentieth century that every compact surface is geometric: more precisely, it is either elliptic, Euclidean or hyperbolic, i.e. can be obtained as the quotient of the round 2-sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{E}^2$, or the hyperbolic plane $\mathbf{H}^2$ by a discrete group of isometries.

Some important properties of surfaces, e.g. linearity of the fundamental group, can be deduced from this fact. Geometric structures on surfaces can also be used to attack more difficult and subtle problems such as studying mapping class groups. Moreover, the Gauss-Bonnet formula provides a strong link between geometry and topology in dimension 2.

In dimension 3, it is fairly easy to see that not every manifold is geometric. However, it was W. Thurston's groundbreaking idea that the situation should be almost as nice: any compact 3-manifold should be uniquely decomposable along a finite collection of disjoint embedded surfaces into geometric pieces. This is the content of his Geometrization Conjecture, formulated in the mid seventies, and which we shall state more precisely in Chapter 1. Positive solutions of many important problems in 3-manifold topology, including the famous Poincaré Conjecture, as well the Universal Cover Conjecture, or residual finiteness of 3-manifold groups, would follow from the Geometrization Conjecture.

Thurston observed that there are only eight 3-dimensional geometries: those of constant curvature $\mathbf{S}^3$, $\mathbf{E}^3$, and $\mathbf{H}^3$; the product geometries $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$; the twisted product geometries **Nil** and $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, and finally **Sol**. Among geometric manifolds, those modelled on $\mathbf{H}^3$ remain the more mysterious. Thus the Geometrization Conjecture reduces in principle any problem on 3-manifolds to combination theorems and understanding hyperbolic manifolds. Hence Thurston's work entailed a shift of emphasis from the purely topological (combinatorial) methods of the 50's and 60's toward geometric methods. It not only offers an approach to old topological problems, but also motivates the study of geometric ones. In particular, it renewed Kleinian group theory, which before Thurston was mainly considered from the point of view of complex analysis, by bringing hyperbolic geometry and topology into it. This is still an active field of research.

The Geometrization Conjecture is known to hold in various cases. The first breakthrough was Thurston's Hyperbolization Theorem, which covers an important and fairly general class of 3-manifolds called Haken manifolds. Since knot exteriors are included in this class, this result had spectacular applications to knot theory, leading for instance to the solution of the Smith Conjecture.

The Geometrization Conjecture is also true for prime 3-manifolds whose fundamental group contains a subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$, by combining the result mentioned above with the full version of the Torus Theorem, including the solution of the Seifert Fiber Space Conjecture. Lastly, it is known for a class of 'manifolds with symmetries', i.e. manifolds with finite group actions satisfying certain properties. The geometrization of these manifolds is reduced to the geometrization of the quotient orbifolds, which is the content of the Orbifold Theorem.

The main purpose of this book is to present those results and some of the ideas and techniques involved in their proofs. Some parts are covered in detail, while others are only sketched. We have tried to give a hint of the various methods and of the various parts of mathematics they draw ideas from: this includes geometric topology, algebraic and differential geometry, and geometric group theory. At several points we indicate connections with other fields in the form of short surveys, references to the literature or open questions. We also supply some background material that is scattered in the literature or missing from it.

The classification of the eight homogeneous 3-dimensional geometries is given in Chapter 1. Chapter 2 provides background material for orbifold theory. The existence of the canonical decomposition is established in Chapter 3, while in Chapter 4 we present the fundamental properties of the class of Haken orbifolds. Chapter 5 is concerned with a homotopic characterization of Seifert fibered orbifolds, which is an important case of the Geometrization Conjecture. Chapter 6 is devoted to hyperbolic orbifolds and Thurston's Hyperbolization Theorem for Haken Orbifolds. In Chapter 7 we discuss the basic properties of representation varieties and the Culler-Shalen theory of ideal points of curves. Chapter 8 deals with Thurston's construction of hyperbolic manifolds by Dehn filling and the structure of the set of volumes of hyperbolic 3-orbifolds. Finally, a proof of the Orbifold Theorem in a special case is outlined in Chapter 9.

We do not present here G. Perelman's recent breakthrough in R. Hamilton's program for proving the Geometrization Conjecture using the so-called Ricci flow equation. This approach relies on techniques from differential geometry and global analysis which are outside the scope of this book.

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Chapter 1

Thurston's eight geometries

In this chapter we present Thurston's Geometrization Conjecture and explain its interaction with some important problems in the topology and geometry of 3-manifolds. We also give the classification of the eight homogeneous 3-dimensional geometries involved in the Geometrization Conjecture.

By Moise's Theorem [163] each topological 3-manifold admits a unique PL or smooth structure. Hence throughout this monograph we will work in the category of differentiable manifolds.

1.1 The Geometrization Conjecture

Recall that a Riemannian manifold X is called *homogeneous* if its isometry group $\mathrm{Isom}(X)$ acts transitively. We call X *unimodular* if it has a quotient of finite volume.

A *geometry* is a simply connected, complete, homogeneous, unimodular Riemannian manifold satisfying the following maximality condition: there is no $\mathrm{Isom}(X)$-invariant Riemannian metric on X whose isometry group is strictly larger than $\mathrm{Isom}(X)$. Two geometries X, X' are *equivalent* if there is a diffeomorphism $\phi : X \to X'$ conjugating $\mathrm{Isom}(X)$ and $\mathrm{Isom}(X')$. Notice that ϕ is not required to be an isometry, nor even a homothety.

Let X be a geometry. If Γ is a discrete subgroup of $\mathrm{Isom}(X)$ acting freely, then the quotient space X/Γ is a smooth manifold with a natural Riemannian metric which is locally isometric to X. If the action is not free, then the quotient has a natural *orbifold* structure, as we will see in Chapter 2.

Let M be a smooth manifold (possibly with boundary). We say that M *admits an X-structure* if $\mathrm{Int}\, M$ is diffeomorphic to some quotient X/Γ as above.

A manifold is *geometric* if it admits an X-structure for some geometry X.

A geometry X is *isotropic* if $\mathrm{Isom}(X)$ acts transitively on the unit tangent bundle T_1X. Intuitively, this means that X looks the same in every direction. This condition is equivalent to requiring that X has constant sectional curvature. A classical result in Riemannian geometry (see e.g. [247]) asserts that in every dimension $n \geq 2$, there are exactly three isotropic geometries up to equivalence. These are the n-sphere $\mathbf{S}^n$, Euclidean n-space $\mathbf{E}^n$ and hyperbolic n-space $\mathbf{H}^n$, with constant sectional curvature equal to respectively +1, 0, and -1. The fact that these spaces are unimodular is obvious for $\mathbf{S}^n$ (which is compact) and $\mathbf{E}^n$ (which for each dimension n admits as compact quotient the n-torus $\mathbf{T}^n = \mathbf{E}^n/\mathbf{Z}^n$). This is nontrivial for $\mathbf{H}^n$ (see e.g. [24, 236]).

A n-manifold is called *spherical* (resp. *Euclidean*, resp. *hyperbolic*) if it has a $\mathbf{S}^n$-structure (resp. a $\mathbf{E}^n$-structure, resp. a $\mathbf{H}^n$-structure). For *closed* manifolds, these three situations are mutually exclusive. Indeed, if M is spherical, then $\pi_1 M$ is finite, so M cannot be Euclidean or hyperbolic. If M is Euclidean, then a theorem of Bieberbach (again see [247]) asserts that $\pi_1 M$ has an abelian subgroup of finite index, which implies that M cannot be hyperbolic.

The situation in dimension 2 is very special. Indeed, the three isotropic geometries are the only ones; furthermore, every closed surface is geometric, i.e. either spherical, Euclidean, or hyperbolic. This last fact can be proved by direct construction once one knows the classification of surfaces, or deduced from the Poincaré-Koebe Uniformization Theorem (see the discussion in [20].)

The fact that a closed surface F cannot have two structures modelled on inequivalent geometries admits a more elementary proof than the one quoted above for isotropic geometries in general dimension n. Indeed, it is a direct consequence of the Gauss-Bonnet formula $\chi(F) = \int_F K\, ds$, where $\chi(F)$ is the Euler characteristic. The situation is therefore particularly nice: F is elliptic if and only if $\chi(F) > 0$ (this gives $\mathbf{S}^2$ and $\mathbf{RP}^2$), Euclidean if and only if $\chi(F) = 0$ (this gives the 2-torus $\mathbf{T}^2$ and the Klein Bottle $\mathbf{K}^2$), and hyperbolic otherwise. We shall see in Chapter 2 a more general statement for 2-dimensional orbifolds (cf. Theorem 2.10).

In dimension 3 the situation is more complicated. Beside the three isotropic geometries $(\mathbf{S}^3, \mathbf{E}^3, \mathbf{H}^3)$, there are five anisotropic 3-dimensional geometries: four geometries are straight line bundles over $\mathbf{S}^2, \mathbf{E}^2$ or $\mathbf{H}^2$ $(\mathbf{S}^2 \times \mathbf{R}, \mathbf{Nil}, \mathbf{H}^2 \times \mathbf{R}, \widetilde{\mathrm{SL}_2}(\mathbf{R}))$, and one geometry is modelled on the only simply connected unimodular Lie group *Sol* which is solvable, but not nilpotent. This classification is explained in Section 1.2.

Thurston's fundamental idea is that geometry should take a central part in the study of compact, orientable 3-dimensional manifolds, through decompositions of these manifolds into canonical geometric pieces. He proposed the following conjecture:

Conjecture 1.1 (Geometrization Conjecture). *The interior of any compact orientable 3-manifold can be split along a finite collection of essential disjoint embedded spheres and tori into a canonical collection of* 3*-submanifolds* $X_1, \ldots, X_n$ *such that for each* i*, the manifold obtained from* X_i *by capping off all sphere components by balls is geometric.*

In the previous statement, an embedding of a closed connected surface in a compact orientable 3-manifold M is called *essential* if it induces an injective homomorphism of fundamental groups and if it does not bound a 3-ball nor cobounds a product with a connected component of ∂M.

A special case of the Geometrization Conjecture is the well-known Poincaré Conjecture. It claims the positive answer to a question raised by Poincaré in 1904 [184], and is one of the leading open problems in low dimensional topology.

Conjecture 1.2 (Poincaré Conjecture). *Any closed, simply-connected* 3*-manifold is homeomorphic to* $\mathbf{S}^3$*.*

More generally, the Geometrization Conjecture would imply that every closed, orientable, aspherical 3-manifold is determined, up to homeomorphism, by its fundamental group. This is a special case of the so-called Borel conjecture and will be discussed further in Section 4.4.

The groups which are fundamental groups of compact surfaces are known. The Poincaré-Koebe Uniformization Theorem shows that the fundamental group of a surface acts isometrically on the round sphere $\mathbf{S}^2$, the Euclidean plane $\mathbf{E}^2$ or the hyperbolic plane $\mathbf{H}^2$. This geometric action is reflected in algebraic properties of the group. For instance, it provides solutions of the word problem and the conjugacy problem. By contrast, any finitely presented group is the fundamental group of some compact 4-manifold.

Characterizing algebraically the class of fundamental groups of compact 3-manifolds is still an open problem. If M is a compact orientable 3-manifold satisfying the conclusion of the Geometrization Conjecture, then $\pi_1 M$ is the fundamental group of a graph of groups whose vertices are discrete subgroups of isometries of the 3-dimensional geometries above, and edges are trivial or isomorphic to $\mathbf{Z}^2$. One can deduce from this the solvability of the word and the conjugacy problems for these groups, see [64, 189]. In general these two questions are still unsolved for the fundamental group of a compact 3-manifold.

The topological background for Thurston's Geometrization Conjecture is given by a splitting of the compact, orientable 3-manifold along a finite collection of disjoint essential spheres and tori into canonical pieces. The existence of this decomposition is a central result in the study of 3-manifolds, which is presented in a more general context in Chapter 3.

An orientable 3-manifold M is *irreducible* if any embedding of the 2-sphere into M extends to an embedding of the 3-ball into M. This notion is crucial for the study of topological properties of 3-manifolds. The *connected sum* of two orientable 3-manifolds is the orientable 3-manifold obtained by pulling out the interior of a 3-ball in each manifold and gluing the remaining parts together along the boundary spheres.

The first stage of the decomposition, due to H. Kneser [128] and J. Milnor [159], expresses any compact, orientable 3-manifold M as the connected sum of 3-manifolds that are either homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^2$ or irreducible. Moreover, the connected summands are unique up to order and orientation-preserving homeomorphism.

The second stage is more subtle. Let M be a compact, orientable, irreducible 3-manifold. An embedded torus in M is called *canonical* if it can be isotoped off any embedded torus. Then a maximal, finite (maybe empty) collection of disjoint, non-parallel, essential, canonical tori exists and is unique up to isotopy. It cuts M into 3-submanifolds that are *homotopically atoroidal* or *Seifert fibered,* where the definitions are as follows.

A compact orientable 3-manifold M is *homotopically atoroidal* if $\pi_1 M$ is not virtually abelian and if every subgroup of $\pi_1 M$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ is conjugated to a subgroup of the fundamental group of a component of ∂M. It is *Seifert fibered* if it admits a foliation by circles such that each circle has a saturated tubular neighborhood. We will see in Section 1.3 that every Seifert fibered manifold has a geometric structure modelled on one of the six geometries: $\mathbf{S}^3, \mathbf{E}^3, \mathbf{S}^2 \times \mathbf{R}, \mathbf{Nil}, \mathbf{H}^2 \times \mathbf{R}, \widetilde{\mathrm{SL}_2(\mathbf{R})}$.

This decomposition along canonical essential tori, also called JSJ-decomposition, was first proved by W. Jaco and P. Shalen [115] and K. Johannson [118] for *Haken manifolds*, i.e. compact, orientable, irreducible 3-manifolds that contain closed essential surfaces or have a non-empty boundary.

The general case is implied by the following result, which is an important step towards the Geometrization Conjecture:

Theorem 1.3 (Torus Theorem). *Let M be a compact, orientable 3-manifold. If M is not homotopically atoroidal, then either M has a non-trivial JSJ-decomposition or M is Seifert fibered.*

This theorem answers a long-standing conjecture and is the conjunction of the works of several authors, using a wide range of techniques (A. Casson and D. Jungreis [41], D. Gabai [77], G. Mess [158], P. Scott [199, 200] and P. Tukia [234]); see Chapter 5.

From the Kneser-Milnor prime decomposition, the JSJ-decomposition and the torus theorem, one sees that Thurston's Geometrization Conjecture reduces to the case of homotopically atoroidal 3-manifolds. It splits into two uniformization conjectures dealing with the spherical or hyperbolic geometries, according to whether the fundamental group is finite or not.

Conjecture 1.4 (Elliptization). *A closed, orientable* 3*-manifold is spherical if and only if its fundamental group if finite.*

Conjecture 1.5 (Hyperbolization). *A compact, orientable* 3*-manifold is hyperbolic if and only if it is homotopically atoroidal and has infinite fundamental group.*

Conjecture 1.4 covers two difficult conjectures, the Poincaré Conjecture 1.2 and the Spherical Space Form Conjecture. The latter asserts that a closed 3-manifold covered by the sphere $\mathbf{S}^3$ is spherical.

Thurston's fundamental contribution to his Geometrization Conjecture is the proof of the Hyperbolization Conjecture for the important class of Haken manifolds, see Chapter 6, Section 6.3:

Theorem 1.6 (Hyperbolization of Haken manifolds). *A Haken* 3*-manifold is hyperbolic iff it is homotopically atoroidal.*

The discussion above implies the following:

Corollary 1.7. *The Geometrization Conjecture holds for Haken manifolds.*

The proof of Thurston's Hyperbolization Theorem is long and difficult. A detailed proof can be found in the monographs [120], [176, 177] and the articles [230, 231, 232], [151, 152], [165]; see [14] and [153] for an overview.

More recently an equivariant version of the Geometrization Conjecture has been established for compact, orientable, irreducible 3-manifolds admitting non-free actions of finite groups of orientation preserving diffeomorphisms, see [16, 17], [43], [229] and Chapter 9.

Theorem 1.8. *Let M be a compact, orientable and irreducible 3-manifold. Let $G \subset \mathrm{Diff}^+(M)$ be a finite, non-trivial subgroup acting on M with non-trivial stabilizers. Then there exists a (possibly empty) G-invariant collection of* *disjoint and non-parallel canonical tori which splits M into geometric pieces on which G acts isometrically.*

By Moise's theorem [163] each 3-manifold can be triangulated. It follows that each closed orientable 3-manifold M is obtained by gluing two handlebodies along their boundaries; for one of the handlebodies one can take a regular neighborhood of the 1-skeleton of a triangulation of M. The *Heegaard genus* of M is the smallest possible genus for the handlebodies in such a decomposition of M. For example M has Heegaard genus 0 iff M is the 3-sphere $\mathbf{S}^3$. It has Heegaard genus 1 iff it is $S^1 \times S^2$ or a lens space (i.e. a quotient of $\mathbf{S}^3$ by a free, cyclic, orthogonal action). In particular, manifolds with Heegaard genus zero or one are geometric.

Since the hyperelliptic involution on a genus 2 surface is central in the mapping class group, it extends on both sides to an involution on M with non-empty fixed point set. Hence a straightforward corollary of Theorem 1.8 is:

Corollary 1.9. *The Geometrization Conjecture holds true for a compact orientable* 3*-manifold of Heegaard genus at most two.*

The natural object to consider for the proof of Theorem 1.8 is the quotient of the manifold M by the group G, equipped with its so-called orbifold structure: this structure records the nonfree group action.

Roughly speaking an orientable n-dimensional *orbifold* is a metrizable space where each point has a neighborhood diffeomorphic to the quotient of $\mathbf{R}^n$ by a finite subgroup of $\mathrm{SO}(n)$. This local finite subgroup varies from point to point, and the set of points where it is not trivial is called the *singular locus.* A precise definition of orbifold is given in Chapter 2.

The notion of orbifold has been introduced by I. Satake [205]. It extends naturally the classical notion of manifold. An orbifold is a manifold iff its singular locus is empty. An orbifold is not necessarily globally the orbit space of a finite group action or even of a properly discontinuous group action on some manifold. If it is the case, the orbifold is called *good*; it is called *bad* otherwise.

All basic notions for manifolds, like map, homotopy, isotopy, covering and fundamental group, extend to the category of orbifolds, see Chapter 2. In the case of good orbifolds, these notions correspond to the equivariant notions in the universal covering, which is a manifold.

If a compact, orientable 3-orbifold does not contain any bad 2-suborbifold, then it admits a splitting along a finite collection of disjoint essential spherical and toric 2-suborbifolds into canonical 3-suborbifolds, where *spherical* and *toric*

2-suborbifolds are defined as finite quotients of a 2-sphere or a 2-torus. This splitting is explained in Chapter 3, see also [22].

These canonical building blocks out of which the 3-orbifold may be constructed are conjectured to be geometric, i.e. to be the orbit spaces of one of the eight geometries by a discrete (but not necessarily torsion free) group of isometries. This is the natural extension of Thurston's Geometrization Conjecture to the setting of orbifolds without bad 2-suborbifold. This version of the Geometrization Conjecture is presented in Chapter 3, Section 3.7.

In the orbifold context, Theorem 1.8 is a special case of a geometrization theorem for orbifolds, called the *Orbifold Theorem.* This theorem is presented in Chapter 9, where a proof is outlined under some simplifying hypotheses.

1.2 3-dimensional geometries

In this section we present the eight 3-dimensional geometries, in decreasing order of the size of the isotropy group. For each geometry, we give a few examples of closed 3-manifolds modelled on it, showing that it is unimodular. General references for this section are [198, 224].

Isotropic geometries

- Spherical geometry $\mathbf{S}^3$.

  The isometry group is the orthogonal group O(4). Manifolds with spherical geometry have finite fundamental group and are classified by their fundamental group and Reidemeister torsion. Classical examples are $\mathbf{S}^3$ itself, lens spaces (see [106]), and the Poincaré homology sphere.

- Euclidean geometry $\mathbf{E}^3$.

  The group of isometries (also called rigid motions of Euclidean space) is the semidirect product

  $$\mathrm{Isom}(\mathbf{E}^3) \cong \mathbf{R}^3 \rtimes \mathrm{O}(3)$$

  where $\mathbf{R}^3$ acts by translations and O(3) by rotations. There are only six closed orientable manifolds with this geometry, and four non-orientable ones. As a consequence of Bieberbach's Theorem (1911), all of those manifolds are finitely covered by the 3-torus $\mathbf{T}^3$.

- Hyperbolic geometry $\mathbf{H}^3$.

This is the richest, and currently least understood of the 8 geometries. In fact, one can give an explicit list of 3-manifolds having a fixed geometry, except for hyperbolic geometry. A geometric manifold admits a collapsing sequence of metrics with pinched curvature (cf. Chapter 9) if and only if it is not hyperbolic. Hyperbolic geometry will be discussed in some detail in Chapter 6.

A well-known example of a closed hyperbolic 3-manifold is the Seifert-Weber dodecahedral space, found in 1933 [244]. Another (not so well-known) example was published by Löbell in 1931 [135]. Examples of hyperbolic 3-manifolds appear also in the context of arithmetic geometry [25].

Trivial products

- $\mathbf{S}^2 \times \mathbf{E}^1$.

  The isometry group of this geometry is the cartesian product $\mathrm{Isom}(\mathbf{S}^2) \times \mathrm{Isom}(\mathbf{E}^1)$. There are only two closed, orientable 3-manifolds modelled on this geometry: $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{RP}^3 \sharp \mathbf{RP}^3$.

- $\mathbf{H}^2 \times \mathbf{E}^1$.

  Again the isometry group is a product $\mathrm{Isom}(\mathbf{H}^2) \times \mathrm{Isom}(\mathbf{E}^1)$. Products of closed hyperbolic surfaces with $\mathbf{S}^1$ are examples of manifolds with this geometry. In fact, each closed $\mathbf{H}^2 \times \mathbf{E}^1$-manifold is a finite quotient of such an example. Alternatively, $\mathbf{H}^2 \times \mathbf{E}^1$-manifolds can be described as Seifert fiber spaces (cf. below).

Twisted products

- **Nil**.

  The geometry **Nil** is a line bundle over the Euclidean plane $\mathbf{E}^2$:

  $$\mathbf{R} \to \mathbf{Nil} \to \mathbf{E}^2$$

  equipped with a connection of constant curvature 1 and a homogeneous Riemannian metric such that:

  - the connection is Riemannian (horizontal directions are perpendicular to vertical ones);
  - the metric on horizontal planes is the pullback of the Euclidean metric on $\mathbf{E}^2$;

– the metric is invariant by the action of $\mathbf{R}$ by vertical translations.

The isometry group is 4-dimensional, generated by lifts of the isometries of $\mathbf{E}^2$ and vertical translations. Since the isometry group preserves the connection, it also preserves the orientation of **Nil** (although it can reverse the orientation of the base $\mathbf{E}^2$). Thus manifolds with geometry **Nil** are all orientable. As examples we have $\mathbf{S}^1$-bundles over $\mathbf{T}^2$ with non-zero Euler class.

An alternative way to describe this geometry is to consider the Heisenberg group:

$$\mathbf{Nil} = \{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R}\}$$

so that the projection $\mathbf{Nil} \to \mathbf{R}^2$ maps each matrix to $(x, y) \in \mathbf{R}^2$. The isometry group is an extension of the Heisenberg group itself acting on the left.

- $\widetilde{\mathrm{SL}_2(\mathbf{R})}$.

  This is a line bundle over the hyperbolic plane

$$\mathbf{R} \to \widetilde{\mathrm{SL}_2(\mathbf{R})} \to \mathbf{H}^2.$$

Again we equip the bundle with a connection of non-zero constant curvature and consider the natural homogeneous metric associated to it: it is the lift of the homogeneous metric on $\mathbf{H}^2$ on horizontal planes, the homogeneous metric on vertical fibers, and fibers are orthogonal to horizontal planes.

We now explain how this bundle can be identified with the universal covering $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ of $\mathrm{PSL}_2(\mathbf{R})$ with the natural metric. The action of $\mathrm{PSL}_2(\mathbf{R})$ on $\mathbf{H}^2$ induces a faithful action on the unit tangent bundle $\mathrm{T}_1\mathbf{H}^2$, hence $\mathrm{PSL}_2(\mathbf{R}) \cong \mathrm{T}_1\mathbf{H}^2$. Thus we have a natural identification between universal coverings, that identifies $\widetilde{\mathrm{PSL}_2(\mathbf{R})} = \widetilde{\mathrm{SL}_2(\mathbf{R})}$ with a line bundle. To see why this bundle is the one we described in the previous paragraph, note that $\mathbf{H}^2$ has constant curvature -1, hence the natural connection on $\mathrm{T}_1\mathbf{H}^2$ has non-zero constant curvature.

The isometry group of this geometry is also 4-dimensional, generated by lifts of isometries of $\mathbf{H}^2$ and "vertical" isometries of the fibers $\mathbf{R}$. More explicitly:

$$\mathrm{Isom}(\widetilde{\mathrm{SL}_2(\mathbf{R})}) \cong (\widetilde{\mathrm{SL}_2(\mathbf{R})} \times \widetilde{\mathrm{O}(2)})/\mathbf{Z}$$

where $O(2) \subset PSL_2(\mathbf{R})$ is the stabilizer of a point and $\pi_1(PSL_2(\mathbf{R})) \cong \pi_1(SO(2)) \cong \mathbf{Z}$. Again the isometry group preserves the orientation, because it preserves the connection, and therefore manifolds with this geometry are orientable.

Typical examples are unit tangent bundles of hyperbolic surfaces.

- **Sol**.

  Sol is a solvable Lie group given by the split extension $\mathbf{R}^2 \to \mathbf{Sol} \to \mathbf{R}$, where $t \in \mathbf{R}$ acts on $\mathbf{R}^2$ by

  $$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

  We equip **Sol** with a left invariant metric, so that the eigenspaces of $\mathbf{R}^2$ are orthogonal, and the fibers are also orthogonal to the section $0 \times \mathbf{R}$. The isometry group has eight components, corresponding to the group generated by the reflections in the $\mathbf{R}$ direction and in the directions of the eigenspaces in $\mathbf{R}^2$, i.e. the group $(\mathbf{Z}/2\mathbf{Z})^3$. The connected component of the identity $\mathrm{Isom}(\mathbf{Sol})_0$ is **Sol** itself, thus:

  $$\mathrm{Isom}(\mathbf{Sol}) \cong \mathbf{Sol} \rtimes (\mathbf{Z}/2\mathbf{Z})^3.$$

  As examples there are torus bundles over $\mathbf{S}^1$ with Anosov monodromies, i.e. given by matrices conjugate to

  $$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda > 1.$$

Proposition 1.10 (Classification of 3-dimensional geometries). *Every 3-dimensional geometry is equivalent to one of the eight geometries described above, and no two geometries on this list are equivalent.*

Sketch of proof. Consider the pair (X, G) where X is a geometry and $G \cong \mathrm{Isom}(X)_0$ is the component of the identity of $\mathrm{Isom}(X)$. Let $G_x \subset G$ be the stabilizer of $x \in X$. Since G_x is connected, there are only three possibilities: $G_x \cong SO(3)$, $G_x \cong SO(2)$ or $G_x \cong SO(1) \cong \{\mathrm{id}\}$.

Case 1 $G_x \cong SO(3)$

Then X is isotropic, so X is equivalent to either $\mathbf{S}^3$, $\mathbf{E}^3$ or $\mathbf{H}^3$.

Case 2 $G_x \cong \mathrm{SO}(2)$

Then for each $x \in X$, $\mathrm{T}_x X$ splits as $\Lambda_x \perp P_x$, where Λ_x is a line invariant by G_x and P_x is an invariant plane orthogonal to Λ_x.

Because X is simply connected, we can choose a coherent orientation on Λ_x. We obtain a unit vector field $\vec{v}$ on X which is G-invariant, i.e. $\vec{v}_{g(x)} = Dg_x(\vec{v}_x)$ for all $g \in G$. Moreover, $\{P_x\}_{x \in X}$ is a G-invariant plane field.

Claim. *The vector field $\vec{v}$ is isometric (i.e. the induced flow Φ_t is isometric).*

Proof. Let L_x denote the trajectory of the induced flow Φ_t through x. Then for each $y \in L_x$ we have $G_y \cong G_x \cong G_L$, because Φ_t commutes with the action of G.

Given $x \in X$ and $\Phi_t(x) \in L_x$, let $g \in G$ be such that $g(\Phi_t(x)) = x$. We want to show that $D(g\Phi_t)_x : \mathrm{T}_x X \to \mathrm{T}_x X$ is an isometry. Now $D(g\Phi_t)_x$ is the identity on Λ_x and commutes with the action of G_x by rotations, so it is a composition of a rotation around Λ_x and a homothety on P_x. We only need to show that the homothety on P_x is the identity. This is a consequence of the existence of a finite volume quotient M. Indeed, the flow Φ_t induces a flow $\overline{\Phi_t}$ on M which preserves the volume and so $\overline{\Phi_t}$ must transversally preserve the area (on the induced plane field $\overline{P_x}$). Therefore, the flow Φ_t cannot expand or contract a direction on the plane field $\overline{P_x}$. □

We remark that the leaves L_x of this vector field do not accumulate, as they are the fixed point set of G_x. Using that the flow Φ_t is isometric, we see that the leaf space $Y := X/\{x \sim \Phi_t(x)\}$ is Hausdorff; in fact one can find a saturated tubular neighborhood of each trajectory. Then Y is a 2-dimensional Riemannian manifold for the induced metric, which is homogeneous. Moreover, Y is simply connected and complete. Hence Y is equivalent to either $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$. The projection $X \to Y$ is a Riemannian line or circle bundle. The plane field $\{P_x\}_{x \in X}$ gives a G-invariant connection for this bundle. Its curvature is constant since X is homogeneous, so after rescaling we may assume it is 0 or +1.

	$\{P_x\}$ integrable, curv. $= 0$	$\{P_x\}$ non-int., curvature $= +1$
$K \equiv 1, Y = \mathbf{S}^2$	$\mathbf{S}^2 \times \mathbf{E}^1$	$\widetilde{\mathrm{T}_1\mathbf{S}^2} = \widetilde{\mathrm{SO}(3)} = \mathbf{S}^3$
$K \equiv 0, Y = \mathbf{E}^2$	$\mathbf{E}^2 \times \mathbf{E}^1 = \mathbf{E}^3$	**Nil** $(\neq \widetilde{\mathrm{T}_1\mathbf{E}^2})$
$K \equiv -1, Y = \mathbf{H}^2$	$\mathbf{H}^2 \times \mathbf{E}^1$	$\widetilde{\mathrm{SL}_2(\mathbf{R})} = \widetilde{\mathrm{T}_1\mathbf{H}^2}$

The "geometries" $\mathbf{E}^2 \times \mathbf{E}^1$ and $\widetilde{\mathrm{T}_1\mathbf{S}^2}$ are clearly not maximal (i.e. the isometry groups, which a priori have dimension 4, can be included in larger ones of dimension 6). Thus they are not geometries in our sense, and we are left with $\mathbf{S}^2 \times \mathbf{E}^1$, **Nil**, $\mathbf{H}^2 \times \mathbf{E}^1$ and $\widetilde{\mathrm{SL}_2(\mathbf{R})}$.

Case 3 G_x is trivial.

Since G acts transitively, $X \cong G/G_x = G$ and X is a unimodular Lie group (i.e. X has a bi-invariant measure).

Proof. Since X is unimodular, there exists a discrete subgroup Γ of G of finite covolume. Let D be a fundamental domain for the action of Γ and μ be the (left-invariant) Haar measure on X. Then $\mu(\gamma D \cap D) = 0$, for every $\gamma \neq \mathrm{id}$. It follows that $\mu(D)$ does not depend on the fundamental domain D. Since Dg is a fundamental domain for all $g \in G$, $\mu(Dg) = \mu(D)$. Now the divergence of right-multiplication by g is constant, so μ is G-right invariant. □

We are looking for the 3-dimensional unimodular Lie groups. Let $\mathfrak{y}$ be the 3-dimensional associated Lie algebra, $[x, y]$ the Lie bracket and $x \times y$ the cross product. There is a unique linear map $L : \mathfrak{y} \to \mathfrak{y}$ such that $L(x \times y) = [x, y]$. The Lie group is unimodular if and only if L is selfadjoint with respect to the left invariant metric (cf. [161]).

Choose an orthonormal positively oriented base $\{e_1, e_2, e_3\}$ consisting of eigenvectors for L, i.e. $Le_i = \lambda_i e_i$. We get $[e_1, e_2] = L(e_1 \times e_2) = L(e_3) = \lambda_3 e_3$ and analogously $[e_2, e_3] = \lambda_1 e_1$, $[e_3, e_1] = \lambda_2 e_2$. After rescaling and normalization: $\lambda_i \in \{-1, 0, 1\}$, $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

The list of possible groups is the following:

$(\lambda_1, \lambda_2, \lambda_3)$	$X = G$
$(+1, +1, +1)$	$\mathrm{SU}(2)$
$(-1, +1, +1)$	$\widetilde{\mathrm{SL}_2(\mathbf{R})}$
$(0, +1, +1)$	$\widetilde{\mathrm{Isom}(\mathbf{E}^2)} \cong \mathbf{E}^3$
$(-1, 0, +1)$	$\mathbf{Sol} = \mathrm{Isom}(E(1,1))$
$(0, 0, +1)$	**Nil**
$(0, 0, 0)$	$\mathbf{R}^3$

Here $E(1, 1)$ denotes the Minkowski plane: $\mathbf{R}^2$ with the pseudometric of signature (1,1).

Except for $G = \mathbf{Sol}$, all these Lie groups G have a metric with an isometry group of dimension ≥ 4. For instance $\mathrm{SU}(2) \cong \mathbf{S}^3$ has an isometry group of dimension 6. We also mention that the isomorphism $\widetilde{\mathrm{Isom}(\mathbf{E}^2)} \cong \mathbf{E}^3$ gives a Lie group structure on $\mathbf{E}^3$ which is not the usual one. It consists of isometries that preserve a fixed foliation by parallel lines. Hence **Sol** is the only geometry we get in this last case. □

1.3 Seifert fibered manifolds

Definition. A *Seifert fibration* on a compact, orientable 3-manifold M is a partition of M into circles, called fibers, such that each fiber has a saturated tubular neighborhood. A *Seifert fibered manifold* is a manifold that admits a Seifert fibration.

This notion generalizes that of circle fibration, allowing the existence of exceptional fibers around which the nearby fibers wind.

A key observation for the classification of spherical 3-manifolds is that if Γ is a finite subgroup of $\mathrm{O}(4)$ acting freely on $\mathbf{S}^3$, then Γ must preserve the orientation and commute with the action of some one-parameter orthogonal group of transformations ([222]; see also [221] or [224, section 4.4]). It follows that these manifolds admit a Seifert fibration.

In fact, it was this observation that led Seifert [203] to investigate in full generality the manifolds that bear his name. Beside $\mathbf{S}^3$, five other geometries yield Seifert fibered 3-manifolds.

Proposition 1.11. *Every closed manifold with geometry* $\mathbf{S}^3$, $\mathbf{E}^3$, $\mathbf{Nil}$, $\mathbf{S}^2 \times \mathbf{E}^1$, $\mathbf{H}^2 \times \mathbf{E}^1$ *or* $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ *is Seifert fibered.*

Sketch of proof. This is not so hard for geometries $\mathbf{Nil}$, $\mathbf{S}^2 \times \mathbf{E}^1$, $\mathbf{H}^2 \times \mathbf{E}^1$ and $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, because these geometries are fibered. For an elementary proof, see [198]. Here we sketch a more conceptual proof: any closed, connected manifold M with one of these geometries has a natural isometric flow. Since $\mathrm{Isom}(M)$ is compact, the closure G of this flow is a compact abelian group of isometries, and therefore isomorphic to a torus. If $\dim G = 1$, then one easily sees that M is Seifert fibered. If $\dim G = 2$, then M is a lens space or a torus bundle over $\mathbf{S}^1$, according to whether the orbit space of the flow is an interval or a circle. If $\dim G = 3$, then the orbit space of the flow is a point and $M \cong \mathbf{T}^3$.

For spherical and Euclidean 3-manifolds, the existence of a Seifert fibration can be deduced from the classification of spherical and Euclidean 3-manifolds [174, 198]. $\square$

Remark. There are orbifolds with geometry $\mathbf{E}^3$ and $\mathbf{S}^3$ which are not orbifold-Seifert fibered [54, 57].

To prove the converse of the previous proposition, we need some material from Chapter 2. A Seifert fibration may be viewed as an orbifold fibration by circles over a two dimensional orbifold, called the base. The following proposition will be proved in Chapter 2 for the general case of orbifolds, see Proposition 2.13 in Section 2.4.

Proposition 1.12. *If M is a Seifert fibered manifold, then it has geometry* $\mathbf{E}^3$*,* $\mathbf{S}^3$*,* $\mathbf{Nil}$*,* $\mathbf{S}^2 \times \mathbf{E}^1$*,* $\mathbf{H}^2 \times \mathbf{E}^1$ *or* $\widetilde{\mathrm{SL}_2(\mathbf{R})}$.

□

1.4 Large scale geometry

Definition. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that a map $f\colon X_1 \to X_2$ is a *quasi-isometry* if there exist $\lambda \geq 1$ and $C \geq 0$ such that:

i. The inequality

$$\lambda^{-1}\, d_1(x, x') - C \leq d_2(f(x), f(x')) \leq \lambda\, d_1(x, x') + C$$

holds for any $x, x' \in X_1$.

ii. Every point of X_2 is C-close to the image of f.

We say that (X_1, d_1) is *quasi-isometric* to (X_2, d_2) if there exists a quasi-isometry $f\colon X_1 \to X_2$.

This defines an equivalence relation between metric spaces. (To prove that the relation is symmetric one needs the axiom of choice.)

Two equivalent geometries are quasi-isometric.

Let Γ be a finitely generated group and S a generating set. We define a metric on Γ, called the *word metric*, by setting $d_S(\gamma_1, \gamma_2)$ equal to the least integer $n \geq 0$ such that $\gamma_1^{-1}\gamma_2$ can be written as a product of n elements of $S \cup S^{-1}$. It is easy to check that if S, S' are two generating sets for Γ, then (Γ, d_S) and $(\Gamma, d_{S'})$ are quasi-isometric.

Thus it makes sense to say that a finitely generated group is quasi-isometric to a geometry. In fact more is true. Recall that two groups Γ and Λ are *weakly commensurable* if there is a finite sequence of groups $\Gamma = \Gamma_1, \ldots, \Gamma_n = \Lambda$ such that Γ_{i+1} is isomorphic to a finite index subgroup of Γ_i, or to the quotient of Γ_i by a finite normal subgroup. Then one can show that two weakly commensurable, finitely generated groups are quasi-isometric.

The link between groups and geometries is provided by the following fundamental proposition due independently to Efremovič, Švarc and Milnor. (See [160] and also [86, 120, 32].)

Proposition 1.13. *Let X be a complete Riemannian manifold. Let Γ be a group acting properly and cocompactly on X by isometries. Then Γ is finitely generated and quasi-isometric to X.*

Hence if X is a geometry and M is a closed manifold with an X-structure, then $\pi_1 M$ is quasi-isometric to X. Thus we are led to two basic questions: classify geometries up to quasi-isometry, and determine which groups are quasi-isometric to a given geometry.

In dimension 2, the situation is as rigid as can be expected:

Proposition 1.14. *The three* 2*-dimensional geometries* $\mathbf{S}^2$, $\mathbf{E}^2$, $\mathbf{H}^2$ *are pairwise not quasi-isometric.*

Theorem 1.15. *Let* X *be a* 2*-dimensional geometry and* Γ *be a finitely generated group. Then* Γ *is quasi-isometric to* X *if and only if* Γ *is weakly commensurable to some cocompact discrete subgroup of* $\mathrm{Isom}(X)$.

Proposition 1.14 is not difficult. It is obvious that $\mathbf{S}^2$, being bounded as a metric space, is quasi-isometric to neither $\mathbf{E}^2$ nor $\mathbf{H}^2$. It is slightly less obvious that $\mathbf{E}^2$ and $\mathbf{H}^2$ are not quasi-isometric. To see this one may use Gromov hyperbolicity (cf. Chapter 6) or the notion of growth function of a group, defined below. By contrast, Theorem 1.15 is a deep result following from work of Gromov [93] for $\mathbf{E}^2$ and Tukia [234], Gabai [77], Casson-Jungreis [41] for $\mathbf{H}^2$. We shall have to say more about this in Chapters 5 and 6.

The *growth function* of a finitely generated group Γ with generating set S is the function $n \mapsto \#B_S(n)$ where $B_S(n)$ is the ball of radius n around the identity in the word metric associated to S. By finding an equivalent of the growth functions of $\mathbf{Z}^2$ and $\pi_1 F_g$, one can deduce that $\mathbf{E}^2$ and $\mathbf{H}^2$ are not quasi-isometric.

In dimension 3, things are not quite so nice.

Theorem 1.16 ([84]). *The eight* 3*-dimensional geometries fall into seven quasi-isometry classes. The only inequivalent geometries that are quasi-isometric are* $\mathbf{H}^2 \times \mathbf{R}$ *and* $\widetilde{\mathrm{SL}_2(\mathbf{R})}$.

Theorem 1.17 ([38, 178]). *Let* X *be a* 3*-dimensional geometry different from* $\mathbf{H}^2 \times \mathbf{R}$, $\widetilde{\mathrm{SL}_2(\mathbf{R})}$, **Sol**, *and* Γ *a finitely generated group. Then* Γ *is quasi-isometric to* X *if and only if* Γ *is weakly commensurable to some cocompact discrete subgroup of* $\mathrm{Isom}(X)$.

This theorem provides an approach to the hyperbolization conjecture: let M be an irreducible 3-manifold whose fundamental group is infinite and has no $\mathbf{Z}^2$ subgroup. Then M is aspherical, so $\pi_1 M$ is torsion free. Suppose that we manage to prove that $\pi_1 M$ is quasi-isometric to $\mathbf{H}^3$. Then Theorem 1.17 tells us that $\pi_1 M$ is weakly commensurable to a Kleinian group. In fact, since it is torsion free, it must be a Kleinian group. Then a theorem of Gabai-Meyerhoff-Thurston [80] implies that M is hyperbolic.

For groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$, there is a rigidity result due to E. Rieffel [191]. For other results on the theme of quasi-isometries and 3-manifolds, see the series of papers by M. Kapovich and B. Leeb [121, 122, 123, 124]. The quasi-isometry rigidity of **Sol** is a major open question (cf. [67].)

Chapter 2

Orbifolds

In this chapter, we give an exposition of the theory of orbifolds, with a bias towards low dimensions. We have tried to give a rather complete treatment of the general theory, which seemed to be missing from the literature. However, to keep the size of the book down, we leave many details as exercises.

2.1 Definitions

2.1.1 Orbifolds

We restrict our attention to smooth orbifolds. *Throughout the book, actions of discrete groups are smooth, unless stated otherwise.*

Definition. A (smooth) *n-orbifold* is a metrizable topological space $\mathcal{O}$ endowed with a collection $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_i$, called an *atlas,* where for each i, U_i is an open subset of $\mathcal{O}$, $\tilde{U}_i$ is an open subset of $\mathbf{R}^{n-1} \times [0, \infty)$, $\phi_i : \tilde{U}_i \to U_i$ is a continuous map (called a *chart*) and Γ_i is a finite group of diffeomorphisms of $\tilde{U}_i$ satisfying the following conditions:

i. The U_i's cover $\mathcal{O}$.

ii. Each ϕ_i factors through a homeomorphism between $\tilde{U}_i/\Gamma_i$ and U_i.

iii. The charts are compatible in the following sense: for every $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there is a diffeomorphism ψ between a neighborhood V of x and a neighborhood W of y such that $\phi_j(\psi(z)) = \phi_i(z)$ for all $z \in V$.

For convenience, we will always assume that the atlas is maximal.

Note that this definition extends the classical definition of a manifold. Thus we say that the orbifold $\mathcal{O}$ *is a manifold* if all the Γ_i's are trivial. Sometimes it will be necessary to distinguish between the orbifold $\mathcal{O}$ and its *underlying space,* i.e. the topological space obtained by forgetting the orbifold structure. When we want to make the distinction clear, we will denote this underlying space by $|\mathcal{O}|$. In many cases, $|\mathcal{O}|$ will be a manifold, even when $\mathcal{O}$ is not a manifold (in the sense above). We say that $\mathcal{O}$ is connected (resp. compact) if $|\mathcal{O}|$ is connected (resp. compact).

The *local group* of $\mathcal{O}$ at a point $x \in \mathcal{O}$ is the group Γ_x defined as follows: let $\phi : \tilde{U} \to U \ni x$ be a chart. Then Γ_x is the stabilizer of any point of $\phi^{-1}(x)$ under the action of Γ. It is well-defined up to isomorphism. If Γ_x is trivial, we say that x is *regular*, otherwise it is *singular*. The *singular locus* is the set $\Sigma_{\mathcal{O}}$ of singular points of $\mathcal{O}$. Notice that $\Sigma_{\mathcal{O}} = \emptyset$ if and only if $\mathcal{O}$ is a manifold. Since every smooth action of a finite group on a manifold is locally conjugate to an orthogonal action, local groups are isomorphic to subgroups of $\mathrm{O}(n)$. This fact can be used to study the structure of the singular locus.

Example. We start with an example $\mathcal{O}$ with underlying space a triangle $|\mathcal{O}| \cong T$. Points inside the edges of T are locally modelled on the quotient of $\mathbf{R}^2$ by a reflection. The vertices of T are modelled on the quotient of $\mathbf{R}^2$ by a dihedral group (generated by two reflections along two lines). In particular $\Sigma_{\mathcal{O}} = \partial T$.

Example. Here is a generic example that will appear repeatedly in this chapter. Fix a knot K in $\mathbf{S}^3$ and a natural number $n \geq 2$. Then there is a unique orbifold with underlying space $\mathbf{S}^3$, singular locus K and such that non-trivial local groups are cyclic of order n. The local model for all singular points is $\mathbf{R}^3$ with a cyclic group of rotations of order n.

More generally, if K is a link and each component K_i is marked with a number $n_i \geq 2$, then we can talk about the orbifold with underlying space $\mathbf{S}^3$, singular locus K, and such that points of K_i have local group a cyclic group of rotations of order n_i.

Several notions can be defined for orbifolds by extending the definition for manifolds in a rather straightforward way. The *boundary* of $\mathcal{O}$, denoted by $\partial\mathcal{O}$, is the set of points $x \in \mathcal{O}$ such that there is a chart $\phi_i : \tilde{U}_i \to U_i \ni x$ with $\phi_i^{-1}(x) \subset \mathbf{R}^{n-1} \times \{0\}$.[1] The orbifold $\mathcal{O} - \partial\mathcal{O}$ is called the *interior* of $\mathcal{O}$ and denoted by $\operatorname{Int}\mathcal{O}$.

[1]Note that the boundary of the underlying space $|\mathcal{O}|$ (when this makes sense, e.g. if $|\mathcal{O}|$ is a manifold) is in general different from the underlying space of $\partial\mathcal{O}$. For instance, the quotient of $\mathbf{R}^2$ by a reflection has empty boundary in the orbifold sense, but its underlying space is a half-plane.

When $\partial\mathcal{O} = \emptyset$, we say that $\mathcal{O}$ is *closed* if it is compact and *open* otherwise. We call $\mathcal{O}$ *orientable* if it has an atlas $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_i$ such that each ϕ_i and all elements of each Γ_i are orientation preserving.

2.1.2 Orbifold coverings

Let M be a manifold and Γ be a discrete group acting properly on M by diffeomorphisms. Then, as announced in the previous chapter, the quotient space M/Γ has a natural orbifold structure. Here, natural means that the projection map $M \to M/\Gamma$ is a covering map in the orbifold sense (defined below). An orbifold is called *good* if it is obtained in this way, and *bad* otherwise. It is *very good* if it is the quotient of a manifold by a *finite* group.

Example. We consider the involution τ with 4 fixed points in the 2-torus $\mathbf{T}^2$ as in Figure 2.1. The quotient $\mathcal{O} = \mathbf{T}^2/\tau$ has underlying space the 2-sphere and $\Sigma_{\mathcal{O}}$ consists of 4 points. The local group of each singular point is a group of rotations with two elements.

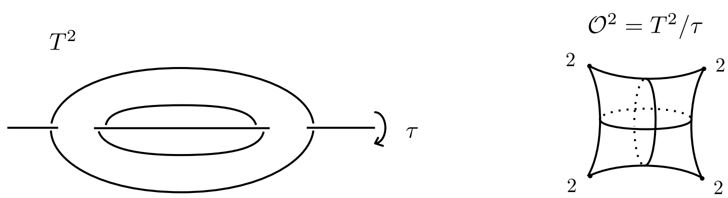

Figure 2.1: The action of the involution on the torus and its quotient.

An orbifold is called *spherical* (resp. *discal*, resp. *annular*, resp. *toric*) if it is a quotient of a sphere (resp. a disk, resp. an annulus, resp. a torus) by an isometric action. One defines similarly *Euclidean* (e.g. the previous example) and *hyperbolic* orbifolds, extending the definitions of the previous chapter.

Here is the list of all discal 2-orbifolds (see Figure 2.2):

i. A 2-disk without singular points.

ii. A 2-disk with a single singular point modelled on a group of rotations.

iii. A triangle whose singular locus is the union of two edges. The local group of each point interior to a singular edge is a two-element reflection group. The vertex where the two singular edges meet has local group a dihedral group.

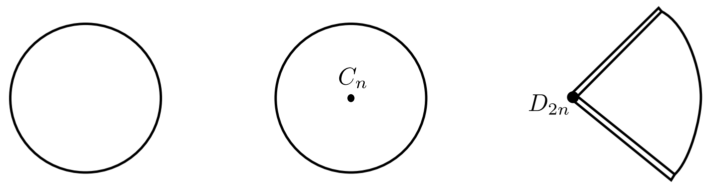

Figure 2.2: The discal 2-orbifolds

Example. Let $\mathcal{O}$ be the orbifold with underlying space $|\mathcal{O}| \cong \mathbf{S}^3$, singular locus $\Sigma_{\mathcal{O}}$ the trefoil knot and nontrivial local groups of order five. Then $\mathcal{O}$ is the quotient of the Poincaré dodecahedral space by a cyclic orthogonal action of order five. Hence $\mathcal{O}$ is spherical.

Definition. A *covering* of an orbifold $\mathcal{O}$ is an orbifold $\hat{\mathcal{O}}$ with a continuous map $p : |\hat{\mathcal{O}}| \to |\mathcal{O}|$, called a *covering map*, such that every point $x \in \mathcal{O}$ has a neighborhood U with the following property: for each component V of $p^{-1}(U)$ there is a chart $\phi : \tilde{V} \to V$ such that $p \circ \phi$ is a chart.

Using the language of orbifold coverings, an orbifold is good when it is covered by a manifold, and very good if it is finitely covered by a manifold.

Example. Let $\mathcal{O}$ be the orbifold with underlying space $\mathbf{S}^3$, singular locus a link with k components, so that the local groups of the components have order $m_1, \ldots, m_k$. A branched covering of $\mathbf{S}^3$ branched along the link $\Sigma_{\mathcal{O}}$ with the corresponding branching indices induces an orbifold covering of $\mathcal{O}$.

When the underlying spaces are manifolds, an orbifold covering induces a branched covering of the underlying manifolds.

Example. Let $\mathcal{O}(m,n)$ denote the orbifold with underlying space $\mathbf{S}^3$, singular locus the Hopf link and local groups of orders m and n (depending of the component). If n divides n' and m divides m', then obviously $\mathcal{O}(m,n)$ covers $\mathcal{O}(m',n')$. We leave as an exercise to determine for which coefficients $m,n,m',n' \in \mathbf{N}$ the orbifold $\mathcal{O}(m,n)$ covers $\mathcal{O}(m',n')$.

2.1.3 Maps and suborbifolds

A *map* between two orbifolds $\mathcal{O}$ and $\mathcal{O}'$ is a continuous map $f : |\mathcal{O}| \to |\mathcal{O}'|$ such that for every $x \in \mathcal{O}$ there are charts $\phi_i : \tilde{U}_i \to U_i \ni x$ and $\phi'_j : \tilde{U}'_j \to U'_j$ such that $f(U_i) \subset U'_j$ and the restriction $f|_{U_i}$ can be lifted to a smooth map

$\tilde{f} : \tilde{U}_i \to \tilde{U}'_j$ which is equivariant with respect to some homomorphism $\Gamma_i \to \Gamma'_j$.[2]

A map $f : \mathcal{O} \to \mathcal{O}'$ is *proper* if $f^{-1}(\partial\mathcal{O}') = \partial\mathcal{O}$. It is an *immersion* (resp. a *submersion*) if in the definition the lifts $\tilde{f}$ are immersions (resp. submersions). An *embedding* is an immersion whose underlying map is a homeomorphism with the image. A *diffeomorphism* is a surjective embedding. A (proper) *suborbifold* is the image of a (proper) embedding.

Let $\mathcal{O}$, $\mathcal{O}'$ be two orbifolds. The *product orbifold* $\mathcal{O} \times \mathcal{O}'$ is defined in the natural way. For $y \in \mathcal{O}'$, the map $f : \mathcal{O} \ni x \mapsto (x, y) \in \mathcal{O} \times \mathcal{O}'$ is an embedding.

Let $\mathcal{O}$ be an orbifold. Let Γ be a discrete group acting properly by diffeomorphisms on $\mathcal{O}$. Then the orbit space has an orbifold structure such that the canonical map from $\mathcal{O}$ is a covering map. This orbifold will be denoted by $\mathcal{O}/\Gamma$.

2.1.4 Local models for low-dimensional orbifolds

We give a list of local models for orbifolds of dimension up to 3. For simplicity, we will only consider orbifolds without boundary, and in dimension 3, we restrict attention to *orientable* orbifolds. For a more thorough discussion, see [198, 51] or [225, chap. XIII].

As noted before, what we have to do is essentially interpret the classification of finite subgroups of O(1), O(2) and SO(3) in terms of orbifolds by describing the quotient spaces and the isotropy information.

In dimension 1, the local models for orbifolds are given by finite subgroups of O(1). There are two models: regular points (with local group the trivial group) and singular points with local group $\mathbf{Z}_2$ acting by the reflection $\mathbf{R} \ni x \mapsto -x \in \mathbf{R}$. Therefore there are only, up to homeomorphism, two closed 1-dimensional orbifolds. They are both spherical: the circle $\mathbf{S}^1$, which is orientable, and the nonorientable orbifold $\mathbf{S}^1/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts by a reflection through the x-axis (thinking of $\mathbf{S}^1$ as the unit circle in $\mathbf{R}^2$). We call this last one *mirrored interval*; its underlying space is the closed interval $\mathbf{I}$ and its singular points are the endpoints.

Let us proceed to dimension 2. It is well-known that any (nontrivial!) finite subgroup of O(2) has one of the following types:

- A cyclic group C_n of order n generated by a rotation of angle $2\pi/n$.
- A reflection group R of order 2 generated by a reflection through a line.

[2] We do not assume this homomorphism to be injective or surjective. Thus our definition is less restrictive than Kapovich's [120].

- A dihedral group D_{2n} of order $2n$, generated by two reflections through lines making an angle π/n.

From this, we get four classes of local models for singular points of 2-orbifolds: the three classes of groups acting on the whole plane $\mathbf{R}^2$ and the group $R \cong \mathbf{Z}/2\mathbf{Z}$ acting on a half plane $\mathbf{R} \times [0, +\infty)$. It follows that the underlying space of a 2-orbifold is always a 2-manifold (possibly with boundary not coming from the boundary of the orbifold.) If we restrict our attention to orientable orbifolds, all local groups must be of type C_n, so the singular locus is discrete. Singular points of this type will be called *cone points.* In general, the singular locus consists of a discrete set with local groups of type C_n together with a polygon in the boundary of the underlying space, with local groups of type R inside the edges, D_n in the vertices and R at the endpoint (if any).

In dimension 3, we only consider the orientable case. Any finite subgroup of $SO(3)$ is cyclic, dihedral, or isomorphic to one of the so-called *platonic groups* T_{12}, O_{24}, I_{60}, which are the isometry groups of the regular tetrahedron, octahedron and icosahedron respectively.

It follows from this that the underlying space is a manifold. The singular locus is a graph, whose vertices, if any, are trivalent: the local groups of edges are cyclic, local groups of the vertices are of type D_{2n}, T_{12}, O_{24} or I_{60}. The order of the local groups of the edges concurrent to a singular vertex determines the local group of the vertex, as in Figure 2.3.

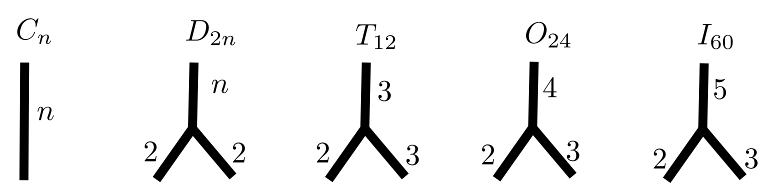

Figure 2.3: The possible configurations of the singular locus in the 3-dimensional orientable case.

The boundary has to be transverse to the singular locus.

We let the reader check that quotients of $\mathbf{R}^3$ by finite subgroups of SO(3) are indeed as in Figure 2.3.

2.2 Coverings and the Seifert-van Kampen Theorem

In this section, all orbifolds are *connected* unless stated explicitly otherwise. We discuss a generalization of the classical theory of coverings of topological spaces, so that in particular we can define the fundamental group and the universal covering of an orbifold. Since it would be too long to develop the theory from scratch, we assume some familiarity with the classical theory (see e. g. [87].)

2.2.1 General theory

We first give definitions completely analogous to the classical definitions for topological spaces. In the following, $\mathcal{O}$ is a fixed (connected) orbifold.

Definition. The *deck transformation group* of a covering $p : \mathcal{O}' \to \mathcal{O}$ is the group of all self-diffeomorphisms $f : \mathcal{O}' \to \mathcal{O}'$ such that $p \circ f = p$. It is denoted by $\mathrm{Aut}(\mathcal{O}', p)$, or simply $\mathrm{Aut}(\mathcal{O}')$ if the map p is understood.

A *universal covering* of $\mathcal{O}$ is a covering $p : \tilde{\mathcal{O}} \to \mathcal{O}$ such that for every covering $q : \hat{\mathcal{O}} \to \mathcal{O}$, there is a covering $r : \tilde{\mathcal{O}} \to \hat{\mathcal{O}}$ such that $q \circ r = p$.

Two coverings $p_1 : \mathcal{O}_1 \to \mathcal{O}$, $p_2 : \mathcal{O}_2 \to \mathcal{O}$ are *equivalent* if there is a diffeomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that $p_2 \circ f = p_1$.

Theorem 2.1 (Thurston [225]). *Every (connected) orbifold $\mathcal{O}$ has a unique (up to equivalence) universal covering, which will be denoted by $\tilde{\mathcal{O}}$.*

The proof of existence given in [225] goes roughly like this: there is a "fiber product" construction which to a pair of coverings $p_1 : \mathcal{O}_1 \to \mathcal{O}$, $p_2 : \mathcal{O}_2 \to \mathcal{O}$ associates a covering $p : \mathcal{O}_1 \times_{\mathcal{O}} \mathcal{O}_2 \to \mathcal{O}$ which factors through both p_1 and p_2. The universal covering is then defined as an inverse limit. We will give a different proof closer to the usual construction by homotopy classes of paths starting at a fixed basepoint. Notice however that orbifolds are not locally simply-connected in any reasonable sense at singular points, so we will need a refinement to take care of that.

Definition. The *fundamental group* of $\mathcal{O}$, denoted by $\pi_1\mathcal{O}$, is the deck transformation group of its universal covering.

In general, $\pi_1\mathcal{O}$ is different from $\pi_1(|\mathcal{O}|)$. In fact there is a natural epimorphism $\pi_1\mathcal{O} \to \pi_1(|\mathcal{O}|)$ obtained by forgetting the orbifold structure, see [6]. If $\Sigma_{\mathcal{O}}$ is empty, then this homomorphism is an isomorphism, which means that the definition above extends the classical definition for manifolds.

Since the group of deck transformations of a covering acts properly and hence the quotient has a natural orbifold structure, the following definition makes sense.

Definition. A covering $p : \mathcal{O}' \to \mathcal{O}$ is *regular* (or *Galois*) if $\mathcal{O}' / \operatorname{Aut}(\mathcal{O}', p) = \mathcal{O}$.

Theorem 2.2. *Let $\mathcal{O}$ be an orbifold.*

i. There is a one-to-one correspondence between conjugacy classes of subgroups of $\pi_1\mathcal{O}$ and equivalence classes of (connected) coverings of $\mathcal{O}$. A covering corresponds to a normal subgroup if and only if it is regular. (In particular, the universal covering is regular, i.e. $\tilde{\mathcal{O}}/\pi_1\mathcal{O} = \mathcal{O}$.)

ii. There is a one-to-one correspondence between equivalence classes of possibly disconnected *coverings of $\mathcal{O}$ and actions of $\pi_1\mathcal{O}$ on discrete topological spaces.*

From part (ii) of Theorem 2.2 and its proof, one can derive the Seifert-van Kampen theorem as in the classical case (see e.g. [87].)

Corollary 2.3 (Seifert-van Kampen Theorem). *Let $\mathcal{O}$ be an orbifold and $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ two open suborbifolds such that $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2$ are connected. If $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ then $\pi_1\mathcal{O}$ is the amalgamated product:*

$$\pi_1\mathcal{O} \cong \pi_1\mathcal{O}_1 *_\Gamma \pi_1\mathcal{O}_2$$

where $\Gamma = \pi_1(\mathcal{O}_1 \cap \mathcal{O}_2)$.

Here is a version of the Seifert-van Kampen Theorem in the non-separating case:

Corollary 2.4. *Let $\mathcal{O}$ be an orbifold and $\mathcal{O}_1 \times [0,1] \subset \mathcal{O}$ be a suborbifold such that $\mathcal{O}_1$ and $\mathcal{O}' = \mathcal{O} \setminus \mathcal{O}_1 \times [0,1]$ are connected. Then $\pi_1\mathcal{O}$ is a HNN-extension: $\pi_1\mathcal{O} \cong \pi_1\mathcal{O}' *_{\pi_1\mathcal{O}_1}$.*

The remainder of this subsection is devoted to a sketch of the proof of Theorem 2.2. We need an interpretation of $\pi_1\mathcal{O}$ in terms of homotopy classes of loops in $\mathcal{O}$. Obviously, defining a path in $\mathcal{O}$ as just a continuous map $\alpha : \mathbf{I} \to |\mathcal{O}|$ will not do, because lifts are not unique when α crosses the singular locus. If we restrict attention to orbifolds whose singular locus has codimension at least 2, then it is possible to use only paths that do not meet $\Sigma_{\mathcal{O}}$. But even there, defining the correct notion of homotopy would be an issue. For instance, a 2-disk $D(n)$ with one cone point of order n has universal covering a nonsingular 2-disk, with deck transformation group $\mathbf{Z}/n\mathbf{Z}$ acting by rotations. Thus intuitively, a loop in $D(n)$ winding n times around the cone point should be null-homotopic, but a null-homotopy must pass through the singular locus.

There are several equivalent ways to overcome this difficulty. For a slightly different approach, see [86, Chap. 11], and for a generalization, see [32, p. 604].

Definition. A *path* in an orbifold $\mathcal{O}$ is given by the following data:

i. A continuous map $\alpha : \mathbf{I} \to |\mathcal{O}|$ such that there are at most finitely many t such that $\alpha(t)$ is singular, and

ii. For each t such that $\alpha(t)$ is singular, a triple (ϕ, V, l), where $\phi : \tilde{U} \to U \ni \alpha(t)$ is a chart, V is a neighborhood of t in $\mathbf{I}$ such that for all $u \in V(t) - \{t\}$, $\alpha(u)$ is regular and lies in U, and l is a lift of $\alpha|_{V(t)}$ to $\tilde{U}$. We shall call l a *local lift* of α around t.

By abuse of notation, we write α for both the path and the underlying map $\mathbf{I} \to |\mathcal{O}|$.

Let $p : \hat{\mathcal{O}} \to \mathcal{O}$ be a covering. Let α be a path in $\hat{\mathcal{O}}$. Define the *projection* of α, denoted by $p(\alpha)$, as follows: $p(\alpha)$ is a path in $\mathcal{O}$ whose underlying map is the composition of p and the underlying map of α; for each t such that $p \circ \alpha(t) \in \Sigma_{\mathcal{O}}$, look at $\alpha(t)$: if it is regular, then there is a neighborhood V of $\alpha(t)$ such that $p|V$ is a chart at $p \circ \alpha(t)$, and this can be used to define the local lift of $p \circ \alpha$ around t; if it is singular, then there is a chart $\phi : \tilde{V} \to V$ at α_t and a local lift to $\tilde{V}$ of α around t; if $\tilde{V}$ is chosen small enough, then $\phi \circ p$ is a chart at $p \circ \alpha(t)$ which provides the local lift of $p \circ \alpha$ around t.

If α is a path in $\hat{\mathcal{O}}$ and β is a path in $\mathcal{O}$, then α is a *lift* of β if $p(\alpha) = \beta$. It is not difficult to show that if $\hat{*} \in \hat{\mathcal{O}}$ is a regular point such that $p(\hat{*})$ is regular, then any path α such that $\alpha(0) = p(\hat{*})$ has a unique lift $\hat{\alpha}$ such that $\hat{\alpha}(0) = \hat{*}$.

Definition. Let α be a path in an orbifold $\mathcal{O}$. Let U be an open subset of $\mathcal{O}$ and $\phi : \tilde{U} \to U$ be a chart. Let $[a, b]$ be a subinterval of $\mathbf{I}$ such that $\alpha([a, b]) \subset U$. Let β be a lift of $\alpha|_{[a,b]}$ to $\tilde{U}$, which always exists. Replace $\alpha|_{[a,b]}$ by the projection of a path in $\tilde{U}$ which is homotopic (in the classical sense) to β with fixed endpoints. The result is a path in $\mathcal{O}$, which is said to be obtained from α by an *elementary homotopy*. We define *homotopy of paths* as the equivalence relation generated by elementary homotopies.

The following basic facts about homotopies of paths can be easily proved:

i. If $\Sigma_{\mathcal{O}} = \emptyset$, this is the same as homotopy of the underlying maps.

ii. If two paths are homotopic, then projections of those paths (resp. lifts of those paths with same initial point) are homotopic.

Definition. Let $*$ be a regular point of $\mathcal{O}$. A *loop based at* $*$ is a path $\alpha : \mathbf{I} \to |\mathcal{O}|$ such that $\alpha(0) = \alpha(1) = *$. We define $\pi_1(\mathcal{O}, *)$ as the set of homotopy classes of loops based at $*$ with the usual composition law.

We collect the fundamental properties of $\pi_1(\mathcal{O}, *)$ in a proposition, whose proof is left as an exercise.

Proposition 2.5. *Let $p : (\hat{\mathcal{O}}, \hat{*}) \to (\mathcal{O}, *)$ be a covering of based orbifolds.*

i. $\pi_1(\mathcal{O}, *)$ is a group, and change of basepoint results in an isomorphic group.

ii. There is an natural epimorphism $\pi_1(\mathcal{O}, *) \to \pi_1(|\mathcal{O}|, *)$ defined by forgetting the extra structure.

iii. There is a natural monomorphism $p_* : \pi_1(\hat{\mathcal{O}}, \hat{*}) \to \pi_1(\mathcal{O}, *)$ defined by projecting loops.

iv. Let N be the normalizer of $p_*(\pi_1(\hat{\mathcal{O}}, \hat{*}))$ in $\pi_1(\mathcal{O}, *)$. There is a natural homomorphism $h : N \to \mathrm{Aut}(\hat{\mathcal{O}})$ such that for every $[\alpha] \in N$, the lift $\hat{\alpha}$ of α starting at $\hat{*}$ ends at $h([\alpha])(\hat{*})$.

We now come to a key result.

Theorem 2.6. *Let $(\mathcal{O}, *)$ be a based orbifold. There exists a covering $(\tilde{\mathcal{O}}, \tilde{*})$ such that $\pi_1(\tilde{\mathcal{O}}, \tilde{*})$ is trivial. This covering is regular and universal. It is unique up to equivalence. Its deck transformation group (which we previously called $\pi_1\mathcal{O}$) is isomorphic to $\pi_1(\mathcal{O}, *)$.*

Sketch of proof. First we prove existence. Let $\tilde{\mathcal{O}}$ be the set of homotopy classes of paths in $\mathcal{O}$ with initial point $*$. Let $\tilde{*} \in \tilde{\mathcal{O}}$ be the homotopy class of the constant path at $*$. Let $p : \tilde{\mathcal{O}} \to \mathcal{O}$ be the set-theoretic map which sends $[\alpha]$ to $\alpha(1)$. This map is onto. Our goal is to find a natural orbifold structure on $\tilde{\mathcal{O}}$ such that p is a covering map. The triviality of $\pi_1(\tilde{\mathcal{O}}, \tilde{*})$ will then follow easily as in the classical case.

Let $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_i$ be an atlas for the orbifold structure on $\mathcal{O}$. Without loss of generality we assume that each $\tilde{U}_i$ is simply-connected. For each i, choose a basepoint $*_i \in U_i$, set $F_i := p^{-1}(*_i)$, and choose a point $\tilde{*}_i \in F_i$. Our goal is to define a map $\psi_i : \tilde{U}_i \times F_i \to p^{-1}(U_i)$ and use the collection $\{\psi_i\}_i$ to construct the topology and the orbifold structure on $\tilde{\mathcal{O}}$.

Let $x \in \tilde{U}_i, y \in F_i$. Choose a path $\tilde{\lambda}$ in $\tilde{U}_i$ from $\tilde{*}_i$ to x. Project it to a path $\lambda \in U_i$. By construction, y is a homotopy class of paths from $*$ to $*_i$. Let γ be a representative of y. Let α be the path obtained by composing γ with λ. Since $\tilde{U}_i$ is simply-connected, the homotopy class of α does not depend on the choices of γ and λ. Hence we can define the map $\psi_i : \tilde{U}_i \times F_i \to p^{-1}(U_i)$ by setting $\psi_i(x, y) := [\alpha]$. It is not hard to see that this map is onto.

Give F_i the discrete topology and $\tilde{U}_i \times F_i$ the product topology. Then ψ induces a topology on $p^{-1}(U_i)$. Its connected components are the images of sets of the form $\tilde{U}_i \times \{y\}$. As charts for the orbifold structure on $\tilde{\mathcal{O}}$, we use the restrictions $\psi_i|(U_i \times \{y\})$. Then one can check that these charts induce a well-defined topology and orbifold structure on $\tilde{\mathcal{O}}$, which make p a covering map, and that $\pi_1(\tilde{\mathcal{O}}, \tilde{*})$ is trivial.

It follows easily from Proposition 2.5 (iv) that $(\tilde{\mathcal{O}}, p)$ is regular and $\pi_1(\mathcal{O}, *)$ is isomorphic to $\mathrm{Aut}(p)$. The proofs that a simply-connected covering is universal and unique are similar to the classical case and left to the reader. □

At this point it is an exercise to prove Theorem 2.2 by adapting standard proofs for semi-locally simply-connected topological spaces. We have already seen how to associate a subgroup of $\pi_1\mathcal{O}$ to a covering. In the opposite direction, associate to a subgroup Γ of $\pi_1\mathcal{O}$ the orbifold $\tilde{\mathcal{O}}/\Gamma$. The lifting property ensures that this correspondence is bijective. Then use Proposition 2.5 (iv) to prove that normal subgroups correspond to regular coverings. For part (ii) of Theorem 2.2, associate to a covering the action of the deck transformation group on the fiber of the basepoint; for the opposite direction, associate to an action of $\pi_1\mathcal{O}$ on a discrete space F the quotient of $\tilde{\mathcal{O}} \times F$ by the diagonal action, where the (left) action on $\tilde{\mathcal{O}}$ is obtained from the right action (defined by lifts of loops) by $g \cdot x := x \cdot g^{-1}$.

Remark. The reader might wonder how one can *compute* fundamental groups of orbifolds. One first remark is that the homomorphism $\pi_1\mathcal{O} \to \pi_1(|\mathcal{O}|)$ is not very useful for this in general, since much information is lost (consider the example of knots in $\mathbf{S}^3$ with cyclic local groups.) It is more interesting to use the complement of $\Sigma_{\mathcal{O}}$. Indeed, if $\Sigma_{\mathcal{O}}$ has codimension at least two (which is always the case if $\mathcal{O}$ is orientable), then $\mathcal{O} - \Sigma_{\mathcal{O}}$ is connected, and we have a surjective homomorphism $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}}) \to \pi_1\mathcal{O}$ induced by inclusion. (The surjectivity comes from the fact that any loop can be perturbed to avoid the singular locus.) To compute $\pi_1\mathcal{O}$, we need only know which elements of $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}})$ get killed. This can be seen in a cellular decomposition adapted to the orbifold structure. In the next paragraph, we make this more precise in the case where $\mathcal{O}$ has dimension 2 or 3, but this is valid in all dimensions.

2.2.2 The orientable low-dimensional case

Let F be a connected 2-orbifold. In particular Σ_F is a discrete subset of $|F|$. Using the Seifert-van Kampen theorem, $\pi_1 F$ can be computed from $\pi_1(F - \Sigma_F)$ and discal neighborhoods of the cone points. More precisely, for each $x \in \Sigma_F$, let $\mu_x \in \pi_1(|F| - \Sigma_F)$ denote the element represented by the boundary of a discal neighborhood of x. By Seifert-van Kampen we have:

Proposition 2.7. *The group $\pi_1 F$ is the quotient of $\pi_1(|F| - \Sigma_F)$ by the group normally generated by the elements $\mu_x^{m_x}$, for $x \in \Sigma_F$, where m_x is the order of the local group of x and μ_x is a meridian around x.*

Let F be the orbifold with underlying space $\mathbf{S}^2$ and singular locus a single point. Then $\pi_1 F$ is trivial. Hence F is bad, since it is its own universal covering.

In the 3-dimensional orientable case we make a similar construction. Given a 3-orbifold $\mathcal{O}$, for every edge or circle e of $\Sigma_{\mathcal{O}}$ we consider a meridian $\mu_e \in \pi_1(|\mathcal{O}| - \Sigma_{\mathcal{O}})$ around e. Let m_e be the order of the local group of the interior points in the edge e.

Proposition 2.8. *The group $\pi_1 \mathcal{O}$ is the quotient of $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}})$ by the group normally generated by the set of elements $\mu_e^{m_e}$, where e runs over all edges of $\Sigma_{\mathcal{O}}$.*

This proposition is proved again using Seifert-Van Kampen theorem. Notice that if v is a vertex of $\Sigma_{\mathcal{O}}$ and $\mathcal{N}(v)$ is a tubular neighborhood of v, then $\pi_1 \mathcal{N}(v) \cong \pi_1 \partial \mathcal{N}(v)$, and therefore $\pi_1 \mathcal{O} \cong \pi_1(\mathcal{O} - \mathcal{N}(v))$.

It follows that for $n \geq 3$, if $\mathcal{O} = B^n/\Gamma$ is a discal n-orbifold, then $\pi_1 \mathcal{O}^n = \pi_1 \partial \mathcal{O}^n = \Gamma$.

For an orientable 3-orbifold $\mathcal{O}$ let $\widetilde{|\mathcal{O}| - \Sigma_{\mathcal{O}}}$ be the regular covering of $|\mathcal{O}| - \Sigma_{\mathcal{O}}$ associated to the surjective homomorphism $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}}) \twoheadrightarrow \pi_1 \mathcal{O}$ given by Proposition 2.8. The deck transformation group of this covering is $\pi_1 \mathcal{O}$. Then one can show that the metric completion of the metric space $\widetilde{|\mathcal{O}| - \Sigma_{\mathcal{O}}}$ is the universal covering of the orbifold $\mathcal{O}$.

2.3 The geometric classification of 2-orbifolds

Apart from a few exceptions, all closed 2-dimensional orbifolds are geometric, hence good (in fact very good). For simplicity, we only consider orientable orbifolds.

A *teardrop* is a 2-sphere with one cone point. A *spindle* is a 2-sphere with two cone points of different orders. A *football* is a 2-sphere with two cone points of the same order. A *turnover* is a 2-sphere with three cone points. A *pillow* is a 2-sphere with four cone points of order two (see Figure 2.4).

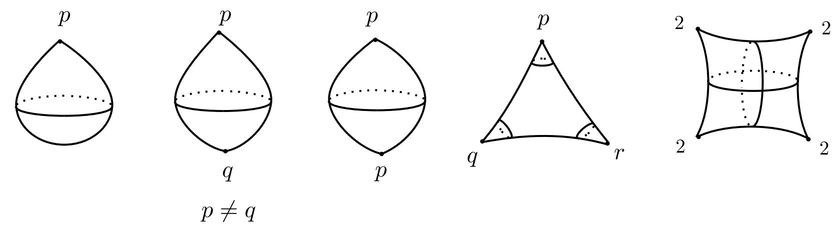

Figure 2.4: From left to right: a teardrop, a spindle, a football, a turnover and a pillow.

We now turn to a discussion of the Gauss-Bonnet formula. We first need to define Euler characteristic and Riemannian metric for orbifolds.

Definition. Given an orbifold $\mathcal{O}$, let K be a CW-complex decomposition of $\mathcal{O}$ such that $\Sigma_{\mathcal{O}}$ is a subcomplex and the local group is constant along each cell. The *Euler characteristic* of $\mathcal{O}$ is defined as:

$$\chi(\mathcal{O}) = \sum_{\Delta} (-1)^{\dim(\Delta)} \frac{1}{|\Gamma_e|}$$

where the sum is taken over the cells of K and $|\Gamma_\Delta|$ denotes the order of the local group of the cell Δ.

Notice that the Euler characteristic is multiplicative on coverings and that it extends the usual definition for manifolds.

A *Riemannian metric* on an orbifold $\mathcal{O}$ is a set of Riemannian metrics on a covering of $\mathcal{O}$ by uniformizing charts, such that the local group actions on the charts are by isometries as well as the transition maps.

In the 2-dimensional case, we have:

Proposition 2.9 (Gauss-Bonnet). *Let F be a closed 2-orbifold endowed with a Riemannian metric of Gauss curvature*[3] *K. Then:*

$$\int_F K = 2\pi\chi(F)$$

[3]Note that the definition of Gauss curvature is clear at nonsingular points. Since Σ_F has measure zero, this is enough for our purposes. One can also define curvature at singular points, using orbifold tangent bundles.

Proposition 2.9 can be proved using the Gauss-Bonnet formula for triangles.

The following proposition is proved in [225, Chap. 13] (see also [164, Appendix A], [198]).

Proposition 2.10. *The only bad closed orientable* 2*-orbifolds are teardrops and spindles. All other closed orientable* 2*-orbifolds are geometric. More precisely, a good, closed, orientable* 2*-orbifold* $\mathcal{O}$ *is spherical (resp. Euclidean, resp. hyperbolic) if and only if* $\chi(\mathcal{O})$ *is positive (resp. zero, resp. negative.)*

Compact orientable 2-orbifolds will play an important role in the sequel as suborbifolds of 3-orbifolds, so it is convenient to get more familiar with them. We collect basic facts in the following proposition, whose proof is left as an exercise.

Proposition 2.11. *Let* $\mathcal{O}$ *be a compact, orientable* 2*-orbifold.*

i. $\mathcal{O}$ *is discal if and only if* $\mathcal{O}$ *is either a nonsingular disk or a disk with one cone point.*

ii. $\mathcal{O}$ *is annular if and only if* $\mathcal{O}$ *is either a nonsingular annulus or a disk with two cone points of order* 2*.*

iii. $\mathcal{O}$ *is spherical if and only if* $\mathcal{O}$ *is either a nonsingular sphere, a football, or a turnover with orders* $(2,2,n)$*,* $(2,3,3)$ *,*$(2,3,4)$*, or* $(2,3,5)$*.*

iv. If $\mathcal{O}$ *is bad, then* $\chi(\mathcal{O}) > 0$*.*

v. $\mathcal{O}$ *is toric if and only if* $\mathcal{O}$ *is either a nonsingular torus, a pillow with orders* $(2,2,2,2)$*, or a turnover with orders* $(2,3,6)$*,* $(2,4,4)$ *or* $(3,3,3)$*.*

Hence $\mathcal{O}$ *is Euclidean if and only if it is toric or annular.*

2.4 Fibered 3-orbifolds

Definition. A *fiber bundle* is a 4-tuple $(\mathcal{O}, B, F, p)$, where $\mathcal{O}, B, F$ are orbifolds and $p : \mathcal{O} \to B$ is a submersion, such that for every point $x \in B$ there is a chart $\phi : \tilde{U} \to U \ni x$, an action of the local group Γ_x on F, and a submersion $\psi : \tilde{U} \times F \to \mathcal{O}$ inducing a diffeomorphism between $(\tilde{U} \times F)/\Gamma_x$ (for the diagonal action) and $p^{-1}(U)$ such that $p \circ \psi = \phi \circ \pi_1$ (where π_1 is the canonical projection $\tilde{U} \times F \to \tilde{U}$).

We say that $\mathcal{O}$ *fibers over* B with *generic fiber* F, or that $\mathcal{O}$ is an F*-bundle with base* B.

A fiber bundle is *twisted* if it is not fiber-preserving diffeomorphic to the trivial bundle $F \times B$.

One can show that, up to fiber preserving diffeomorphism, there is only one twisted **I**-bundles over a given 2-orbifold B. It is orientable if and only if B is nonorientable.

As mentionned before, one can define the tangent bundle of an orbifold, but we will not do this here (see [164]).

A 3-orbifold $\mathcal{O}$ is *Seifert fibered* if it fibers over a 2-orbifold B with generic fiber a circle or a mirrored interval. For *orientable* manifolds, this definition coincides with the original definition of Seifert [203]. For nonorientable manifolds, it is slightly more general (see [198]).

Proposition 2.12. *Let $\mathcal{O}$ be a Seifert fibered 3-orbifold with base B. The projection $p\colon \mathcal{O} \to B$ induces an exact sequence*

$$1 \to C \to \pi_1\mathcal{O} \to \pi_1 B \to 1,$$

where C is cyclic or dihedral (either finite or infinite.) In addition, C is finite if and only if $\pi_1\mathcal{O}$ is finite.

Sketch of proof. Since we interpret the fundamental group by means of homotopy classes of loops, the proof for the exact sequence of a topological fibration applies here. For instance $\pi_1\mathcal{O} \twoheadrightarrow \pi_1 B$ is a surjection because loops in B can be lifted to $\mathcal{O}$. The kernel C of $\pi_1\mathcal{O} \to \pi_1 B$ is the image of the fundamental group of the generic fiber. Now this fiber is either a circle or a mirrored interval, so its fundamental group is either the infinite cyclic group $\mathbf{Z}$ or the infinite dihedral group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. Hence C is cyclic or dihedral.

We discuss now the order of C. When the base B is aspherical (Euclidean, hyperbolic or nonclosed), C is infinite, because the term corresponding to $\pi_2 B$ is trivial. (In other words, one can prove that C is infinite using the fact that every map from $\mathbf{S}^2$ to B can be extended to a 3-ball bounded by $\mathbf{S}^2$.) Thus both $\pi_1\mathcal{O}$ and C are infinite in this case. Otherwise B is bad or spherical, and therefore $\pi_1 B$ is finite. In this last case, it is clear that $\pi_1\mathcal{O}$ is infinite if and only if $\pi_1 C$ is infinite. □

Proposition 2.13. *Every compact Seifert fibered 3-orbifold without bad 2-suborbifold admits a geometric structure modelled on $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{E}^3$, **Nil**, $\mathbf{H}^2 \times \mathbf{R}$, or $\widetilde{\mathrm{SL}_2(\mathbf{R})}$.*

Proof. [226] Let B be the base of the Seifert fibration. Assume first that B is very good: $B = F/\Gamma$ for some compact surface F. Since the natural map

$\mathcal{O} \to B$ induces a surjection $\pi_1\mathcal{O} \twoheadrightarrow \pi_1 B$ and B is finitely covered by a surface F, $\mathcal{O}$ is the quotient of a $\mathbf{S}^1$-bundle M over F by a finite fiber-preserving group isomorphic to Γ.

By Proposition 2.10, B has a metric of constant curvature. Hence F has a Γ-invariant metric of constant curvature. We want to show that the $\mathbf{S}^1$-bundle M has a Γ-invariant connection of constant curvature, so that the universal covering has a fibration by lines or circles over $\widetilde{B}$ with a $\pi_1\mathcal{O}$-invariant connection of constant curvature. Then the geometry of $\mathcal{O}$ depends on $\widetilde{B}$ (which can be $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$) and the fact that the connection has curvature zero or not.

To construct a Γ-invariant connection of constant curvature, we first pick an arbitrary connection ω on M, i.e. a 1-form on M that is invariant by the action of $\mathbf{S}^1$, and such that on each $x \in M$, $\ker \omega_x$ is a horizontal plane in $T_x M$. The curvature of this connection is a closed 2-form Ω on F such that $d\omega = \pi^*\Omega$. Let vol_F be a Γ-invariant volume form on F obtained by lifting a volume form on B. Then ω is homologous to a multiple of the volume form vol_F:

$$\Omega = \lambda\, vol_F + d\beta$$

for some $\lambda \in \mathbf{R}$ and some 1-form β on F. We make the connection $\omega' = (\omega - \pi^*\beta)$ Γ-invariant by averaging. By construction, the connection $\omega' = (\omega - \pi^*\beta)$ has curvature $\lambda\, vol_{F^2}$, i.e. constant curvature λ.

If B is not good, then one can prove that $\mathcal{O}$ admits another Seifert fibration with a good base. More precisely, B is a teardrop, a spindle or a quotient of one by an involution. When B is a teardrop or a spindle, then $\mathcal{O}$ is a generalized lens space: the underlying space $|\mathcal{O}|$ is a lens space and the singular locus $\Sigma_{\mathcal{O}}$ may be one or two circles, corresponding to the cores of the two solid tori glued together to give $|\mathcal{O}|$. In this case, it is easy to find a new Seifert fibration of $\mathcal{O}$ with a good base, except in the case where $\mathcal{O}$ is a product of a bad orbifold with $\mathbf{S}^1$. When B is the quotient of a spindle or a teardrop by an involution, then we use the fact that involutions on lens spaces and solid tori are standard. □

Equally important is the class of 3-orbifolds that fiber over a 1-orbifold. It includes the well-known class of 3-manifolds that fiber over the circle.

Proposition 2.14. *Every compact* 3*-orbifold that fibers over a* 1*-orbifold with toric generic fiber admits a geometric structure modelled on* $\mathbf{E}^3$*,* **Nil***, or* **Sol**.

We leave the proof of Proposition 2.14 as an exercise. For a classification of fibered 3-orbifolds see [21] and [56]. As a warm-up, the reader may give the list of all compact, orientable 2-orbifolds fibering over a 1-orbifold and classify those fibrations up to isotopy. In particular, a $\mathbf{S}^1$-fibration on the torus or the pillow is determined up to isotopy by the choice of an essential curve as the generic fiber.

2.4.1 Basic facts about Seifert fibered orbifolds

We collect below some general results on Seifert fibered 3-orbifolds that will be needed in the sequel.

We start with a description of a saturated tubular neighborhood of a fiber in the interior of a Seifert fibered 3-orbifold (cf. [21] for more details). We call a 3-orbifold *solid-toric* if it is finitely covered by $\mathbf{S}^1 \times \mathbf{D}^2$.

Let $\mathbf{D}^2 \times \mathbf{I}$ be the solid cylinder, where the disk $\mathbf{D}^2$ is identified with the unit disk in $\mathbf{C}$. Given integers $\alpha, \beta \in \mathbf{Z}$ with $0 \leq \beta < \alpha$ and $\alpha \geq 1$, we define a solid-toric 3-orbifold $V(\alpha, \beta)$ as follows: its underlying space $|V(\alpha, \beta)|$ is the foliated solid torus obtained from the product I-fibration on $\mathbf{D}^2 \times \mathbf{I}$, by identifying the disk $\mathbf{D}^2 \times \{0\}$ to the disk $\mathbf{D}^2 \times \{1\}$ via the gluing map $z \mapsto e^{2\pi i\beta/\alpha} z$. If α and β are coprime, then $V(\alpha, \beta)$ is a manifold; otherwise its singular set consists of the core of the solid torus with cyclic local group of order $k = gcd(\alpha, \beta)$. The fraction $\beta/\alpha \in \mathbf{Q}/\mathbf{Z}$ is an invariant of this core fiber. The integer α/k can be interpreted as $card(\partial \mathbf{D}^2 \pitchfork \text{fiber})$. The leaf space of this foliation has a natural orbifold structure. In fact, it is discal with a single cone point of order α (or no cone point if $\alpha = 1$). This gives a model for saturated tubular neighborhoods of fibers homeomorphic to circles.

Let us consider a fiber f that is a mirrored interval $\mathbf{S}^1/\mathbf{Z}_2$. A saturated tubular neighborhood of f can be obtained as the quotient $W(\alpha, \beta)$ of a fibered solid-toric 3-orbifold $V(\alpha, \beta)$ above by the involution τ which is fiber-preserving and reverses both the orientation of the fibers and of the base. This is a solid pillow with possibly a singular core with cyclic local group of order $k = gcd(\alpha, \beta)$. A *solid pillow* is a 3-ball with two unknotted singular arcs with local group of order 2 (see Figure 2.5). Its boundary is a pillow.

A solid pillow contains a unique, properly embedded *meridian disk*, up to isotopy, whose boundary separates $\partial W(\alpha, \beta)$ into two discs with two singular points each: it is the projection of a meridian disk of the solid torus $|V(\alpha, \beta)|$ which does not meet the fixed point set of the involution τ.

The fibers of the fibration of the solid toric 3-orbifold $W(\alpha, \beta)$ are projections of the fibers of $V(\alpha, \beta)$. Hence a fiber of $W(\alpha, \beta)$ is either a circle (a $\mathbf{S}^1$-fiber) or a mirrored interval (a $\mathbf{I}$-fiber) according to whether its preimage in $V(\alpha, \beta)$ meets the fixed point set of τ. In particular, the fiber f is the projection of the

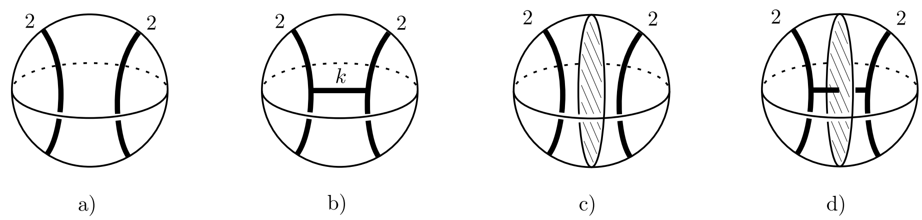

Figure 2.5: The solid pillow a), and the solid pillow with singular soul b). The boundary of both is the pillow. Figures c) and d) represent a meridian disk.

central fiber of $V(\alpha,\beta)$, which meets the fixed point set of the involution τ in two points: it is a core of the solid pillow, and is singular when α and β are not coprime; in this case the singular set is a trivalent graph with two vertices with local group a dihedral group of order 2α. The fraction $\beta/\alpha \in \mathbf{Q}/\mathbf{Z}$ is an invariant of f, and the integer α/k can be interpreted as *card*(∂meridian disk $\pitchfork$ I-fiber). Moreover the leaf space of $W(\alpha,\beta)$ is a non-orientable discal 2-orbifold obtained as the quotient of a discal 2-orbifold by an orientation reversing involution: it is a triangle with two mirrored edges that meet at a vertex whose local group is dihedral of order 2α.

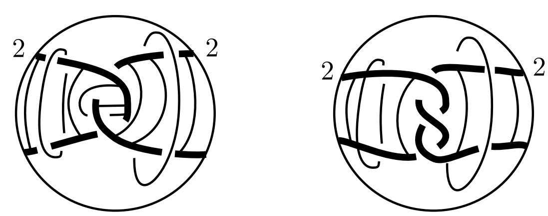

Figure 2.6: Two fibered solid pillows: $W(2,1)$ and $W(3,1)$.

Definition. The fiber f is *exceptional* if its saturated tubular neighborhood is isomorphic to $V(\alpha,\beta)$ or $W(\alpha,\beta)$ with $\alpha > 1$ (*isomorphic* means that there is a fiber-preserving diffeomorphism). The fraction $\beta/\alpha \in (\mathbf{Q}/\mathbf{Z})^\star$ is called the *type* of the exceptional fiber.

The fibered solid torus $V(\alpha,\beta)$ is the quotient of the solid torus $V(1,0) = \mathbf{D}^2 \times \mathbf{S}^1$ with the product fibration by the following finite fiber-preserving action:

$$\begin{aligned}\mathbf{D}^2 \times \mathbf{S}^1 &\to \mathbf{D}^2 \times \mathbf{S}^1\\ (z,x) &\mapsto (e^{2\pi i\beta/\alpha} z, e^{2\pi i\alpha} x).\end{aligned}$$

The quotient of the solid torus $V(1,0) = \mathbf{D}^2 \times \mathbf{S}^1$ with the product fibration by the finite group generated by the previous action together with the Weierstrass involution:

$$\begin{aligned} \mathbf{D}^2 \times \mathbf{S}^1 &\to \mathbf{D}^2 \times \mathbf{S}^1 \\ (z,x) &\mapsto (\bar{z},\bar{x}) \end{aligned}$$

gives the fibered solid pillow $W(\alpha,\beta)$ (here the bar denotes complex conjugation, as we assume $\mathbf{S}^1$ and $\mathbf{D}^2$ are the unit circle and disk in the complex line $\mathbf{C}$ respectively.)

Let $\mathcal{O}$ be a compact, Seifert fibered 3-orbifold. Since exceptional fibers are isolated, there are only finitely many of them. The boundary of $\mathcal{O}$ consists of fibered tori and pillows, and has therefore zero Euler characteristic.

H. Seifert [203] associated to a Seifert fibration on a 3-manifold a finite set of invariants which determines the fibration up to fiber-preserving homeomorphism (see also [174] and [198]). An analogous set of invariants can be defined for a Seifert fibration on a 3-orbifold, see [21]. These are not always topological invariants: there are 3-manifolds with nonisomorphic Seifert fibrations, such as $\mathbf{S}^3$ or lens spaces. The complete topological classification of compact Seifert manifolds follows from the work of several authors: H. Seifert and W. Threlfall [222] for spherical 3-manifolds different from lens spaces (1934), E. J. Brody [35] for lens spaces (1960), F. Waldhausen [238] for Haken Seifert 3-manifolds (1967), and P. Orlik, E. Vogt and H. Zieschang [175] for Seifert 3-manifolds with infinite fundamental group (1968).

Given any pair of coprime integers (p,q), one can construct infinitely many nonisomorphic Seifert fibrations on $\mathbf{S}^3$, whose generic fiber is a (p,q)-torus knot.

Another non-uniqueness phenomenon for Seifert fibrations occurs in manifolds that have several Seifert fibrations which are isomorphic, but not isotopic (e.g. $\mathbf{T}^3$). However, this situation is in some sense nongeneric, as shown by the following theorem.

Theorem 2.15. *Let $\mathcal{O}$ be a compact, orientable, Seifert fibered 3-orbifold with infinite fundamental group. If $\mathcal{O}$ is not covered by $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{T}^3$ or $\mathbf{T}^2 \times \mathbf{I}$, then the Seifert fibration on $\mathcal{O}$ is unique up to isotopy.*

When the base 2-orbifold is sufficiently large (i.e. is not a turnover or a quotient of a turnover) the proof is analogous to the one for 3-manifolds given by F. Waldhausen in [238] (see F. Bonahon and L. Siebenmann [22, Thm.2]). The proof in this case runs as follows. One splits $\mathcal{O}$ along a succession of *essential* (see Chapter 3 for a definition) saturated 2-suborbifolds for the first Seifert fibration, in order to get finitely many saturated tubular neighborhoods of the fibers. Since $\mathcal{O}$ is not covered by $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{T}^3$ or $\mathbf{T}^2 \times \mathbf{I}$, the second Seifert fibration can be isotoped so that these 2-suborbifolds are saturated for this fibration too, and that the two fibrations coincide near them, by Proposition 2.16 below (cf. [238], [22, Thm. 4], see also [103]). Thus the proof reduces to show that a Seifert fibration on a 3-orbifold with base a discal 2-orbifold is determined, up to an isotopy fixing the boundary, by the restriction of the fibration to the boundary; this follows easily from the description of the saturated tubular neighborhood of a fiber given above.

Proposition 2.16. *Let $\mathcal{O}$ be a compact, orientable, Seifert fibered 3-orbifold which is not covered by $\mathbf{S}^2 \times \mathbf{R}$. Up to isotopy, any orientable, essential 2-suborbifold F in $\mathcal{O}$ is* vertical *(i.e. saturated) or* horizontal *(i.e. everywhere transverse to the fibers).*

When the base of the fibration is not sufficiently large, the proof of Theorem 2.15 for 3-manifolds follows from [202] for most of the cases and from [15] for the remaining ones. For Seifert 3-orbifolds with non-empty singular locus the result can be deduced from the case of manifolds. Let M be a compact orientable Seifert 3-manifold satisfying the hypothesis of Theorem 2.15, and let G be a finite group of orientation preserving diffeomorphisms of M. If two G-equivariant Seifert fibrations on M are isotopic, then they are G-equivariantly isotopic [253].

2.5 Dehn filling on 3-orbifolds

In this section, we consider a compact 3-orbifold $\mathcal{O}$ whose boundary components are Euclidean 2-orbifolds. This is the case of a compact, orientable Seifert fibered 3-orbifold; this will apply also to finite volume hyperbolic 3-orbifolds, as we will see in Chapter 6. Here we work in a purely topological setting. Thus each component of $\partial\mathcal{O}$ is either a nonsingular torus, a pillow, or a turnover of type $S(n_1, n_2, n_3)$ with $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$.

If $\mathcal{O}$ is a manifold, then its boundary is a union of tori $T_1, \ldots, T_n$. A *Dehn filling* consists in gluing a solid torus V_i to each T_i. It is uniquely determined by choosing for each i the isotopy class of a simple closed curve $\mu_i \subset T_i$, called the *meridian* curve, and requiring that μ_i bounds a disk in V_i. Then any simple closed curve that bounds a disk in V_i but not in T_i must be isotopic to μ_i (see for example [192]). Hence, Dehn fillings on $\mathbf{T}^2$ are determined by primitive elements in $H_1(\mathbf{T}^2) \cong \mathbf{Z} \oplus \mathbf{Z}$ up to sign.

This construction was first used in 1910 by M. Dehn [52] to produce an infinite family of homology 3-spheres by removing a knotted solid torus in $\mathbf{S}^3$ and gluing it back with a different meridian curve. If one allows to do this surgery along finitely many solid tori in $\mathbf{S}^3$ instead of only one, this construction turns out to be quite general: R. Lickorish [133] and A. D. Wallace [243] proved in the early 1960's that every closed, orientable, connected 3-manifold may be obtained in this way; see [192, Chap.9] for more details on this topic.

Let us turn to orbifolds. A turnover $S^2(n_1, n_2, n_3)$ cannot bound the quotient of a solid torus, hence we cannot do any Dehn filling on it.

For a torus $T \subset \partial\mathcal{O}$, *Dehn filling* consists in gluing along T a solid torus V *whose core may be singular* (i.e. the quotient of a solid torus by a finite rotation around its core). As in the previous case, it is uniquely determined by the isotopy class of the simple closed curve on T which bounds a discal 2-orbifold in V. The underlying space of the new orbifold is obtained by Dehn filling on the underlying space of $\mathcal{O}$, but a new circle component may be added to the singular locus.

For a pillow $P \subset \partial\mathcal{O}$, *Dehn filling* consists in gluing a solid pillow (see Figure 2.5) *whose core may also be singular.* As for a (possibly singular) solid torus, a singular solid pillow V contains a unique meridian disk, up to isotopy, defined as "the" properly embedded discal 2-orbifold whose boundary (called the *meridian curve*) is not a torsion element in $\pi_1(\partial V)$. This meridian curve separates ∂V into two disks with two singular points each. Conversely, let P be a pillow and $\mu \subset P$ be a simple closed curve that separates P into two disks with two singular points each; then there is a diffeomorphism between P and the boundary of a solid pillow that sends μ to the meridian curve.

Summarizing, a Dehn filling consists in gluing a solid-toric orbifold to a toric component of the boundary, which is either a pillow or a smooth 2-torus. Moreover, like in the smooth 2-torus case, the Dehn filling on a pillow is determined by the isotopy class of the meridian curve.

To parametrize such isotopy classes, we use the exact sequence:

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \to \pi_1 S^2(2,2,2,2) \to \mathbf{Z}/2\mathbf{Z} \to 1,$$

coming from the fact that $S^2(2,2,2,2) = \mathbf{T}^2/(\mathbf{Z}/2\mathbf{Z})$.

The kernel $\mathbf{Z}\oplus\mathbf{Z}$ is the unique maximal torsion-free subgroup of $\pi_1 S^2(2,2,2,2)$ and the isotopy classes of meridians are determined by primitive elements in this torsion-free subgroup $\mathbf{Z} \oplus \mathbf{Z}$ up to sign.

It is an observation due to Seifert [203] that a Dehn filling on a compact orientable Seifert 3-orbifold, along a boundary torus or pillow T, always yields a Seifert fibered 3-orbifold if the meridian curve is not isotopic to a fiber. Moreover the core of the glued solid torus or pillow is an exceptional fiber iff the algebraic intersection number between the isotopy classes of the meridian curve and the fiber in the maximal torsion-free subgroup of $\pi_1 T$ is strictly larger than 1.

M. Dehn [52] showed that the Poincaré homology sphere [184] (i.e. the spherical dodecahedral space) can be obtained by Dehn filling on the exterior of the trefoil knot, which is Seifert fibered. This gives a purely topological way to see that this space is Seifert fibered.

A Seifert fibration, with an orientable base, on a compact, orientable 3-orbifold $\mathcal{O}$ can be obtained by Dehn fillings on a product orbifold, also compact and oriented, equipped with a product fibration. This is the original construction by H. Seifert [203]. Let $f_1, \ldots, f_r$ be the singular fibers of the Seifert fibration of $\mathcal{O}$. Pull out saturated open tubular neighborhoods V_i around these fibers plus one saturated open regular neighborhood V_0 around a regular fiber f_0. The Seifert fibration on $\mathcal{O} \setminus \mathrm{Int}(V_0 \cup \ldots \cup V_r)$ is a product fibration. One chooses sections $s_0, s_1, \ldots, s_r$ to the Seifert fibration on each torus or pillow ∂V_i, $i = 0, \ldots r$. Then $\mathcal{O}$ is obtained by Dehn filling on $\mathcal{O} \setminus \mathrm{Int}(V_0 \cup \ldots \cup V_r)$ in the following way:

- For $i = 1, \ldots r$ the meridian curve on the component V_i is $s_i^{\alpha_i} f^{\beta_i}$, if the exceptional fiber f_i is of type $\beta_i/\alpha_i \in (\mathbf{Q}/\mathbf{Z})^\star$. Beware that it is a simple closed curve iff α_i and β_i are coprimes, otherwise you glue a singular solid torus or pillow and the core of the Dehn filling belongs to the singular locus.

- The meridian curve on the component V_0 is $s_0 f^e$, where $e \in \mathbf{Z}$ is the obstruction to find a horizontal 2-suborbifold in $\mathcal{O} \setminus \mathrm{Int}(V_0 \cup \ldots \cup V_r)$ with boundary components isotopic to the given sections $s_0, s_1, \ldots, s_r$.

The integer Euler class e depends on the choice of sections, but the rational Euler number

$$e_0 := e - \sum_{i=1}^{r} \beta_i/\alpha_i$$

is a well-defined invariant of the Seifert fibration. It is the obstruction to find a multifold section of the projection $p : \mathcal{O} \to B$ onto the base B of the Seifert fibration (see [21], [198], [225, Chap.13]).

Chapter 3

Decompositions of orientable 3-orbifolds

In this chapter, all 2-orbifolds and 3-orbifolds are assumed to be connected and orientable unless mentioned otherwise. In general, 2-suborbifolds of 3-orbifolds are assumed to be either properly embedded or suborbifolds of the boundary.

The main goal of this chapter is to establish existence and properties of the topological decomposition of a 3-orbifold along spherical 2-suborbifolds, toric 2-suborbifolds, and hyperbolic turnovers involved in Thurston's Geometrization Conjecture. This conjecture is precisely stated in the last Section 3.7.

In Section 3.1 we state the results which will be proved in Sections 3.3, 3.4 and 3.5, using the theory of normal 2-suborbifolds presented in Section 3.2. In Section 3.6 we discuss some equivariant theorems, and in Section 3.7 we present the orbifold version of Thurston's Geometrization Conjecture and state the Orbifold Theorem which will be discussed in Chapter 9.

3.1 General discussion

For convenience, we introduce some terminology. A *system* (of 2-suborbifolds) in a 3-orbifold $\mathcal{O}$ is a finite collection $\mathfrak{F} = \{F_1, \ldots, F_n\}$ of pairwise disjoint, properly embedded (orientable) 2-suborbifolds. The 2-orbifolds $F_1, \ldots, F_n$ are called the *components* of $\mathfrak{F}$. If each component of $\mathfrak{F}$ is spherical (resp. toric), we call $\mathfrak{F}$ a *spherical system* (resp. a *toric system*).

We shall denote by $\mathcal{O}\backslash\mathfrak{F}$ the orbifold obtained from $\mathcal{O}$ by removing a disjoint union of open product neighborhoods of the components of $\mathfrak{F}$. The operation of removing such neighborhoods is called *splitting* $\mathcal{O}$ *along* $\mathfrak{F}$.

Let $F_0, F_1 \subset \mathcal{O}$ be 2-suborbifolds (either properly embedded or contained in $\partial\mathcal{O}$). We say that F_0, F_1 are *parallel* if they cobound in $\mathcal{O}$ a suborbifold $F \times [0,1] \subset \mathcal{O}$, called a *product region* such that $F \times \{0\} = F_0$, $F \times \{1\} = F_1$ and $\partial F \times [0,1] \subset \partial M$. A properly embedded 2-suborbifold $F \subset M$ is *∂-parallel* (boundary-parallel) if F is parallel to a suborbifold of ∂M.

A 2-suborbifold $F \subset \mathcal{O}$ is *compressible* if either F is spherical and bounds a discal 3-suborbifold, or there exists a discal 2-suborbifold $D \subset \mathcal{O}$, called a *compression disk,* such that $\partial D = D \cap F \subset \operatorname{Int} \mathcal{O}$ and ∂D does not bound a discal 2-suborbifold in F. Otherwise it is *incompressible*. Note that the term "compression disk" is a slight abuse of language since it might be a disk with a singular point.

It is obvious that discal 2-orbifolds, bad 2-orbifolds and turnovers of non-positive Euler characteristic are always incompressible, since they do not have essential curves. A spherical turnover is compressible if and only if it surrounds a vertex in the singular locus.

A 2-suborbifold $F \subset \mathcal{O}$ is *∂-compressible* if either F is a discal 2-suborbifold which is ∂-parallel, or if there exists a discal 2-suborbifold $D \subset \mathcal{O}$, called a *∂-compression disk,* such that ∂D is the union of two arcs α, β with $\partial\alpha = \partial\beta = \alpha \cap \beta$, $\alpha \subset F$, $\beta \subset \partial\mathcal{O}$, and α does not cobound a discal suborbifold of F with an arc in ∂F. Otherwise it is *∂-incompressible.*

A 2-suborbifold $F \subset \mathcal{O}$ is *essential* if it is incompressible, ∂-incompressible and not ∂-parallel.

Let $F \subset \mathcal{O}$ be a compressible 2-orbifold and $D \subset \mathcal{O}$ be a compression disk. A *compression surgery* on F along D consists in replacing a tubular neighborhood of ∂D in F by two parallel copies of D. Notice that this process increases the Euler characteristic of the 2-orbifold by $\frac{2}{n}$, where n is the order of the cone point in D. Similarly one can define *∂-compression surgery.*

Definition. A 3-orbifold $\mathcal{O}$ is *irreducible* if $\mathcal{O}$ contains no bad 2-suborbifold, and every (orientable) spherical 2-suborbifold of $\mathcal{O}$ is compressible.[1]

A 3-orbifold is *atoroidal* if it is irreducible and contains no essential toric 2-suborbifold.

Here is a fundamental result, due to Alexander [2] in the case of manifolds.

Theorem 3.1. *Any 2-suborbifold of a discal 3-orbifold (or a spherical 3-orbifold) is compressible. In particular, discal and spherical 3-orbifolds are irreducible.*

[1] Note however that $\mathcal{O}$ may contain 1-sided, nonorientable 2-suborbifolds.

Theorem 3.1 implies that any discal 2-suborbifold of a discal 3-orbifold is ∂-parallel.

An important step in the study of compact 3-orbifolds without bad 2-suborbifold is given by the following splitting result, which is proved in Section 3.3:

Theorem 3.2. *Let $\mathcal{O}$ be a compact 3-orbifold without bad 2-suborbifold. There exists a spherical system $\mathfrak{S}$ in $\mathcal{O}$ such that for every component X of $\mathcal{O} \setminus \mathfrak{S}$, the 3-orbifold obtained from X by gluing a discal 3-orbifold along each spherical component of ∂X is irreducible.*

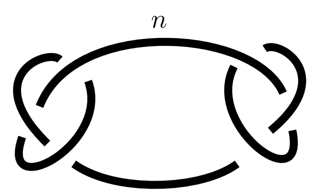

Figure 3.1: When $\mathcal{O}$ is $\mathbf{S}^3$ with singular locus a knot, Theorem 3.2 corresponds to the decomposition of the knot as a connected sum of prime knots. The picture is the sum of two factors.

For 3-manifolds Theorem 3.2 goes back to H. Kneser [128], whose article contains most of the essential ideas for the general case.[2] Our statement focuses on the existence of the decomposition. There is no uniqueness of the spherical system $\mathfrak{S}$ up to isotopy, even if it has the minimal number of elements. However the 3-orbifold summands, obtained from the non-discal components X of $\mathcal{O} \setminus \mathfrak{S}$ by gluing a discal 3-orbifold along each spherical component of ∂X, are unique up to diffeomorphism. For the uniqueness of these nonspherical summands, we refer to [159] for 3-manifolds and to [194, 102] for 3-orbifolds with underlying space S^3 and singular set a link (see also [36, Chap. 7] and [126, Chap. 3]). The general case can be worked out in a similar way. In fact it follows from the Orbifold Theorem (see Section 3.7 and Chapter 9) that a compact 3-orbifold without bad 2-suborbifold is finitely covered by a manifold and hence the uniqueness of

[2]Kneser's formulation is slightly different from ours: he considers only connected sum decompositions, i.e. separating spheres, and allows the summands to be homeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$, whereas we allow splitting along nonseparating spheres. The link between both statements is provided by the fact that a self-connected sum of an orientable 3-manifold M is homeomorphic to a connected sum of M with $\mathbf{S}^2 \times \mathbf{S}^1$.

the nonspherical summands can be deduced from the manifold case by using the Equivariant Sphere Theorem (see Section 3.6, Theorem 3.21).

Thus the study of compact 3-orbifolds without bad 2-suborbifolds is (at least theoretically) reduced to the study of irreducible 3-orbifolds, with which Theorem 3.3 below is concerned.

Theorem 3.3. *Let $\mathcal{O}$ be a compact, irreducible* 3*-orbifold. There exists a system* $\mathfrak{C}$ *of essential, pairwise nonparallel toric* 2*-suborbifolds of* $\mathcal{O}$ *such that every component of* $\mathcal{O}\backslash\mathfrak{C}$ *is Seifert fibered or atoroidal. A minimal such system is unique up to isotopy.*

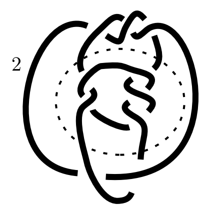

Figure 3.2: When $\mathcal{O}$ is S^3 with singular set a knot and branching index 2, Theorem 3.3 corresponds to the tangle decomposition along Conway spheres, as in the picture. The toric suborbifold is indicated with dotted lines.

The toric splitting, also called the JSJ-splitting, has been first proved by W. Jaco and P. Shalen [115] and K. Johannson [118] for 3-manifolds in the mid 1970's (see also [173] for a simpler proof). It has been generalized to 3-orbifolds by F. Bonahon and L. Siebenmann [22]. This natural splitting gives the topological basis for Thurston's geometrization program; it is a fundamental result for the study of compact irreducible 3-orbifolds.

Compact, irreducible, atoroidal 3-orbifolds may be further decomposed along hyperbolic turnovers. Recall that a turnover is a 2-orbifold with underlying space a sphere and three cone points. Turnovers are characterized among closed Euclidean and hyperbolic 2-orbifolds by the fact that they contain no essential curves, and therefore they have a special behavior in many respects. That is why it is sometimes useful to get rid of essential turnovers in the 3-orbifold under consideration. This will become clearer in the next chapter when we introduce *Haken* and *small* orbifolds.

Theorem 3.4. *Any compact, irreducible atoroidal* 3*-orbifold can be split along a system of essential, pairwise nonparallel hyperbolic turnovers such that the* *resulting pieces contain no essential turnovers. This system is unique up to isotopy.*

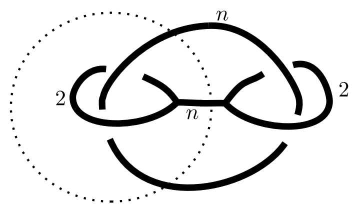

Figure 3.3: Assume that $n \geq 4$. Then this orbifold is atoroidal, and the hyperbolic turnover shown in dotted lines splits it into two orbifolds that do not contain essential turnovers (in fact no essential suborbifolds at all.)

3.2 Normal 2-suborbifolds

One important tool to prove Theorem 3.2 and the existence part of Theorems 3.3 and 3.4 is the notion of normal 2-suborbifold, which extends the notion of normal surface in a 3-manifold introduced by H. Kneser [128] and W. Haken [99]. This notion is very useful for studying decision problems, cf. [114, 193].

Definition. Let $\mathcal{O}$ be a 3-orbifold. A *triangulation* $\mathcal{T}$ of $\mathcal{O}$ is a triangulation of the underlying manifold $|\mathcal{O}|$ for which $\Sigma_{\mathcal{O}}$ is a subcomplex of the 1-skeleton and each 3-simplex meets $\Sigma_{\mathcal{O}}$ in either one vertex, one edge, or the empty set. We call $(\mathcal{O}, \mathcal{T})$ a *triangulated orbifold*.

A 2-suborbifold F is *in general position* with respect to $\mathcal{T}$ if the underlying surface $|F|$ misses the 0-skeleton and intersects transversely the 1-skeleton and the 2-skeleton.

A 2-suborbifold F is *normal* if $|F|$ (considered as an embedded surface in $|\mathcal{O}|$) is in general position with respect to $\mathcal{T}$ and for each 3-simplex σ, $\sigma \cap |F|$ is a union of disks intersecting each edge of σ in at most one point and intersecting at least one edge of σ.

A system of 2-suborbifolds is *normal* if each component is normal.

We will also need a notion of general position for pairs of 2-suborbifolds. If F_1, F_2 are orientable 2-suborbifolds of $\mathcal{O}$, we say that they are *in general position* if their singular loci $\Sigma_{F_1}, \Sigma_{F_2}$ are disjoint and $F_1 - \Sigma_{F_1}$ intersects $F_2 - \Sigma_{F_2}$ transversally. The intersection is then a disjoint union of curves and arcs avoiding the singular loci. Given two suborbifolds F_1, F_2, it follows from standard theorems that one can always perform small isotopies to make F_1, F_2 in general position. This allows us to prove facts inductively on the number of intersection components.

General position can also be defined for nonorientable suborbifolds. For example, to deal with suborbifolds with silvered boundary, one would require the silvered curves and arcs to intersect transversally. This will not be needed in this book.

Our main interest for normal 2-suborbifolds comes from the following finiteness result for normal systems.

Theorem 3.5. *Let $(\mathcal{O}, \mathcal{T})$ be a compact triangulated 3-orbifold. There is an integer $h(\mathcal{O}, \mathcal{T}) > 0$ such that if $\mathfrak{F}$ is any normal system of 2-suborbifolds of $\mathcal{O}$ with more than $h(\mathcal{O})$ components, then some component of $\mathcal{O} \backslash \mathfrak{F}$ is a product region between two components of $\mathfrak{F}$.*

This fact was discovered by H. Kneser and is the prototype for all finiteness theorems. The rest of this section is devoted to the proof of Theorem 3.5.

A *product region* in a 3-simplex σ is a 3-ball $(B^2 \times I, \partial B^2 \times I) \subset (\sigma, \partial\sigma)$ avoiding the vertices, and such that $B^2 \times \{0\}$ and $B^2 \times \{1\}$ are properly embedded.

Remark. Let $(\mathcal{O}, \mathcal{T})$ be a compact triangulated 3-orbifold and $\mathfrak{F}$ be a normal system of 2-suborbifolds. Let σ be a 3-simplex. Then $\sigma \backslash \mathfrak{F}$ has at most 6 components that are not product regions.

Lemma 3.6. *Let $(\mathcal{O}, \mathcal{T})$ be a compact triangulated 3-orbifold. Let t be the number of 3-simplices of $\mathcal{T}$. Let $\mathfrak{F}$ be a normal system of 2-suborbifolds and set $u := \#\pi_0(\mathcal{O} \backslash \mathfrak{F})$. Assume that $u > 6t$. Then except for at most $6t$ of them, each component of $\mathcal{O} \backslash \mathfrak{F}$ is a product region between two components of $\mathfrak{F}$ or a twisted* **I**-*bundle.*

Proof. By the previous remark, except for at most $6t$ of them, each component X of $\mathcal{O} \backslash \mathfrak{F}$ meets each 3-simplex in product regions. The **I**-bundle structures of these product regions are compatible, so X is an **I**-bundle. If it is trivial, then it must be a product region between two distinct components of $\mathfrak{F}$, for otherwise $\mathfrak{F}$ would consist of a single nonseparating 2-suborbifold, contradicting the hypothesis that $u > 6t$. □

Lemma 3.7. *Let X be a compact 3-orbifold. Let $\mathfrak{F} = \{F_1, \ldots, F_n\}$ be a system of 2-suborbifolds of X. Then $X \backslash \mathfrak{F}$ has at least $n + 1 - \dim H_2(|X|, \partial|X|, \mathbf{Z}_2)$ components that are not twisted* **I**-*bundles.*

Proof. The proof is by induction on $\dim H_2(|X|, \partial|X|, \mathbf{Z}_2)$. If this number vanishes, then every 2-suborbifold of X separates and X contains no twisted I-bundle, so the conclusion holds. Otherwise, the conclusion might fail only if some F_i does not separate or bounds a twisted I-bundle. If this happens, apply the induction hypothesis to $X\backslash\{F_i\}$. □

Lemmas 3.6 and 3.7 show that the conclusion of Theorem 3.5 holds with the integer $h(\mathcal{O}, \mathcal{T}) = 6t + \dim H_2(|X|, \partial|X|, \mathbf{Z}_2)$, where t is the number of 3-simplices of the triangulation $\mathcal{T}$ of $\mathcal{O}$.

3.3 The spherical decomposition

For convenience, we call an orbifold $\mathcal{O}$ *punctured spherical* (resp. *punctured irreducible*) if it is diffeomorphic to $\mathcal{O}' - \bigcup_i \operatorname{Int} B_i$, where $\mathcal{O}'$ is a spherical (resp. irreducible) 3-orbifold and $\{B_1, \ldots, B_n\}$ is a collection of disjoint discal 3-suborbifolds.

Definition. A spherical system $\mathfrak{S}$ is *essential* if no component of $\mathcal{O}\backslash\mathfrak{S}$ is a punctured spherical 3-orbifold. (In particular, no component of $\mathcal{O}\backslash\mathfrak{S}$ is a product region.)

Theorem 3.8 (Finiteness Theorem for spherical suborbifolds). *Let $\mathcal{O}$ be a compact 3-orbifold without bad 2-suborbifolds. There is an integer $s(\mathcal{O}) > 0$ such that any essential spherical system in $\mathcal{O}$ has at most $s(\mathcal{O})$ components.*

Theorem 3.8 is immediate from Theorem 3.5 and the following lemma.

Lemma 3.9. *For any triangulation $\mathcal{T}$ of $\mathcal{O}$ and for every integer $n > 0$, if $\mathcal{O}$ admits an essential spherical system of cardinal n, then $\mathcal{O}$ admits a normal, essential spherical system of cardinal n.*

Proof. Let $F \subset \mathcal{O}$ be a general position 2-suborbifold. The *complexity* of F is the triple $\mathrm{c}(F) = (\mathrm{sing}(F), \mathrm{c}_1(F), \mathrm{c}_2(F))$, where $\mathrm{sing}(F)$ is the number of cone points of F, $\mathrm{c}_1(F)$ is the cardinal of $|F| \cap \mathcal{T}^{(1)}$ and $\mathrm{c}_2(F)$ is the sum over all 2-simplices σ of $\#\pi_0(|F| \cap \sigma)$. The *complexity* of a system of 2-suborbifolds is the sum of the complexities of its components.

Let $\mathfrak{S} = \{S_1, \ldots, S_n\}$ be a general position, essential spherical system. Suppose that $\mathfrak{S}$ has minimal complexity among all such systems of cardinality n. If $\mathfrak{S}$ is not normal, then some 3-simplex σ intersects $\mathfrak{S}$ in a wrong way. By a case-by-case analysis somewhat similar to the proof of Lemma 3.10, we shall produce an essential spherical system of the same cardinality and smaller complexity, thereby obtaining a contradiction. There are three cases to consider (details are left as exercises.)

Case 1 Some component U of $\sigma \cap \bigcup_i S_i$ is not a disk. Then U cannot be a sphere. (Hint: use Alexander's Theorem.) Thus U admits a compression disk, and after compression surgery, we get a new system with smaller complexity.

Case 2 Some component U of $\sigma \cap \bigcup_i S_i$ is a disk that does not meet $\mathcal{T}^{(1)}$. Then there is an isotopy that decreases c_2, by pushing this disk outside σ.

Case 3 Some component U of $\sigma \cap \bigcup_i S_i$ is a disk that meets the same edge twice. Then a ∂-compression along some innermost arc with both endpoints in the same edge produces an essential system with sing no greater and c_1 smaller.

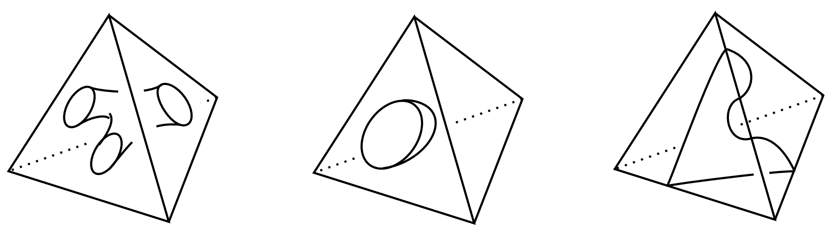

Figure 3.4: Examples of the different cases of "bad" components U of $\sigma \cap \bigcup_i S_i$ in the proof of Lemma 3.9.

This completes the proof of Lemma 3.9, hence of Theorem 3.8. □

Lemma 3.10. *Let X be a compact 3-orbifold. If X is not a punctured irreducible 3-orbifold, then either X contains a nonseparating spherical 2-suborbifold, or X contains a separating spherical 2-suborbifold S such that no component of $X \setminus \{S\}$ is a punctured spherical 3-orbifold.*

Proof. Let $\{S_1, \ldots, S_n\}$ be the spherical components of ∂X. Let $\hat{X}$ be a 3-orbifold obtained from X by gluing a discal 3-orbifold B_i along each S_i. By hypothesis, $\hat{X}$ is not irreducible. Let $S \subset \hat{X}$ be an incompressible spherical 2-suborbifold. Assume that S is in general position with respect to the S_i's and intersects them in a minimal number of components. We are going to show that S is in fact disjoint from the S_i's.

Suppose by contradiction that $S \cap S_{i_0} \neq \emptyset$ for some i_0. If some component F of $S \cap B_{i_0}$ is nondiscal, then by Theorem 3.1, F is compressible. Choose a compression disk D for F whose interior does not meet S. The result of compression surgery along D is a pair of disjoint separating spherical 2-suborbifolds, at least one of which is incompressible. Thus we can replace S by another incompressible, separating, spherical 2-suborbifold, still denoted by S, such that $\#\pi_0(S \cap \bigcup S_i)$ has not increased, and the Euler characteristic of $S \cap S_{i_0}$ has increased.

Thus after a finite number of such modifications, we can assume that S intersects B_{i_0} only in discal 2-suborbifolds. Let F be a component of $S \cap B_{i_0}$. By Theorem 3.1, F is ∂-parallel in B_{i_0}. Choose F such that some product region between F and a discal 2-suborbifold of S_{i_0} contains no other component of $S \cap B_{i_0}$. Then observe that $\#\pi_0(S \cap \bigcup S_i)$ can be decreased by an isotopy of S, giving a contradiction.

We have shown that S is disjoint from the S_i's. Theorem 3.1 implies that $S \subset X$. Either S is nonseparating or it separates X in two components. These components are not punctured spherical because otherwise S would compress in $\hat{X}$. This completes the proof of Lemma 3.10. □

Proof of Theorem 3.2. Let $\mathcal{O}$ be a compact 3-orbifold without bad 2-suborbifold. By Theorem 3.8, $\mathcal{O}$ contains a (possibly empty) finite maximal essential spherical system $\mathfrak{S}'$. Let X be a component of $\mathcal{O} \backslash \mathfrak{S}'$. By Lemma 3.10 and maximality of $\mathfrak{S}'$, we see that either X is punctured irreducible or it contains a nonseparating spherical 2-suborbifold S. Again by maximality, X split along S is punctured spherical. Since there are only finitely many components X, we may enlarge $\mathfrak{S}'$ if necessary by adding a finite system of nonseparating spherical 2-suborbifolds to get a (possibly inessential) spherical system $\mathfrak{S}$ such that each component of $\mathcal{O} \backslash \mathfrak{S}$ is punctured irreducible. This completes the proof of Theorem 3.2. □

3.4 The toric splitting of an irreducible 3-orbifold

Definition. Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. A (nonspherical) system $\mathfrak{F} = \{F_1, \ldots, F_n\}$ of 2-suborbifolds of $\mathcal{O}$ is *essential* if each component is essential and no two components are parallel.

We shall need a finiteness theorem for essential systems of (nonspherical) 2-suborbifolds in compact, irreducible 3-orbifolds. Its proof from Theorem 3.5 is very similar to that of Theorem 3.8 and left as an exercise in manipulating normal suborbifolds.

Theorem 3.11 (Finiteness theorem for irreducible 3-orbifolds). *Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. There is an integer $h(\mathcal{O}) > 0$ such that any essential system in $\mathcal{O}$ has at most $h(\mathcal{O})$ components.*

The following useful notion has been used by W. Neumann and G. Swarup [173] in the case of tori in 3-manifolds.

Definition. Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. An essential toric 2-suborbifold $F \subset \mathcal{O}$ is *canonical* if it can be isotoped off any essential toric 2-suborbifold.

Remark. Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. Let $T \subset \mathcal{O}$ be an incompressible turnover and $F \subset \mathcal{O}$ be an incompressible 2-suborbifold. Since every simple closed curve on T bounds a discal 2-suborbifold in T, every intersection curve in $T \cap F$ bounds a discal 2-suborbifold in F. Hence T can be isotoped off F. It follows that Euclidean turnovers are always canonical.

In general, tori and pillows are not canonical. For example, by using the intersection pairing between the first and the second homology groups, one easily shows that a torus $\mathbf{T}^2 \times \{*\} \subset \mathbf{T}^2 \times \mathbf{S}^1$ is not canonical. More generally, let F be a closed surface and γ be a simple closed curve in F which does not bound a disk. Then the torus $\mathbf{S}^1 \times \{\gamma\} \subset \mathbf{S}^1 \times F$ is not canonical.

Proposition 3.12. *Let F be a closed hyperbolic or Euclidean 2-orbifold. Any incompressible compact 2-suborbifold $(S, \partial S) \subset (F \times \mathbf{I}, F \times \{0\})$ is ∂-parallel. In particular if $\partial S = \emptyset$, then S is isomorphic to F.*

Proof. Assume that S is not ∂-parallel. After finitely many ∂-compressions, at least one component S' of the surgered 2-suborbifold must be essential in $F \times \mathbf{I}$ and satisfy $\partial S' \subset F \times \{0\}$.

The orbifold $F \times I$ is the quotient of the Seifert fibered orbifold $F \times \mathbf{S}^1$ by the orientation reversing and fiber preserving involution τ acting by a reflection in the S^1-factor. Then the closed 2-suborbifold $S' \cup \tau(S')$ is essential in $F \times \mathbf{S}^1$. Otherwise the existence of a compression disk for $S' \cup \tau(S')$ would imply the existence of a τ-equivariant compression disk (cf. Theorem 3.19 in Section 3.6), contradicting the fact that S' is essential in $F \times \mathbf{I}$. However, by construction $S' \cup \tau(S')$ cannot be isotopic in $F \times \mathbf{S}^1$ to a vertical or a horizontal surface. This is incompatible with Theorem 2.16. For a different proof based on Stallings' 3-dimensional h-cobordism theorem, we refer to [55]. □

Corollary 3.13. *Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. A system $\mathfrak{F}$ of closed 2-suborbifolds is essential if and only if each component is essential and no component of $\mathcal{O} \backslash \mathfrak{F}$ is a product region between two components of $\mathfrak{F}$.*

Proof. Necessity is clear. For sufficiency, assume that two components of $\mathfrak{F}$ are parallel. Let $X = F \times \mathbf{I}$ be a product region between them. If some component F' of $\mathfrak{F}$ lies in $F \times (0,1)$, then by Theorem 3.12, F' is parallel to $F \times \{0\}$. By induction, one sees that some component of $\mathcal{O}\backslash\mathfrak{F}$ is a product region. □

We recall that a 3-orbifold is called solid-toric if it is finitely covered by $\mathbf{S}^1 \times \mathbf{D}^2$.

Proposition 3.14. *Let $\mathcal{O}$ be a compact, irreducible 3-orbifold. Let T be a compressible toric 2-suborbifold of $\mathcal{O}$. Then either T lies in a discal 3-orbifold or T bounds a solid-toric 3-orbifold.*

Proof. An easy consequence of the irreducibility of $\mathcal{O}$ is that T must separate $\mathcal{O}$ (Exercise: prove it.)

Let $\mathcal{O}'$ be the closure of a component of $\mathcal{O}\backslash T$ which contains a compression disk D for T. A surgery of T along D yields a spherical 2-suborbifold S in $\mathcal{O}'$. If S bounds a discal 3-suborbifold in $\mathcal{O}'$, then $\mathcal{O}'$ is a solid-toric 3-suborbifold of $\mathcal{O}$ bounded by T. Otherwise S must bound in $\mathcal{O}$ a discal 3-suborbifold containing T, since $\mathcal{O}$ is irreducible. □

We now state and prove a more precise version of Theorem 3.3.

Theorem 3.15 (Toric splitting). *Let $\mathcal{O}$ be a compact, irreducible 3-orbifold.*

i. $\mathcal{O}$ admits a (possibly empty) maximal essential system $\mathfrak{C}$ of canonical toric 2-suborbifolds, which is unique up to isotopy.

ii. Every component of $\mathcal{O}\backslash\mathfrak{C}$ is atoroidal or Seifert fibered.

iii. Seifert fibrations on adjacent pieces never match up.

Lemma 3.16. *Let W be a compact, irreducible 3-orbifold. If W is atoroidal and contains an essential annular 2-suborbifold A that meets only toric components of ∂W, then W admits a Seifert fibration for which A is vertical.*

Proof. Let N be a regular neighborhood of the union of A and of the component(s) of ∂W that it meets. It is easy to construct a Seifert fibration on N such that A is vertical.

Let T be a component of ∂N. Then by an easy Euler characteristic argument, T is toric. Since W is atoroidal, T is either compressible or boundary-parallel. Now T cannot be compressible or boundary-parallel in N, so it is compressible or boundary-parallel on the other side. Hence T separates.

Let X be the closure of the component of $W - T$ that does not meet $\operatorname{Int} N$. If T is boundary-parallel, then X is a product region between T and some component of ∂W. If T is compressible, then by Proposition 3.14 X is solid toric, because T, containing an essential annular 2-suborbifold, cannot lie in a discal 3-orbifold. It is now easy to extend the Seifert fibration on N to all of W. □

Lemma 3.17. *Let X be a compact, irreducible 3-orbifold. If X is not atoroidal, but does not contain any canonical toric 2-suborbifold, then X is Seifert fibered.*

Proof. Let $\mathfrak{T} \subset X$ be a maximal essential toric system. By Theorem 3.11, such a system exists, and by hypothesis it is not empty. Moreover, no component of $\mathfrak{T}$ is canonical.

Let T, T' be essential toric 2-suborbifolds that are not isotopically disjoint. Then one can always isotope T' so that $T \cap T'$ is a disjoint union of finitely many transverse curves, and the number of such curves is minimal. We shall refer to this operation as "putting T' in minimal position with respect to T". Note that all intersection curves must then be essential, because T, T' are incompressible and X is irreducible. (Exercise: prove it.)

Special Case $\mathfrak{T}$ consists of a single nonseparating 2-suborbifold T.

Since T is not canonical, there exists an essential toric 2-suborbifold T' such that T, T' are not isotopically disjoint. Put T' in minimal position with respect to T and set $W := X \backslash \mathfrak{T}$. Then the trace of T' in W in a union of essential annuli. Let A be one of these annuli such that ∂A meets two distinct components of ∂W. Lemma 3.16 gives a Seifert fibration on W such that A is vertical. When we glue back two components of ∂W to obtain X, the two components of ∂A become isotopic curves on T. Hence X is Seifert fibered.

Generic Case For each component $T \in \mathfrak{T}$, choose an essential toric 2-suborbifold T' in minimal position such that T, T' are not isotopically disjoint. For each component F of $\partial(X \backslash \mathfrak{T})$ coming from T, choose a component of $(X \backslash \mathfrak{T}) \cap T'$ that is an essential annular 2-suborbifold meeting F. Let $\mathfrak{A}$ be the collection of these annular 2-suborbifolds.

Let W be a component of $X \backslash \mathfrak{T}$. If only one component of ∂W comes from $\mathfrak{T}$, then W contains one element A of $\mathfrak{A}$. Lemma 3.16 gives a Seifert fibration on W such that A is vertical. If at least two components of ∂W come from $\mathfrak{T}$, then W may contain several elements $A_1, \ldots, A_k \in \mathfrak{A}$. Lemma 3.16 gives (*a* *priori* nonisotopic) Seifert fibrations $p_1, \ldots, p_k$ on W such that A_i is vertical with respect to p_i.

Since $\mathfrak{T}$ is essential and we are not in the Special Case, W cannot be finitely covered by $\mathbf{T}^2 \times \mathbf{I}$. Otherwise, since ∂W is orientable, the classification of finite group actions on $\mathbf{T}^2 \times \mathbf{I}$ would imply that W is a product region between two components of $\mathfrak{T}$, see [154].

Hence by Theorem 2.15, the fibrations $p_1, \ldots, p_k$ are pairwise isotopic and we can find a Seifert fibration such that $A_1, \ldots, A_k$ are all vertical. Doing this for every component of $X \backslash \mathfrak{T}$ and gluing back, we find a Seifert fibration on X. □

Proof of Theorem 3.15. (i) By Theorem 3.11, there exists a maximal essential system $\mathfrak{C}$ of canonical toric 2-suborbifolds. Uniqueness follows from Corollary 3.13; we leave it as an exercise.

(ii) follows from Lemma 3.17.

(iii) Let T be a component of $\mathfrak{C}$ and W_1, W_2 two components of $\mathcal{O} \backslash \mathfrak{C}$ which are adjacent to T. Assume that W_1 and W_2 are both Seifert fibered and that the Seifert fibrations match up. Since T is not a turnover and is incompressible in both W_1 and W_2, there are two vertical essential annular 2-suborbifolds $A_i \subset W_i$. After an isotopy, one can glue A_1 and A_2 along their boundaries to get an essential toric 2-suborbifold $T' \subset \mathcal{O}$ such that T and T' are not isotopically disjoint. This finishes the proof of Theorem 3.15. □

3.5 The turnover splitting of an irreducible, atoroidal 3-orbifold

The following is a restatement of Theorem 3.4.

Theorem 3.18 (Turnover splitting). *Let $\mathcal{O}$ be a compact, irreducible and atoroidal* 3*-orbifold.*

i. A (possibly empty) maximal essential system $\mathfrak{H}$ of hyperbolic turnovers in $\mathcal{O}$ is unique up to isotopy.

ii. Components of $\mathcal{O} \backslash \mathfrak{H}$ do not contain any essential turnover.

Note that since $\mathcal{O}$ is irreducible and atoroidal, it does not contain any essential spherical or Euclidean turnover. Hence (ii) follows from (i). The proof of (i) follows the same outline as the proof of the toric splitting Theorem and is simpler, because as we already remarked, hyperbolic turnovers are always *canonical* in the sense of the previous section.

As in the proof of Theorem 3.15, the existence of a maximal family of hyperbolic turnovers follows from Theorem 3.11 and the uniqueness from Corollary 3.13. We leave the details to the reader as an exercise.

3.6 Equivariant Theorems

Two classical results in 3-manifold topology are the Loop Theorem and the Sphere Theorem (see [106, 113] and the references therein). These theorems provide a connection between the homotopy groups of a 3-manifold and existence and properties of incompressible surfaces in M. One may wonder whether they can be generalized to orbifolds.

Let $\mathcal{O}$ be a *good*, compact 3-orbifold. Any statement about $\mathcal{O}$ can be translated into an "equivariant" statement for the universal cover of $\mathcal{O}$, which is a manifold. For instance, the existence of a compression disk for a given 2-suborbifold $F \subset \mathcal{O}$ is equivalent to the existence of an equivariant compression disk D for the preimage of F in $\tilde{\mathcal{O}}$, where *equivariant* means that for every covering transformation g, either $gD = D$ or $gD \cap D = \emptyset$. Thus one can hope to prove a theorem on orbifolds by proving an equivariant theorem for manifolds. Of course, this approach fails for orbifolds that are not *a priori* supposed to be good.

It turns out that equivariant versions of the Loop Theorem and the Sphere Theorem were obtained by Meeks and Yau [156], motivated by the Smith Conjecture and other questions about group actions on 3-manifolds. Their proofs were based on minimal surfaces, i.e. surfaces that locally minimize area. They originally dealt with finite group actions on compact manifolds. We shall give more general statements (cf. [116]) together with useful corollaries that will be used in later chapters.

A 2-suborbifold F of an orbifold $\mathcal{O}$ is π_1-*injective* if the morphism $\pi_1 F \to \pi_1 \mathcal{O}$ induced by inclusion is injective. Given a proper action of a discrete group Γ on $\mathcal{O}$, we say that F is Γ-*equivariant* if for all $g \in \Gamma$, either $gF = F$ or $gF \cap F = \emptyset$.

Theorem 3.19 (Equivariant Loop Theorem). *Let M be a 3-manifold with a proper (smooth) action of a discrete group Γ. Let F be an equivariant subsurface of ∂M. If F is not π_1-injective, then it admits an equivariant compression disk.*

Here is a straightforward corollary:

Corollary 3.20. *Let $\mathcal{O}$ be a good 3-orbifold. Let $F \subset \mathcal{O}$ be a 2-suborbifold which is either properly embedded or a suborbifold of $\partial\mathcal{O}$. If F is not π_1-injective, then F is compressible.*

Theorem 3.21 (Equivariant Sphere Theorem). *Let M be a 3-manifold. Let Γ be a discrete group acting properly on M. If $\pi_2 M$ is nontrivial, then M contains a Γ-equivariant 2-sphere representing a nontrivial element of $\pi_2 M$.*

By using Hurewicz's Theorem one can deduce the following corollary (see [106]):

Corollary 3.22. *Let $\mathcal{O}$ be a good, irreducible 3-orbifold. Then one of the following holds:*

i. $\mathcal{O}$ is closed and has finite fundamental group, or

ii. The universal cover of $\mathcal{O}$ is a contractible manifold.

Meeks, Simon and Yau [157] used minimal surfaces to prove the long standing conjecture that universal covers of irreducible manifolds are irreducible. A combinatorial approach to equivariant theorems was then given by Dunwoody, and later by Jaco and Rubinstein [116, 117]. The latter authors first used least weight normal surfaces (i.e. normal surfaces that minimize the number of intersection points with the 1-skeleton, which can be thought of as a combinatorial version of area) and then introduced PL minimal surfaces, which are normal surfaces locally minimizing a half-combinatorial, half-differential geometric functional called PL area. PL minimal surfaces share enough properties with (analytic) minimal surfaces to serve as tools to prove equivariant theorems, but their existence is easier to establish. (See however [104] for a simplified approach to the existence of minimal surfaces in 3-manifolds.)

To illustrate this, we shall sketch a proof of the following generalization of the main result of Meeks-Simon-Yau.

Theorem 3.23. *Let $\mathcal{O}$ be a 3-orbifold. If $\mathcal{O}$ is irreducible, then its universal cover $\tilde{\mathcal{O}}$ is irreducible.*

Remark. The converse is true: if $\tilde{\mathcal{O}}$ is irreducible, then $\mathcal{O}$ is irreducible. This is an easy exercise when $\mathcal{O}$ is a manifold. For the general case the proof is similar, but it uses the nontrivial fact that smooth finite group actions on the 3-ball are conjugated to orthogonal actions. In the cyclic case, it follows from the solution of the Smith Conjecture (cf. [166]) and in the general case from the works of Meeks and Yau [155] and of Kwasik and Schultz [132]. That follows also from the Orbifold Theorem, see Section 3.7 and Chapter 9.

Note that unlike Theorems 3.20 and 3.22, we do not assume in Theorem 3.23 that $\mathcal{O}$ is good. Thus we do not know *a priori* that $\tilde{\mathcal{O}}$ is a manifold, and we will have to adapt the techniques of [117] to orbifolds. We follow [141]. The same result was established in [220] by similar methods, but note that these authors use a different terminology.

Let $(\mathcal{O}, \mathcal{T})$ be a triangulated 3-orbifold. Let $f : F \to \mathcal{O}$ be a general position immersed 2-suborbifold. The *PL area* of f is the ordered triple $\|f\| := (\mathrm{sing}(f), \mathrm{wt}(f), \mathrm{lg}(f)) \in \mathbf{N}^2 \times \mathbf{R}_+$ (with the lexicographical order) whose coordinates are defined below.

The *singular weight* $\mathrm{sing}(f)$ is the number of singular points of F. The *total weight* $\mathrm{wt}(t)$ is the number of points of $|f(F)| \cap \mathcal{T}^{(1)}$ counted with multiplicities. To define the *length* $\mathrm{lg}(f)$, we fix an ideal triangle $\Delta \subset \mathbf{H}^2$. We map each 2-simplex σ (minus its vertices) homeomorphically onto Δ using barycentric coordinates and put on σ the induced metric, so that the gluing maps are well-defined (they belong to the isometry group of Δ). We call this (singular Riemannian) metric the *regular Jaco-Rubinstein metric* on $(\mathcal{O}, \mathcal{T})$. We define $\mathrm{lg}(f)$ as the sum of the lengths of the components of $f(F) \cap \mathcal{T}^{(2)}$ for this metric.

We now discuss an important operation called *smoothing out intersection curves.* (This has also been called "exchange/roundoff".) Let F_1, F_2 be two 2-suborbifolds in general position with respect to $\mathcal{T}$ and to each other. Let γ be a curve in $F_1 \cap F_2$. Choose collars U_1, U_2 of γ in F_1, F_2 respectively and write $\partial U_i = \gamma_i^+ \cup \gamma_i^-$. Then to *smooth out* γ means to remove U_1, U_2, glue disjoint smooth annuli U^+, U^- connecting γ_1^+ with γ_2^+ (resp. γ_1^- with γ_2^-), and perform small isotopies in neighborhoods of U^+, U^- away from γ. This construction yields two immersed (possibly nonembedded) 2-suborbifolds $\tilde{F}_1, \tilde{F}_2$. It can always be done so that the sum of the PL areas of $\tilde{F}_1, \tilde{F}_2$ is *strictly* less than the sum of the PL areas of F_1, F_2.

This important fact provides restrictions on the way least PL area suborbifolds may intersect. To illustrate this, assume that F_1 and F_2 minimize PL area in their respective isotopy class and the curve γ bounds on each F_i a disk D_i. Assume further that $\mathrm{Int}\, D_1 \cap F_2 = \mathrm{Int}\, D_2 \cap F_1 = \emptyset$, and that the 2-sphere $D_1 \cup D_2$ bounds a ball whose interior is disjoint from F_1 and F_2. Then there is a way of smoothing out γ so that the disks D_1, D_2 get swapped. For $i = 1, 2$ we have $\tilde{F}_i$ isotopic to F_i, hence $|\tilde{F}_i| \geq |F_i|$. But we just saw that $|\tilde{F}_1| + |\tilde{F}_2| < |F_1| + |F_2|$. This is a contradiction. Thus F_1 and F_2 cannot in fact have such an "inessential" intersection curve. This principle, first exploited by Meeks and Yau [156] in the context of analytic minimal surfaces, is fundamental for all applications to 3-dimensional topology, see also Freedman-Hass-Scott [74].

Sketch of proof of Theorem 3.23. We fix a $\pi_1\mathcal{O}$-invariant triangulation $\mathcal{T}$ of $\tilde{\mathcal{O}}$. Seeking a contradiction, we assume that $\tilde{\mathcal{O}}$ contains an incompressible suborbifold that is bad or spherical.

Step 1 There is a normal 2-suborbifold $F_0 \subset \tilde{\mathcal{O}}$ that has least PL area among bad or spherical incompressible 2-suborbifolds.

Start with any bad or spherical incompressible 2-suborbifold $F \subset \tilde{\mathcal{O}}$. Note that the moves to attain normality reduce PL area (cf. the proof of Lemma 3.9). Thus if a minimizing element exists, it has to be normal. To prove existence of a minimizing element, first observe that it is certainly possible to minimize weight. Now once you have a bound on the weight of a normal 2-suborbifold, you get a bound on the number of simplices they meet.

Hence finding a length minimizing element among weight minimizing elements reduces to a purely local variational argument similar to the one proving existence of geodesics in Riemannian manifolds.[3]

Step 2 This minimal 2-suborbifold F_0 is $\pi_1\mathcal{O}$-equivariant.

Since $\mathcal{T}$ is $\pi_1\mathcal{O}$-invariant, each translate of F_0 by the covering group is also of least PL area among bad or spherical incompressible 2-suborbifolds. The proof that F_0 is equivariant is by contradiction. Since the action of the covering group is properly discontinuous, there are at most finitely many elements $g_1, \ldots, g_p \in \pi_1\mathcal{O}$ such that $g_iF_0 \cap F_0 \neq \emptyset$. Using a perturbation trick (due to Meeks-Yau in the analytic case), we may further assume that for each i, F_0 intersects g_iF_0 transversely in finitely many simple closed curves that avoid the singular locus. Each intersection curve separates F_0 (resp. g_iF_0) in two disks with at most three singular points. There are finitely many such disks. Let D be of least PL area among them.

If $D \subset F_0$, set $F_1 := F_0$ and $F_2 := g_iF_0$, and otherwise set $F_1 := g_iF_0$ and $F_2 := F_0$, so that $D \subset F_1$ anyway. We now perform a smoothing out operation. Let D_1, D_2 be the 2-suborbifolds such that $D_1 \cup D_2 = F_2$ and $\partial D_1 = \partial D_2 = \partial D$. Let F_i' be a 2-suborbifold obtained by rounding the corner of $D \cup D_i$ for $i = 1, 2$. By choice of D, the interior of D avoids F_2, so F_1' and F_2' are embedded 2-suborbifolds. Moreover, F_1' and F_2' have strictly smaller PL area than F_0.

There are several cases to consider, according to how many singular points the various suborbifolds contain. For instance, if there are no singular points at

[3]In general, a minimizing sequence may converge to a double cover of a nonorientable 1-sided 2-suborbifold. Here this cannot happen because $\tilde{\mathcal{O}}$, and therefore $|\tilde{\mathcal{O}}|$, are simply-connected.

all (e. g. if $\tilde{\mathcal{O}}$ is a manifold), then F_1' and F_2' are both nonsingular spheres, and by an argument we have already seen, at least one of them is incompressible. This contradicts the minimality of F_0 and completes the sketch of Step 2.

We leave the other cases as an exercise to the reader. The idea is to prove that D has either 0 or 1 singular point. In each case, there are several subcases, according to the repartition of singular points between D_1 and D_2.

Step 3 The end.

Projecting down F_0 to $\mathcal{O}$, we get a 2-suborbifold $F \subset \mathcal{O}$, which is bad or spherical, but possibly nonorientable.

If F is orientable, then it is a compressible spherical 2-suborbifold because $\mathcal{O}$ is irreducible. Any discal 3-suborbifold of $\mathcal{O}$ bounded by F lifts to a discal 3-suborbifold of $\tilde{\mathcal{O}}$ bounded by F_0, contradicting the incompressibility of F_0.

If F is nonorientable, then it is 1-sided. Let F_ϵ be the boundary of a small regular neighborhood of F. Some lift of F_ϵ is a small perturbation of F_0, hence incompressible, and we get a contradiction as in the previous paragraph. □

A similar argument (with a little more work) gives:

Theorem 3.24 ([141]). *Let $p : \hat{\mathcal{O}} \to \mathcal{O}$ be a regular covering of 3-orbifolds. If $\mathcal{O}$ is irreducible, then $\hat{\mathcal{O}}$ is irreducible. If $\mathcal{O}$ is irreducible and contains no incompressible turnovers, then $\hat{\mathcal{O}}$ contains no incompressible turnovers.*

Basically, what makes the proof work is the fact that incompressible turnovers are canonical, i.e. can always be isotoped off one another.[4]

We end this chapter by stating an important theorem of W. Meeks and P. Scott [154] whose proof uses the same techniques and relies on work of Freedman-Hass-Scott [74].

Theorem 3.25 ([154]). *Let M be a compact, irreducible Seifert fibered 3-manifold with infinite fundamental group. Let Γ be a finite group acting on M respecting the normal subgroup generated by the generic fiber. Then M admits a Γ-invariant Seifert fibration. Hence the orbifold M/Γ is Seifert fibered.*

3.7 The Orbifold Geometrization Conjecture

Thurston's Geometrization Conjecture states that the canonical pieces given by the spherical and toric decomposition of a compact 3-orbifold without bad 2-suborbifold are geometric:

[4] A similar statement for e. g. tori or pillows does not hold since there are small Seifert manifolds finitely covered by Haken ones. See next chapter.

Conjecture 3.26 (Orbifold Geometrization Conjecture). *The interior of any compact 3-orbifold that does not contain any bad* 2*-suborbifold can be split along a finite collection of disjoint, non-parallel, essential, embedded spherical and toric* 2*-suborbifolds into canonical* 3*-suborbifolds* $X_1, \ldots X_n$*, such that for each* i*, the* 3*-orbifold obtained from* X_i *by capping off all spherical boundary components by discal* 3*-orbifolds is geometric.*

The following case of Thurston's Geometrization Conjecture has been settled, see [16, 17, 18, 43, 228, 229] and Chapter 9:

Theorem 3.27 (Orbifold Theorem). *The Geometrization Conjecture holds for a compact, irreducible 3-orbifold with non-empty singular locus.*

The spherical decomposition (Theorem 3.2), the Orbifold Theorem (Theorem 3.27) and the fact that compact 3-orbifolds with a geometric decomposition are finitely covered by a manifold [149] imply the following characterization of orbits spaces of finite group actions on compact 3-manifolds:

Corollary 3.28. *A compact* 3*-orbifold is the quotient of a compact* 3*-manifold by an orientation preserving finite group action if and only if it does not contain a bad* 2*-suborbifold.*

Chapter 4

Haken orbifolds

In this chapter we introduce an important class of 3-orbifolds, called *Haken orbifolds*, and we present some of their fundamental properties.

Again in this chapter all 2-orbifolds and 3-orbifolds are assumed to be connected and orientable unless mentioned otherwise. In general 2-suborbifolds of 3-orbifolds are assumed to be either properly embedded or suborbifolds of the boundary.

We start by recalling this notion in the case of manifolds.

4.1 Haken manifolds

Homotopy theory plays an important role in the study of 3-dimensional manifolds. A long standing conjecture concerning universal coverings of compact, irreducible 3-manifolds is:

Conjecture 4.1 (Universal Covering Conjecture). *The universal covering of the interior of a compact, irreducible 3-manifold is homeomorphic to* $\mathbf{R}^3$ *or* $\mathbf{S}^3$.

This conjecture implies the Poincaré Conjecture. Indeed, assume by way of contradiction that there exists a fake 3-sphere M (i.e. a closed, simply-connected 3-manifold which is not homeomorphic to $\mathbf{S}^3$). Applying the Spherical Decomposition Theorem 3.2 to M, we get a connected sum decomposition $M = M_1 \# \cdots \# M_m$ where the M_i's are irreducible. By the Seifert-Van Kampen theorem and elementary group theory, each Mi is a fake 3-sphere. In particular, there exists an irreducible fake 3-sphere, contradicting the Universal Covering Conjecture.

Remark. A *Whitehead manifold* is an *open,* irreducible, contractible 3-manifold not homeomorphic to $\mathbf{R}^3$. (We recall that open means non-compact and without boundary.) The first such example was constructed by J. H. C. Whitehead [246] in 1935. It is known that there are uncountably many Whitehead manifolds, but it is still an unsolved problem if a Whitehead manifold can cover a compact 3-manifold. In dimension $n > 3$, M. Davis [50] has given examples of contractible open n-manifolds not homeomorphic to $\mathbf{R}^n$ which cover compact manifolds.

Beside the case of geometric manifolds, the Universal Covering Conjecture is known to be true mainly for Haken manifolds, by work of F. Waldhausen [237]. We discuss this result in Section 4.3 in the more general setting of Haken orbifolds.

Definition. A *Haken manifold* is a compact, irreducible 3-manifold which contains an essential surface, or is a 3-ball.

Haken manifolds were studied by W. Haken in the early 60's [99, 100]. In the late 60's, F. Waldhausen [237] established their fundamental properties and showed their central role in the study of 3-dimensional manifolds.

The following proposition gives many important examples of Haken manifolds. A typical example is the exterior of a knot in $\mathbf{S}^3$.

Proposition 4.2. *If M is compact, irreducible and $H^1(M;\mathbf{Q}) \neq \{0\}$, then M is Haken.*

Proof. Since $H^1(M;\mathbf{Q}) \neq \{0\}$, $\pi_1 M$ has infinite abelianization. Hence there is a surjective homomorphism $\phi : \pi_1 M \to \mathbf{Z}$. This homomorphism ϕ can be realized by a continuous map $h : M \to \mathbf{S}^1$. Approximating h by a C^∞ map and taking the preimage of a regular value, we get an embedded surface. After surgery on compressing disks we obtain an essential non-separating surface in M. □

Any non-separating properly embedded surface F in a compact manifold M defines a morphism $\phi\colon \pi_1 M \to \mathbf{Z} \to 0$, by considering the algebraic intersection number of any homology class of loops in $H_1(M;\mathbf{Z})$ with F.

Let M be compact irreducible and $\partial M \neq \emptyset$. Then either M is a 3-ball, or the inclusion $i : \partial M \to M$ induces a non-trivial homomorphism $i_* : H_1(\partial M;\mathbf{Z}) \to H_1(M;\mathbf{Z})$. Therefore in any case M is Haken.

Example. One can construct Haken 3-manifolds M with trivial homology starting with the exterior $E = \mathbf{S}^3 \setminus \mathrm{Int}(\mathcal{N}(k))$ of any non-trivial knot $k \subset \mathbf{S}^3$. One defines $M := E \cup_{\partial E} E$ by gluing two copies of E along their boundaries by means of a homeomorphism $f : \partial E \to \partial E$ such that the induced homomorphism $f_* : H_1(\partial E; \mathbf{Z}) \to H_1(E; \mathbf{Z})$ has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the basis given by a meridian and a longitude.

The following lemma will be useful in the next chapter:

Lemma 4.3. *Let M be an irreducible compact Seifert fibered space with non-empty boundary. Then any nonzero element of $H_2(M; \mathbf{Q})$ can be represented by a family of pairwise disjoint, embedded, incompressible tori.*

Proof. M is a Haken manifold, so any nonzero element of $H_2(M; \mathbf{Q})$ can be represented by a family of incompressible closed surfaces, which must be vertical tori by Proposition 2.16 (see [238]). They can be made pairwise disjoint by simple cut and paste operations. □

Till the 70's, few examples of non-Haken 3-manifolds were known. Those were mostly Seifert fibered manifolds, and hence finitely covered by Haken manifolds. Then the work of W. Thurston in the mid 70's allowed one to construct infinitely many non-Haken hyperbolic 3-manifolds ([225, Chap. 4], cf. Section 8.3 of this book). However, the following important conjectures are still wide open:

Conjecture 4.4 (Finite Covering Conjectures).

1 *The fundamental group of any closed, irreducible 3-manifold is finite or contains the fundamental group of a closed orientable surface.*

2 *Any closed, irreducible, 3-manifold with infinite fundamental group is finitely covered by a Haken 3-manifold.*

3 *Any closed, irreducible, 3-manifold with infinite fundamental group is finitely covered by a 3-manifold with strictly positive first Betti number.*

4 *Any compact, irreducible, 3-manifold with an infinite fundamental group that does not contain a solvable subgroup of finite index is finitely covered by a 3-manifold with arbitrary large first Betti number.*

5 *Any complete hyperbolic 3-manifold of finite volume is finitely covered by a bundle over $\mathbf{S}^1$.*

Remark. Conjectures 1 to 4 are due to F. Waldhausen [242]. The last one is due to W. Thurston [228]. Clearly, Conjecture 4 implies Conjecture 3, which in turn implies Conjecture 2.

Conjecture 2 implies Thurston's Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental group, because a virtually Haken, closed, 3-manifold either is Haken or has the homotopy type of a hyperbolic or Seifert fibered 3-manifold. Thurston's Geometrization Conjecture is true for Haken 3-manifolds. Moreover, the works of D. Gabai, R. Meyerhoff, N. Thurston [80], and of P. Scott [200], show that a closed, 3-manifold with the same homotopy type as a hyperbolic or a Seifert fibered 3-manifold with infinite fundamental group is hyperbolic or Seifert fibered.

Conjecture 3 holds for arithmetic hyperbolic manifolds in many cases, but remains still unknown in general, see [138].

Conjecture 4 holds if $\partial M \neq \emptyset$ [44] or if M contains an essential torus [130, 137].

The existence of an essential surface $(F, \partial F) \subset (M, \partial M)$ allows to split the manifold M along the surface F to obtain a new manifold $M \backslash F$ which is still irreducible. If $M \backslash F$ is not a finite collection of 3-balls, it is Haken, and we can iterate this process. The Haken Finiteness Theorem (Theorem 3.11) shows that, after a finite number of steps, this process yields a finite collection of 3-balls (see [106, 113] and the references therein). Such a finite sequence of 3-manifolds, obtained at each step by splitting along an essential surface, is called a *hierarchy* for M.

The existence of a hierarchy is a fundamental tool in the proof of most important results about Haken manifolds. In the next section, we study hierarchies in the more general context of Haken orbifolds.

4.2 Hierarchies of Haken orbifolds

First we define what it means for an orbifold to be Haken. This notion is more delicate to handle for orbifolds than manifolds because of the existence of turnovers.

Recall that discal means a quotient of a disc (a ball in the 3-dimensional case). A *thick turnover* is the product of a turnover with an interval.

Definition. A 3-orbifold $\mathcal{O}$ is called *Haken* if it is compact, irreducible, and either

i. $\mathcal{O}$ is discal, or a thick turnover, or

ii. $\mathcal{O}$ contains an essential 2-suborbifold, but contains no essential turnover.

Remark. The word Haken may lead to confusion: it is not true that a compact, irreducible 3-orbifold containing an incompressible properly embedded 2-suborbifold is Haken in our sense. Figure 4.1 illustrates examples of 3-orbifolds with non-empty boundary, but containing no essential 2-suborbifold— a phenomenon that does not occur for manifolds and has of course something to do with the presence of boundary turnovers, which are not spherical, but carry no essential closed curves. By doubling any of those examples along its boundary, one gets a closed, irreducible, non-Haken 3-orbifold that contains an essential embedded 2-sided 2-suborbifold.

Definition. A compact 3-orbifold $\mathcal{O}$ is *small* if:

- $\mathcal{O}$ is irreducible,
- $\partial\mathcal{O}$ is empty or a union of turnovers,
- $\mathcal{O}$ does not contain any essential 2-suborbifold.

Note that discal 3-orbifolds and thick turnovers are small. In fact, they are the only 3-orbifolds that are both Haken and small.

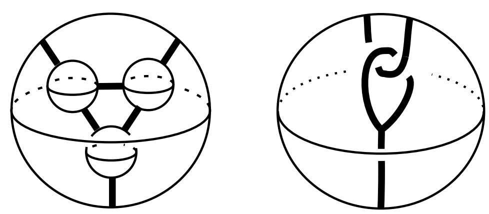

Figure 4.1: Two examples of small orbifolds with boundary, provided that the orders of the local groups are large enough.

Definition. An *orbifoldbody* is a Haken 3-orbifold that can be split along a finite (possibly empty) collection of disjoint properly embedded discal 2-suborbifolds into a disjoint union of discal 3-orbifolds and/or thick turnovers.

A fundamental property of Haken 3-orbifolds is the existence of a hierarchy, due to W. Dunbar [55]:

Theorem 4.5. *Let $\mathcal{O}$ be a compact, Haken 3-orbifold. There is a finite sequence of pairs*

$$(\mathcal{O}_1, F_1) \rightsquigarrow (\mathcal{O}_2, F_2) \rightsquigarrow \ldots \rightsquigarrow (\mathcal{O}_n, \emptyset)$$

such that:

i. $\mathcal{O}_1 = \mathcal{O}$*;*

ii. F_i *is an essential 2-suborbifold of* $\mathcal{O}_i$ *which is not discal, nor a turnover.*

iii. $\mathcal{O}_{i+1}$ *is obtained by splitting* $\mathcal{O}_i$ *along* F_i*;*

iv. $\mathcal{O}_n$ *is a finite collection of orbifoldbodies.*

Such a sequence is called a *hierarchy*, and the integer n its *length*. There is an upper bound on the length depending only on the 3-orbifold $\mathcal{O}$, because of the orbifold version of the Haken Finiteness Theorem (Theorem 3.11; see also [55, Thm. 12]). The greatest possible length for a hierarchy given by Theorem 4.5 is called the *length* of $\mathcal{O}$ and denoted by $\ell(\mathcal{O})$.

Remark. The key point for the construction of a hierarchy is to find at each step an essential 2-suborbifold, which is not discal nor a turnover. In the case of 3-manifolds, this follows from Proposition 4.2. In the orbifold case the proof is much more involved: it uses Thurston's Hyperbolization Theorem for Haken 3-manifolds, which is based on the existence of a hierarchy for such 3-manifolds, and then M. Culler and P. Shalen's technique [49] involving curves of representations. Therefore one has to prove first the existence of a hierarchy for Haken 3-manifolds and the Hyperbolization Theorem for Haken 3-manifolds in order to prove the existence of a hierarchy for Haken 3-orbifolds and then the Hyperbolization Theorem for Haken 3-orbifolds (cf. Chapter 6, Theorem 6.5)

The inductive step of Theorem 4.5 is given by the following proposition:

Proposition 4.6. *Let* $\mathcal{O}$ *be a compact, irreducible 3-orbifold that does not contain any essential turnover. Assume that there is at least one boundary component of* $\mathcal{O}$ *that is not a turnover. Then either* $\mathcal{O}$ *is a discal 3-orbifold or it contains an essential, 2-suborbifold. In particular,* $\mathcal{O}$ *is Haken.*

Proof. Let us assume that $\mathcal{O}$ is not a discal 3-orbifold and does not contain a closed essential 2-suborbifold. The singular locus $\Sigma_{\mathcal{O}}$ is a trivalent graph properly embedded in $|\mathcal{O}|$. Set $M := \mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})$, where $\mathcal{N}(\Sigma_{\mathcal{O}})$ is an open tubular neighborhood of $\Sigma_{\mathcal{O}}$, and let $P \subset \partial\mathcal{N}(\Sigma_{\mathcal{O}})$ be the collection of tori and annuli corresponding to boundaries of neighborhoods of edges and circles in $\Sigma_{\mathcal{O}}$.

The following claim reduces the proof of Proposition 4.6 to the case where M admits a geometrically finite hyperbolic structure, by using Thurston's Hyperbolization Theorem for Haken 3-manifolds [230, 152, 153, 120, 165, 176, 177] (cf. Chapter 6, Theorem 6.5).

Claim. *Either* M *is Seifert fibered or* (M, P) *is a pared manifold.*

A *pared manifold* is a pair (M, P) such that:

- M is a compact, irreducible, atoroidal 3-manifold that is not Seifert fibered.
- $P \subset \partial M$ is a disjoint union of incompressible tori and annuli such that no two components of P are isotopic in M, and P contains all torus components of ∂M.
- There is no essential annulus $(A, \partial A) \subset (M, P)$.

Proof of the claim. Since $\mathcal{O}$ contains no essential 2-suborbifold, it is irreducible and atoroidal. It follows that M is irreducible and atoroidal. Let us assume that M is not Seifert fibered and prove that (M, P) is a pared manifold.

First we show that P is incompressible in M. A compressible annulus in P would give a teardrop in $\mathcal{O}$, contradicting the irreducibility of $\mathcal{O}$. Because of the irreducibility of M, the compressibility of a torus component of P would imply that M is a solid torus, hence Seifert fibered. We check now that the pair (M, P) is acylindrical. Let $(A, \partial A) \subset (M, P)$ be an essential annulus; we distinguish three cases according to whether ∂A is contained in a) torus components of P, b) annulus components of P, or c) a torus and an annulus of P. In the first case, a classical argument using the atoroidality of M implies that M is Seifert fibered (cf. 3, Lemma3.16). In case b), gluing two meridian discal orbifolds to A along ∂A would give a bad or an essential spherical 2-suborbifold, contradicting the irreducibility of $\mathcal{O}$. Case c) reduces to case b), by considering the essential annulus obtained by gluing two parallel copies of A with the annulus $P_0 \setminus \mathcal{N}(\partial A)$, where P_0 is the torus component of P that meets ∂A. □

End of the proof of Proposition 4.6. By the claim, either M is Seifert fibered, or (M, P) is pared. We deal with both cases separately.

When M is Seifert fibered, then the fibration of M extends to a fibration of the orbifold $\mathcal{O}$ such that the components of Σ are fibers, because $\mathcal{O}$ is irreducible. In particular the components of $\partial\mathcal{O}$ are tori and $\mathcal{O}$ contains an essential discal 2-orbifold or an essential annulus.

When (M, P) is a pared manifold, since M is Haken, by Thurston's Hyperbolization Theorem for pared 3-manifolds [230, 152, 153, 120, 165, 176, 177], the interior of M admits a geometrically finite hyperbolic structure whose cusp ends correspond to the circles and edges of $\Sigma_{\mathcal{O}}$ and to the tori of $\partial\mathcal{O}$ (cf. Chapter 6). Then it follows from Culler and Shalen's theory for ideal points of curves of representations in $\mathrm{PSL}_2(\mathbf{C})$ (see Chapter 7, Theorem 7.13 and Corollary 7.14) that $\mathcal{O}$ contains an essential suborbifold with non-empty boundary. □

Let $\mathcal{O}$ be a compact irreducible 3-orbifold with $\partial\mathcal{O} \neq \emptyset$. Then $\mathcal{O}$ can be split along a finite (possibly empty) collection of disjoint properly embedded discal 2-suborbifolds into a disjoint union of compact, irreducible, ∂-incompressible 3-orbifolds and of discal 3-orbifolds: one can argue by induction on the Euler characteristic of $\partial\mathcal{O}$, see [55, Lemma 13].)

Similar arguments show that a Haken 3-orbifold, in which the only essential 2-suborbifolds are discal, is an orbifoldbody.

The proof of Theorem 4.5 follows now by induction, using the orbifold version of Haken's Finiteness Theorem (cf. Theorem 3.11, see also [55, Thm 12]). Let $\mathcal{O} = \mathcal{O}_1$ be a compact, Haken 3-orbifold. The hypothesis allows to split $\mathcal{O}$ along an essential 2-suborbifold which is not a turnover, to get a compact, irreducible, possibly disconnected 3-orbifold $\mathcal{O}_2$. Then by Proposition 4.6 each connected component of the orbifold $\mathcal{O}_2$ is either an orbifoldbody or still contains an essential 2-suborbifold which is not discal, nor a turnover. This process must stop because of the following proposition which can be proved exactly as in [113, Prop. IV.7.].

Proposition 4.7. *The length n of a hierarchy given in Theorem 4.5 is always bounded above by $3h(\mathcal{O})$, where $h_0(\mathcal{O})$ is the maximal number of connected components for an essential system of closed 2-suborbifolds in $\mathcal{O}$.* □

Before closing this section, we remark that Proposition 4.6 implies that an irreducible 3-orbifold without essential turnovers is small or Haken. Hence we can reformulate once more the turnover splitting theorem of the previous chapter:

Theorem 4.8 (Turnover splitting). *Let $\mathcal{O}$ be a compact, irreducible, atoroidal 3-orbifold.*

i. A (possibly empty) maximal essential system $\mathfrak{H}$ of hyperbolic turnovers in $\mathcal{O}$ is unique up to isotopy.

ii. Each component of $\mathcal{O}\backslash\mathfrak{H}$ is Haken or small.

4.3 Universal coverings

The following theorem is a straightforward extension to the case of orbifolds of a theorem due to Waldhausen [237]:

Theorem 4.9. *The universal covering of the interior of a Haken 3-orbifold is homeomorphic to $\mathbf{R}^3$.*

It is sufficient to prove that the universal covering of a Haken 3-orbifold is a manifold (i.e. that the orbifold is good). Then the proof of Theorem 4.9 reduces to Waldhausen's proof (cf. [237]).

Proposition 4.10. *Haken* 3*-orbifolds are good.*

Proposition 4.10 can be proved by induction on the length $\ell(\mathcal{O})$ of $\mathcal{O}$. The inductive step is given by the following lemma [216, Thm. A]:

Lemma 4.11. *Let $\mathcal{O}$ be a compact, irreducible* 3*-orbifold and let $F \subset \mathcal{O}$ be an essential* 2*-suborbifold. If each connected component of $\mathcal{O}' = \mathcal{O} \setminus F$ is good, then $\mathcal{O}$ is good.*

Proof. For good 3-orbifolds, there is a natural version of the Loop Theorem, derived from the equivariant version for 3-manifolds (see Chapter 3, Corollary 3.20; see also [216], [219]). It follows that each copy of the essential 2-suborbifold F is π_1-injective in each component of $\mathcal{O}'$. Therefore F is π_1-injective in $\mathcal{O}$ and each component of $\mathcal{O}'$ is π_1-injective in $\mathcal{O}$.

Let $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ be the universal covering. Since F is π_1-injective in $\mathcal{O}$, $p^{-1}(F) =: \widetilde{F} = \bigcup_{i\in\mathbb{N}} \widetilde{F}_i$ where each $\widetilde{F}_i$ is homeomorphic to the universal covering of F. Moreover:

$$\widetilde{\mathcal{O}} \setminus p^{-1}(\mathrm{Int}(\mathcal{O}')) = p^{-1}(F \times [0,1]) = \bigcup_{i\in\mathbb{N}} \widetilde{F}_i \times [0,1]$$

and

$$p^{-1}(\mathcal{O}') =: \widetilde{\mathcal{O}'} = \bigcup_{i\in\mathbb{N}} \widetilde{\mathcal{O}'}_i.$$

Each $\widetilde{\mathcal{O}'}_i$ is homeomorphic to the universal covering of $\mathcal{O}'$ or of one component of $\mathcal{O}'$ if F is separating.

Since by hypothesis F and $\mathcal{O}'$ are good orbifolds, the universal covering of $\mathcal{O}$ is a union of manifolds, hence a manifold. □

By induction on $\chi(\partial\mathcal{O})$, one shows easily that orbifoldbodies are good.

4.4 Topological Rigidity

The following conjecture, often referred to as Borel's Conjecture, is still wide open:

Conjecture 4.12. *Let M, N be two compact, irreducible 3-manifolds with infinite fundamental groups. Any proper homotopy equivalence $f : M \to N$ such* *that the restriction $f|_{\partial M} : \partial M \to \partial N$ is a homeomorphism, is homotopic rel ∂M to a homeomorphism.*

F. Waldhausen [237] has proved this conjecture for Haken 3-manifolds in the late 1960's. The Geometrization Conjecture implies the Borel Conjecture: when M, N are both hyperbolic, this is a consequence of G. Mostow's Rigidity Theorem [170] (see Chapter 6, Theorem 6.9); when M, N are both Seifert fibered, it is due to P. Orlik, E. Vogt and H. Zieschang [175].

More recently, D. Gabai, R. Meyerhoff, N. Thurston [78, 79, 80] in the hyperbolic case and P. Scott [200] in the Seifert fibered case proved the Borel conjecture assuming only that one of the 3-manifolds M, N is geometric. A consequence is that a compact, irreducible 3-manifold M with infinite fundamental group verifies the Geometrization Conjecture if and only if a finite covering of M does.

Next we discuss an extension of Waldhausen's result (cf. [219, 217]). Since Haken 3-orbifolds are good, a *homotopy equivalence* between such orbifolds can be defined simply as the projection of an equivariant homotopy equivalence between the universal coverings, which are manifolds.

Theorem 4.13. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two compact, Haken 3-orbifolds. Any homotopy equivalence $f : (\mathcal{O}_1, \partial\mathcal{O}_1) \to (\mathcal{O}_2, \partial\mathcal{O}_2)$, whose restriction to $\partial\mathcal{O}_1$ is a homeomorphism, is homotopic to a homeomorphism.*

Remark. The Orbifold Theorem implies Theorem 4.13 for compact, irreducible 3-orbifolds with infinite fundamental group and nonempty singular locus.

A key lemma for the proof of Theorem 4.13 is:

Lemma 4.14. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a homotopy equivalence between two compact, Haken 3-orbifolds. Let $F \subset \mathcal{O}_2$ be an essential, embedded 2-suborbifold. Then after a homotopy of f, $f^{-1}(F) = G$ is a collection of essential suborbifolds of $\mathcal{O}_1$.* □

Sketch of proof. The proof can be done in three steps (cf. [213, Chapter 2B]).

Step 1 Let $p_i : \widetilde{\mathcal{O}_i} \to \mathcal{O}_i$ be the universal covering of $\mathcal{O}_i$, $i = 1, 2$. Define a continuous map h from $\mathcal{O}_2$ to $[0, 1]$ (resp. $\mathbf{S}^1$) if F_2 is separating (resp. nonseparating) such that:

- For any value $y \in (0, 1)$ (resp. $y \in \mathbf{S}^1 - \star$,) the preimage $h^{-1}(y) = F_y$ is an embedded 2-suborbifold isotopic to F.
- the map $h \circ p_2$ is smooth over $(0, 1)$ (resp. $\mathbf{S}^1 - \{\star\}$.)

Step 2 Let $\tilde{f}\colon \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ denote a lift of f. By homotoping f we may assume that $g := h \circ p_2 \circ \tilde{f}$ is smooth over $(0,1)$ (resp. $\mathbf{S}^1 - \{\star\}$.)

Pick a regular value $y \in (0,1)$ (resp. $y \in \mathbf{S}^1 - \{\star\}$) for g. Then $g^{-1}(y) =: \widetilde{G}$ is an embedded 2-submanifold of $\widetilde{\mathcal{O}_1}$, equivariant under the action of $\pi_1\mathcal{O}_1$. Therefore, after an isotopy of F, $f^{-1}(F) = p_1(\widetilde{G}) = G$ is a collection of embedded, compact, orientable 2-suborbifolds.

Step 3 Since $\mathcal{O}_1$ is irreducible, by surgering G along discal 2-orbifolds, one can homotope f further such that each connected component of $G = f^{-1}(F)$ is essential. □

The proof of Theorem 4.13 is again by induction on the length $\ell(\mathcal{O}_1)$ of $\mathcal{O}_1$, using Lemma 4.14 to reduce $\ell(\mathcal{O}_1)$.

We can now sketch the proof of Theorem 4.13.

Proof of Theorem 4.13. First we consider the case where $\partial\mathcal{O}_2 \neq \emptyset$ and there is an essential 2-suborbifold with non-empty boundary $(F, \partial F) \hookrightarrow (\mathcal{O}_2, \partial\mathcal{O}_2)$. By a homotopy of f, we can make $G := f^{-1}(F) \subset \mathcal{O}_1$ a collection of properly embedded, essential 2-suborbifolds. Choose a connected component $G_i \subset G$. Then $f|_{G_i *} : \pi_1 G_i \to \pi_1 F$ is injective. The generalized Baer theorem (cf. [252]) implies that $f|_{G_i}$ is homotopic to a covering map $G_i \to F$. Therefore $f|_{G_i} : G_i \to F$ is homotopic to a homeomorphism, since $f|_{G_i} : \partial G_i \to \partial F$ is a homeomorphism.

It follows that $G = G_i = f^{-1}(F)$ is connected, because $f^{-1}(\partial F) = \partial G_i$ and $f|_{\partial M} : \partial M \to \partial N$ is a homeomorphism. Hence after a homotopy of f in a neighborhood of F, $f|_{G_i} : f^{-1}(F) = G \to F$ is a homeomorphism. By splitting $\mathcal{O}_1$ along G and $\mathcal{O}_2$ along F, we get a map $f| : \mathcal{O}_1 \backslash G \to \mathcal{O}_2 \backslash F$. Since F and G are essential, this map is a homotopy equivalence and a homeomorphism on the boundary.

If $\mathcal{O}_1$ is an orbifoldbody, then the argument above reduces $\chi(\partial\mathcal{O}_1)$ by splitting along an essential discal 2-suborbifold. Otherwise, it reduces the length $\ell(\mathcal{O}_1)$. In both cases a finite induction completes the proof.

The proof of the case where $\mathcal{O}_2$ contains only closed essential 2-suborbifolds can be reduced to the previous case as in S. Gadgil and G. Swarup's proof [83]. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are orientable, f induces a proper degree-one map from the underlying spaces of the orbifolds. Then using Kneser's theorem (cf. [210]) one can homotope f to be a homeomorphism over a non-singular 3-ball $B^3 \subset \mathcal{O}_2$. By an isotopy of F we can assume that $F \cap B^3 = D$ is a properly embedded disk in B^3. Then the argument above can be easily adapted to the restriction

$f| : \mathcal{O}_1 \setminus \mathrm{Int}(f^{-1}(B^3)) \to \mathcal{O}_2 \setminus \mathrm{Int}(B^3)$ (see [83]). We leave the details as an exercise to the reader.

□

4.5 The Torus Theorem

The Torus Theorem gives a homotopic characterization of atoroidal Haken 3-manifolds. It was announced by F. Waldhausen [241] in the late sixties. Subsequently several authors including Feustel [70, 71], Jaco and Shalen [115], Johannson [118], and Scott [199] gave proofs of various forms of this theorem. All these approaches involve quite intricate topological arguments. Among these, Scott's account is the most easily digestible and uses mainly properties of the fundamental group of the manifold. In fact Scott's work ([199, 200]) is the starting point of the proof of the Torus Theorem in the general case. This proof, which is beyond the scope of classical methods in dimension 3, is a consequence of the works of several people: A. Casson and D. Jungreis [41], D. Gabai [77], G. Mess [158], and P. Tukia [234].

In this section, we explain a proof of the Torus Theorem for compact, Haken 3-orbifolds, adapting unpublished arguments of Cassons's. The more general case of compact, irreducible 3-orbifolds which contain no incompressible turnover is presented in the next Chapter 5 (cf. [141]).

Theorem 4.15 (Torus Theorem). *Let $\mathcal{O}$ be a compact, Haken 3-orbifold. If $\pi_1\mathcal{O}$ contains a subgroup $\mathbf{Z} \oplus \mathbf{Z}$, which is not peripheral (i.e. not conjugated to a subgroup of the fundamental group of a boundary component), then either $\mathcal{O}$ contains an essential toric 2-suborbifold or $\mathcal{O}$ is Euclidean or Seifert fibered.*

One can triangulate $\mathcal{O}$ and take a normal immersion $f : \mathbf{T}^2 \to \mathcal{O}$ which is PL-minimal in the sense of Jaco and Rubinstein [117] and which induces the inclusion $\mathbf{Z} \oplus \mathbf{Z} \subset \pi_1\mathcal{O}$. Lifting the immersed torus to the universal covering $\mathbf{R}^3$ of $\mathrm{Int}(\mathcal{O})$, one gets a family of planes $\{P\}$ which is equivariant under the action of $\pi_1\mathcal{O}$ and with the property that two planes intersect in at most one line (see [74, Lemma 6.4 and 6.5]. Then one of the following three cases happens:

(1) $gP = P$ or $gP \cap P = \emptyset$, $\forall g \in \pi_1\mathcal{O}$, then f covers an embedded toric 2-suborbifold (or a one-sided Klein suborbifold). In any case $\mathcal{O}$ contains an essential embedded toric 2-suborbifold.

(2) There are two double lines $\ell_1, \ell_2 \subset P$ with incommensurable stabilizers generated by γ_1 and γ_2 in $\pi_1\mathcal{O}$. Let P_1 and P_2 be the planes in the family $\{P\}$, such that $P \cap P_1 = \ell_1$ and $P \cap P_2 = \ell_2$. Then the double line

$P_1 \cap P_2 = \ell_3$ has a stabilizer generated by $\gamma_3 \in \pi_1\mathcal{O}$ which is not commensurable with the stabilizers of ℓ_1 and ℓ_2. Thus the elements $\gamma_1, \gamma_2, \gamma_3$ generate in $\pi_1\mathcal{O}$ a free abelian group of rank 3. In this case, $\mathcal{O}$ is finitely covered by a closed irreducible 3-manifold, that is homeomorphic to $\mathbf{T}^3$ by Waldhausen's Theorem [237]. Since $\mathcal{O}$ is Haken, by Theorem 4.13, $\mathcal{O}$ is Euclidean.

(3) All double lines in P have commensurable stabilizers. Then there is an infinite primitive element $\gamma \in \pi_1\mathcal{O}$ which stabilizes all double lines in P. In this case γ generates an infinite cyclic normal subgroup in the fundamental group of a regular neighborhood $\mathcal{O}_0$ of $f(T^2) \subset \mathcal{O}$. Then $\partial\mathcal{O}_0$ is a collection of toric or spherical 2-suborbifolds which are incompressible in $\mathcal{O}_0$ by [139, Corollary 4.5]. Moreover each connected component of $\mathcal{O} \setminus \mathcal{O}_0$ is irreducible, since $\mathcal{O}$ is irreducible and $f(T^2) \subset \mathcal{O}_0$ cannot be contained in a discal 3-suborbifold of $\mathcal{O}$. Therefore, either one toric boundary component of $\mathcal{O}_0$ is essential in $\mathcal{O}$ or the inclusion $\mathcal{O}_0 \subset \mathcal{O}$ induces a surjective homomorphism at the level of fundamental groups and γ generates an infinite cyclic normal subgroup of $\pi_1\mathcal{O}$.

In the third case , when $\pi_1\mathcal{O}$ contains an infinite cyclic normal subgroup, up to taking a 2-fold covering, one may assume that $\pi_1\mathcal{O}$ contains an infinite cyclic central subgroup Z. Then an argument analogous to [115, Thm II.6.3] or the equivariant Theorem 3.25 of [154] allows to conclude in the general case.

Therefore the proof of Theorem 4.15 reduces to the proof of the following result which is the orbifold version of a theorem of Waldhausen [239] (see Chapter 5 for a proof in a more general case).

Theorem 4.16. *Let $\mathcal{O}$ be a compact, Haken 3-orbifold. If $\pi_1\mathcal{O}$ has an infinite cyclic central subgroup Z, then $\mathcal{O}$ admits a Seifert fibration such that Z is generated by a power of the regular fiber.*

Proof. If $\partial\mathcal{O} \neq \emptyset$, then either $\partial\mathcal{O}$ is incompressible or $\mathcal{O}$ is a solid torus with possibly a singular core, since the center of a non-trivial free product with amalgamation is contained in the amalgamated subgroup (cf. [139, Corollary 4.5]).

Let $F \subset \mathcal{O}$ be an essential 2-suborbifold. We distinguish two cases:

a) $Z \cap \pi_1 F \neq \{1\}$. Then F is a Euclidean 2-suborbifold. Each component of the orbifold obtained by splitting $\mathcal{O}$ along F is a Haken 3-orbifold having a strictly smaller length than the length of $\mathcal{O}$ and an infinite cyclic central subgroup in its fundamental group.

b) $Z \cap \pi_1 F = \{1\}$. Then the argument above shows that F is nonseparating by [139, Corollary 4.5].

Let $\mathcal{O}_0$ denote $\mathcal{O}$ split open along F. Its boundary contains two copies F_0^+, F_0^- of F. Let $p : \overline{\mathcal{O}} \to \mathcal{O}$ be the infinite cyclic covering, built by taking for each $n \in \mathbf{Z}$ a copy $\mathcal{O}_n$ of $\mathcal{O}_0$ and by identifying to a single 2-suborbifold F_{n+1} the two copies $F_n^+ \subset \partial\mathcal{O}_n$ and $F_{n+1}^- \subset \mathcal{O}_{n+1}$ of F. The group of deck transformations of this covering is infinite cyclic generated by a homeomorphism $t : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ such that $t(\mathcal{O}_n) = \mathcal{O}_{n+1}$ and $t(F_n) = F_{n+1}$. Thus $\pi_1\mathcal{O}$ is an extension of $p_*\pi_1(\overline{\mathcal{O}})$ by $\mathbf{Z}$, where the generator of $\mathbf{Z}$, corresponding to t, can be represented by a loop $\tau \in \pi_1\mathcal{O}$ which crosses F transversally in a single point.

By the Equivariant Loop Theorem (see Corollary 3.20), $p^{-1}(F)$ is incompressible in $\mathcal{O}$. So $p_*\pi_1(\overline{\mathcal{O}})$ is either equal to $\pi_1 F$ or a non-trivial free product amalgamated along $\pi_1 F$. Since $Z \cap \pi_1 F = \{1\}$, it follows that $Z \cap p_*\pi_1\overline{\mathcal{O}} = \{1\}$ by [139, Corollary 4.5]. Then a generator of Z can be written $\alpha\tau^r$ where $r \neq 0$ and $\alpha \in p_*\pi_1\overline{\mathcal{O}}$ can be represented by a loop contained in the union of finitely many $\mathcal{O}_i$, say $\cup_{-m}^{m}\mathcal{O}_i$ for some integer $m > 0$. It follows that for any element $\gamma \in \pi_1\mathcal{O}_0$ one has: $\tau^r\gamma\tau^{-r} = \alpha^{-1}\gamma\alpha$. Iterating this relation shows that any loop in $\mathcal{O}_{nr}$ for $nr > m$ is homotopic to a loop in $\cup_{-m}^{m}\mathcal{O}_i$, and thus to a loop in $F_{nr} \subset \partial\mathcal{O}_{nr}$, since F_{nr} is incompressible in $\overline{\mathcal{O}}$ and separates $\mathcal{O}_{nr}$ from $\cup_{-m}^{m}\mathcal{O}_i$. This shows that the inclusion homomorphism $\pi_1 F_{nr} \to \pi_1\mathcal{O}_{nr}$ is an isomorphism.

By applying Stallings' 3-dimensional h-cobordism theorem [212] (see also [106, Thm.10.2]) to a finite, regular, manifold covering of $\mathcal{O}_{nr}$ and by using the fact that a finite group action on an irreducible, trivial bundle preserves the product structure [63, Thm.4.1], [154, Thm.8.1], one concludes that $\mathcal{O}_{nr}$, and hence $\mathcal{O}_0$, is homeomorphic to $F \times [0,1]$. In particular $\mathcal{O}$ is a surface bundle over $\mathbf{S}^1$ with fiber F and $p_*\pi_1(\overline{\mathcal{O}}) = \pi_1 F$. Then the relation $\tau^r\gamma\tau^{-r} = \alpha^{-1}\gamma\alpha$ for all $\gamma \in \pi_1 F$ shows that the monodromy $\phi\colon F \to F$ of the surface bundle $\mathcal{O}$ has finite order in the mapping class group of F. By Nielsen's Realization Theorem [251], ϕ is isotopic to a periodic homeomorphism $\phi' : F \to F$. Then the mapping torus of an orbit of a point $x \in F$ under ϕ' is a circle, and $\mathcal{O}$ is a disjoint union of circles, with saturated tubular neighborhoods. Therefore $\mathcal{O}$ is Seifert fibered. Moreover the infinite central subgroup Z is generated by a power of the regular fiber. □

For a Haken 3-manifold M with incompressible boundary, Jaco-Shalen [115] and Johannson [118] obtained more precise conclusions with respect to homotopic properties of the toric splitting than just the existence and the uniqueness of this splitting (see also [113, 199]): they proved that any essential map of a torus into M is properly homotopic into a Seifert piece of the toric splitting. Here is the analogous statement for orbifolds:

Theorem 4.17 (Enclosing property).
Let $\mathcal{O}$ be a compact, Haken 3-orbifold. If $f : T \to \mathcal{O}$ is a π_1-injective essential map from a toric 2-orbifold T, then there exists a Euclidean or Seifert component $S \subset \mathcal{O}$ of the toric splitting of $\mathcal{O}$ and a π_1-injective map $g : T \to \mathcal{O}$ homotopic to f such that the image $g(T)$ lies in S.

This result can be proved by using inductive arguments on hierarchies like in [115] and [118], or like in [201] by making first the immersion of the toric 2-orbifold transverse to the JSJ family of toric suborbifolds and then by using a strengthened version of the torus theorem.

In the case of an irreducible 3-manifold with incompressible boundary, Jaco-Shalen [115] and Johannson [118] were considering a richer characteristic splitting along essential tori and annuli. Such a splitting exists also in the case of 3-orbifolds $\mathcal{O}$ with incompressible boundary. Let $D\mathcal{O}$ denote the double of $\mathcal{O}$, obtained by gluing two copies of $\mathcal{O}$ along their boundary. Then the existence of the JSJ-splitting of $\mathcal{O}$ along toric and annular 2-suborbifolds can be established by using [154] to make the toric splitting of $D\mathcal{O}$ equivariant with respect to the orientation reversing involution exchanging the two halves of $D\mathcal{O}$.

4.6 Compact core

The following theorem is important for the study of noncompact 3-orbifolds $\mathcal{O}$ without bad 2-suborbifold and with a finitely generated fundamental group. It enables one to find a compact suborbifold of $\mathcal{O}$ (a *compact core*) which carries the fundamental group $\pi_1\mathcal{O}$. This has been first proved for 3-manifolds by P. Scott [196, 197] and independently by P. Shalen (unpublished). The case of very good, irreducible orbifolds has been handled by M. Feighn and G. Mess [69].

Theorem 4.18. *Let $\mathcal{O}$ be a very good, irreducible* 3*-orbifold. If $\pi_1\mathcal{O}$ is finitely generated, then there is a compact irreducible* 3*-suborbifold $\mathcal{O}_0 \subset \mathcal{O}$ such that the inclusion map induces an isomorphism $\pi_1(\mathcal{O}_0) \to \pi_1(\mathcal{O})$.*

Proof. Since $\mathcal{O}$ is very good, there is a 3-manifold M and a finite subgroup $G \subset \mathrm{Diff}^+(M)$ such that $\mathcal{O} = M/G$. Since $\mathcal{O}$ is irreducible, Theorem 3.23 shows that M is irreducible. Moreover, $\pi_1 M$ is a finite index subgroup of $\pi_1(\mathcal{O})$. Hence it is finitely generated. By Scott's Compact Core Theorem [196, 197] it is finitely presented. Then the proof of [69, Thm.2(b)] shows that M has a G-invariant compact core. □

Theorem 4.18 shows that the fundamental group of a very good, irreducible 3-orbifold is *coherent,* i.e. its finitely generated subgroups are finitely presented. By the Orbifold Theorem, this applies also to good, compact, irreducible 3-orbifolds, since these are in fact very good.

Remark. One difficulty to extend the proof of Scott's Compact Core Theorem to the case of a good, irreducible 3-orbifold $\mathcal{O}$ with finitely generated fundamental group is the accessibility property for $\pi_1\mathcal{O}$. This property means that the process of splitting $\pi_1\mathcal{O}$ over finite subgroups must stop after a finite number of steps. When one splits over *trivial* subgroups, this property always holds by Gruško's Theorem. This fact is used in a crucial way in the proof of Scott's Compact Core Theorem for 3-manifold groups.

M. Dunwoody [59] proved that every finitely presented group is accessible. However, not all finitely generated groups are accessible [60]. For splitting over subgroups of bounded order, there is Linnell's theorem [134]. See [53, Chap. IV] for a general discussion of accessibility.

Chapter 5

Seifert orbifolds

5.1 Introduction

Let $\mathcal{O}$ be a small 3-orbifold. According to the Geometrization Conjecture, it should be spherical, Euclidean, hyperbolic or Seifert fibered. The geometry is expected to reflect the properties of the fundamental group. If $\mathcal{O}$ is Seifert fibered, then by Proposition 2.12, the generic fiber of any Seifert fibration generates a virtually cyclic normal subgroup Z; moreover, when the base of the fibration is Euclidean or hyperbolic, Z is infinite and has infinite index. By contrast, nonelementary Kleinian groups never have normal infinite cyclic subgroups.

The following theorem is a generalization of what has been called "Seifert Fiber Space Conjecture" (and also, which is somewhat misleading, "Seifert Conjecture"). It is an important step towards the Geometrization Conjecture.

Again in this chapter all orbifolds are orientable unless mentioned otherwise.

Theorem 5.1. *Let $\mathcal{O}$ be a closed, irreducible 3-orbifold. Suppose that $\mathcal{O}$ is good or contains no incompressible turnovers. If $\pi_1\mathcal{O}$ has an infinite cyclic normal subgroup, then $\mathcal{O}$ is Seifert fibered.*

A complete proof of this is outside the scope of this book. It uses techniques developed by A. Casson and D. Jungreis [41], D. Gabai [77], G. Mess [158], P. Scott [199, 200] and P. Tukia [234]. An overview will be given in the last section, and the details can be found in [141]. The main goal of this chapter is to prove Theorem 5.1 under some extra hypotheses (which in fact hold a posteriori).

Recall that a group G is *residually finite* if for all $x \in G - \{1\}$, there is a finite index subgroup $H < G$ such that $x \notin H$. All compact geometrizable orbifolds are very good and have residually finite fundamental groups [149]. We shall give a proof of Theorem 5.1 assuming that $\mathcal{O}$ is very good and $\pi_1\mathcal{O}$ is *half-way residually finite* in the sense of P. Shalen [207], which is a weak version of residual finiteness.

The main tool for this is a geometrization theorem for *uniform TMC's*, which are *open* 3-manifolds endowed with a triangulation and a central element of the fundamental group satisfying some large-scale properties.

5.2 Preliminaries

5.2.1 TMC's

Let W be a 3-manifold and $\mathcal{T}$ be a triangulation of W. We say that $\mathcal{T}$ has *bounded geometry* if there is a uniform bound on the number of simplices in the star of a vertex.

We call *TMC* (for "Triangulated Manifold with a Central element) a triple $(W, \mathcal{T}, a)$ where W is a 3-manifold, $\mathcal{T}$ a triangulation of W of bounded geometry and a a nontorsion central element of $\pi_1 W$. A TMC is called open, irreducible, etc. when the 3-manifold W has this property. It is *Seifert fibered* if W admits a *compatible* Seifert fibration, i.e. a Seifert fibration with orientable base such that a is a power of the element of $\pi_1 W$ represented by the generic fiber.

Let $i \in \{1, 2, 3\}$ and P be an i-dimensional simplicial complex. Recall that a map $f : P \to W$ is called *combinatorial* if it maps each i-simplex homeomorphically onto some i-simplex of $\mathcal{T}$. Thus the number of i-simplices of P can be thought of as the number of i-simplices of $f(P)$ "counted with multiplicities", and is called the i-dimensional *volume* of the map f and denoted by $\mathrm{vol}_i(f)$. We use the words *length* and *area* as synonyms for 1-dimensional and 2-dimensional volume, respectively.

Definition. A TMC $(W, \mathcal{T}, a)$ is *uniform* if it satisfies the following properties:

(UR) There exists a constant $C_0 \geq 0$ such that for each vertex x of $\mathcal{T}$, there is a combinatorial loop based at x, of length at most C_0 and representing $a \in \pi_1 W$.

(UIP) There exists a function $C_1 : \mathbf{N} \to \mathbf{N}$ such that every null-homotopic combinatorial map $f : \mathbf{S}^1 \to W$ can be extended to a combinatorial map $\bar{f} : D^2 \to W$ such that $\mathrm{vol}_2(\bar{f}) \leq C_1(\mathrm{vol}_1(f))$.

In the next section, we prove:

Theorem 5.2. *Let $(W, \mathcal{T}, a)$ be an open, irreducible TMC. If $(W, \mathcal{T}, a)$ is uniform, then W is Seifert fibered.*

Corollary 5.3. *Let $\mathcal{O}$ be a good, closed, irreducible 3-orbifold whose fundamental group has an infinite cyclic central subgroup Z. Let $N \subset \pi_1\mathcal{O}$ be an infinite index normal subgroup containing Z. Then the regular covering $\hat{\mathcal{O}}$ with fundamental group N has a Seifert fibration such that Z is contained in the subgroup generated by the generic fiber.*

Remark. The subgroup N may be infinitely generated. This is crucial for our applications.

In the remainder of this chapter we use the following conventions:

i. All homology groups have coefficients in $\mathbf{Q}$.

ii. In a 3-manifold context, *submanifold* means codimension 0 submanifold.

iii. All unlabelled maps are induced by inclusion.

5.2.2 Quasimetrics defined by triangulations

A *quasimetric* on a set X is a function $d : X^2 \to [0, +\infty[$ such that there is a constant $C \geq 0$ such that:

i. $d(x, x) = 0$ for all $x \in X$.

ii. $d(x, y) = d(y, x)$ for all $x, y \in X$.

iii. $d(x, z) \leq d(x, y) + d(y, z) + C$ for all $x, y, z \in X$.

Given $x \in X$ and $R \geq 0$, the *ball* around x of radius R, written $B(x, R)$, is the set of $y \in X$ such that $d(x, y) \leq R$. Given $A \subset X$, the R-*neighborhood* of A is the set $N(A, R) = \bigcup_{x \in A} B(x, R)$. The *diameter* of A is $\operatorname{diam} A := \sup_{x,y \in A} d(x, y) \in [0, +\infty]$.

Let $(W, \mathcal{T})$ be a triangulated 3-manifold. Define the *size* of a subset $A \subset W$, denoted by $\operatorname{size}(A)$, as the minimal cardinal of a set $\mathcal{S}$ of 3-simplices of $\mathcal{T}$ such that $A \subset \bigcup \mathcal{S}$. It may be infinite in general, but a compact subset has always finite size.

Let x, y be points of W. A *minimizing path* between x and y is a (continuous) path $\alpha : [0, 1] \to W$ such that $\alpha(0) = x$, $\alpha(1) = y$ and α has minimal size. Such a path always exists because W is path connected.

We define a quasimetric on W by setting $d_{\mathcal{T}}(x, y) = \operatorname{size}(\alpha) - 1$ where α is a minimizing path between x and y. Note that a subset A has finite diameter iff it has finite size. Moreover, if A is path-connected, then $\operatorname{diam} A + 1 \leq \operatorname{size}(A)$.

5.2.3 Cyclic homotopies

A basepoint in $\mathbf{S}^1$, denoted by 0, will be fixed throughout.

Definition. A *cyclic homotopy* on a TMC $(W, \mathcal{T}, a)$ is a map $H : W \times \mathbf{S}^1 \to W$ such that:

i. $H(\cdot, 0)$ is the identity of W.

ii. For all $x \in W$, the loop $H(x, \cdot)$ represents a (this makes sense without reference to a basepoint because a is central).

Given a point $x \in W$, the loop $H(x, \cdot)$ is called the *track* of x by H.

The *diameter* of H, denoted by $\operatorname{diam} H$, is the supremum of the diameters of its tracks. We say that H is *bounded* if $\operatorname{diam} H$ is finite.

Proposition 5.4. *Let $(W, \mathcal{T}, a)$ be an open, irreducible TMC. If $(W, \mathcal{T}, a)$ is uniform, then $(W, \mathcal{T}, a)$ admits a bounded cyclic homotopy.*

Before proving this proposition, we need a few lemmas.

Lemma 5.5. *Let $(W, \mathcal{T}, a)$ be an open, irreducible, uniform TMC. Then there exists a function $C_2 : \mathbf{N} \to \mathbf{N}$ such that for any compact $K \subset W$, there is a compact, irreducible submanifold L of W such that $K \subset \operatorname{Int} L$ and $\operatorname{diam} L \leq C_2(\operatorname{diam} K)$.*

Proof. Let K be a compact subset of W. By (UR), we can assume without loss of generality that K does not lie in a ball. Let Y be a regular neighborhood of $N(K, 0)$ obtained by taking the star in the second derived triangulation. If Y is irreducible, then we can set $L = Y$. Otherwise, let $S \subset Y$ be an incompressible embedded 2-sphere. Since W is irreducible, there is a ball $B \subset W$ such that $\partial B = S$. At least one component U of $W - \operatorname{Int} Y$ is contained in B. Now U is contained in the C_0-neighborhood of its boundary, because otherwise some point x of U would have the property that any loop based at x of length at most C_0 is contained in U, hence in B, contradicting (UR).

From this, we deduce a bound in terms of $\operatorname{diam} K$ on the diameter of the union of Y and all components of $W - \operatorname{Int} Y$ that lie in balls. Call L this union. It is easy to check that L is irreducible, so Lemma 5.5 follows. □

We call *uniformly irreducible* an open triangulated 3-manifold satisfying the conclusion of Lemma 5.5.

In the following lemma, the diameter of a map is defined to be the diameter of its image.

Lemma 5.6. *Let* $(W, \mathcal{T})$ *be a uniformly irreducible triangulated* 3*-manifold. Then for any* $i > 1$*, every continuous map* $f : S^i \to W$ *can be extended to a continuous map* $\bar{f} : D^i \to W$ *such that* $\operatorname{diam}(\bar{f}) \leq C_2(\operatorname{diam}(f))$.

Proof. Let K be the image of f. Uniform irreducibility gives a compact, irreducible submanifold L containing K of diameter at most $C_2(\operatorname{diam}(f))$. Since W is noncompact, L has non-empty boundary. Hence by Corollary 3.22, L is aspherical and the lemma follows. □

Lemma 5.7. *Let* $(W, \mathcal{T})$ *be a triangulated* 3*-manifold. Let* P *be an* i*-dimensional connected simplicial complex (where* $1 \leq i \leq 3$*) and* $f : P \to W$ *be a combinatorial map. Then* $\operatorname{diam}(f(P)) \leq \operatorname{vol}_i(f)$.

Proof. The i-dimensional volume of f is at least the number of i-simplices of $f(P)$. Since each i-simplex is contained in a 3-simplex, this number is at least the size of $f(P)$. Now $\operatorname{diam}(f(P)) \leq \operatorname{size}(f(P))$ since $f(P)$ is path-connected. □

Proof of Proposition 5.4. The construction of the cyclic homotopy H is by successive extensions over the i-skeleta. At each step, the problem of extending H from the i-skeleton to the $(i+1)$-skeleton can be viewed as a filling problem whose solution is provided by one of our hypotheses.

First use (UR) to construct H on the 0-skeleton. By Lemma 5.7, we have $\operatorname{diam} H \leq C_0$. Then for each edge e between vertices v, v', consider the combinatorial loop α_e based at v defined by first following the track $H(v, \cdot)$, then running through e, following $H(v', \cdot)$ in the opposite direction and coming back through e. By construction, α_e is null-homotopic and has length at most $2C_0+2$. By (UIP), α_e can be filled by a combinatorial disc of area at most $C_1(2C_0+2)$. This allows us to extend H to a cyclic homotopy of diameter at most $C_1(2C_0+2)$ on the 1-skeleton.

By Lemma 5.5, $(W, \mathcal{T})$ is uniformly irreducible. For each 2-simplex c, the restriction of H to ∂c can be viewed as a combinatorial map of an annulus to W. This map can be extended to a combinatorial map $f : S^2 \to W$ by adding two copies of c. Now by Lemma 5.7, $\operatorname{diam}(f) \leq \operatorname{vol}_2(f) \leq 3C_1(2C_0+2)+2$, so by Lemma 5.6, the map f can be filled by a map of the 3-ball of diameter at most $C_2(3C_1(2C_0+2)+2)$.

The extension of H to the 3-skeleton is analogous to the last paragraph, again using uniform irreducibility via Lemma 5.6. □

5.3 Geometrization of uniform TMC's

In this section, we prove Theorem 5.2. Let $(W, \mathcal{T}, a)$ be an open, irreducible, uniform TMC. By Proposition 5.4, $(W, \mathcal{T}, a)$ admits a bounded cyclic homotopy H. This cyclic homotopy can be viewed as a singular Seifert fibration, which we have to turn into a true Seifert fibration.

The following technical notion is inspired by [34].

Definition. Let L be a compact, connected submanifold of W. We say that L is *regular* if $W - \operatorname{Int} L$ has no compact components and L contains a loop representing the central element $a \in \pi_1 W$. If in addition L admits a compatible Seifert fibration, then it is called *S-regular.*

We collect in the next lemma some immediate consequences of the definitions.

Lemma 5.8. *Let L be a regular submanifold of W. Then*

i. L and $W - \operatorname{Int} L$ are irreducible.

ii. Every embedded torus $T \subset W - L$ is incompressible in $W - L$ or null-homologous in $W - L$.

iii. If L is S-regular, then it is a solid torus or incompressible.

Lemma 5.9. *Each compact $K \subset W$ is contained in the interior of some regular submanifold L.*

Proof. Let Y be the 0-neighborhood of K. Pick a combinatorial loop α of diameter $\leq C_0$ based at some vertex of Y. Let U be a regular neighborhood of $Y \cup \alpha$ in $N(Y \cup \alpha, 0)$ whose boundary is a disjoint union of surfaces $F_1, \ldots, F_n$ combinatorially embedded into $\mathcal{T}''$. Let L be the union of U and all compact components of $W - \operatorname{Int} U$.

Then L is a compact, connected submanifold of W, $W - \operatorname{Int} L$ has no compact components and $\alpha \subset L$, so L is regular. □

Lemma 5.10. *Each compact $K \subset W$ is contained in the interior of some S-regular submanifold $V \subset W$.*

Proof. First by Lemma 5.9, K is contained in the interior of some regular submanifold L. By applying Lemma 5.9 to $N(L, \operatorname{diam} H + 1)$, we obtain a regular submanifold K' such that $K \subset \operatorname{Int} K'$ and $d_{\mathcal{T}}(K, \partial K') > \operatorname{diam} H$. Applying again Lemma 5.9, we find regular submanifolds K'', K''' of W such that $K' \subset \operatorname{Int} K''$, $K'' \subset K'''$, $d_{\mathcal{T}}(\partial K', \partial K'') > \operatorname{diam} H$, and $d_{\mathcal{T}}(\partial K'', \partial K''') > \operatorname{diam} H$.

Define $X := K''' - \operatorname{Int} K'$ and let $X_1, \dots, X_p$ be the components of X. By the usual argument, we see that each X_i is irreducible. For each $1 \leq i \leq p$, let $F_{i,1}, \dots, F_{i,n_i}$ be the components of $\partial K''$ that lie in X_i. Define a class $\omega_i \in H_2(X_i)$ by $\omega_i := \sum_j [F_{i,j}]$.

Claim. *For each i, there is a finite collection of pairwise disjoint embedded tori $T_{i,1}, \dots, T_{i,m_i}$, none of which is null-homologous in X_i, such that the following formula holds in $H_2(X_i)$:*

$$\omega_i = \sum_k [T_{i,k}].$$

Let us assume this claim for the moment. Then the union of all tori $T_{i,k}$ for all i and all k bounds a compact submanifold $V \subset K'''$ containg K'. Hence $K \subset \operatorname{Int} V$ and $\operatorname{diam} V$ is bounded by a function of $\operatorname{diam} K$. By Lemma 5.8(ii), each boundary torus of V is incompressible in $W - L$ since it cannot be null-homologous in $W - L$.

If every component of ∂V is incompressible in W, then the map $\pi_1 V \to \pi_1 W$ is injective. Now V contains a loop representing the central element a, so $\pi_1 V$ has nontrivial center, and by Theorem 4.16 it has a compatible Seifert fibration. The proof of Lemma 5.10 is complete in this case.

Suppose now that some component T of ∂V is compressible in W. Then since L does not lie in a 3-ball, V is a solid torus, hence S-regular. This completes the proof of Lemma 5.10 modulo the claim.

Our next task is to prove the claim. Let i be fixed throughout.

Lemma 5.11. *For all j, there is a finite collection of (possibly singular) tori $T'_{j,1}, \dots, T'_{j,p_j}$ such that the following formula holds in $H_2(X_i)$:*

$$F_{i,j} = \sum_l [T'_{j,l}].$$

Proof. To simplify notation, set $F := F_{i,j}$. Recall that F is a boundary component of K'', so the condition $d_{\mathcal{T}}(\partial K'', \partial X) > \operatorname{diam} H$ implies that the image of $F \times \mathbf{S}^1$ by H is contained in X_i. Thus we may consider the restriction $H : F \times \mathbf{S}^1 \to X_i$. Let $x_0 \in F$ be a basepoint and $G := H_*(\pi_1(F \times \mathbf{S}^1, (x_0, 0))) \subset \pi_1(X_i, x_0)$. Let α be the track of x_0. Then the homotopy class of α in $\pi_1(W, x_0)$ is a, so the homotopy class of α in $\pi_1(X_i, x_0)$ is a nonzero element, say a', which belongs to G by definition. In fact, a' is central in G because $\pi_1 \mathbf{S}^1$ is central in $\pi_1(F \times \mathbf{S}^1)$. We have just shown that G has nontrivial center.

Let X' be the covering of X_i with $\pi_1 X' = G$. By Corollary 3.22, since X_i is irreducible, both X_i and X' are aspherical, so $H_2(X') = H_2(G)$. Furthermore, X' is irreducible by Theorem 3.24. Consider the inclusion $j : F \to X$. Since

G contains $j_*(\pi_1 F)$, j can be factored through a map $j' : F \to X'$. Now the homology class $[F]$ is nonzero in $H_2(X)$. Taking the image of the homology class $[F] \in H_2(X)$ under j'_*, we get a nontrivial element ω of $H_2(G)$.

Scott's compact core theorem [196] (cf. Theorem 4.18) provides a submanifold $X'_0 \subset X'$ such that $\pi_1 X'_0 = G$. Since G has nontrivial center, Theorem 4.16 tells us that X'_0 is Seifert fibered. Hence G is the fundamental group of a compact Seifert fibered manifold with nonempty boundary. Therefore, the class $\omega \in H_2(G)$ can be represented by $\mathbf{Z}^2$ subgroups by Lemma 4.3, so we get the desired singular tori in X_i. □

Theorem 4.17 gives a Seifert fibered (possibly disconnected) submanifold $S \subset X$ which contains singular tori homotopic to the $T'_{j,l}$'s. Take the collection for all j and all l, and call it $T''_1, \ldots, T''_q$. Let ω' be their sum in $H_2(S)$. Apply Lemma 4.3 again to obtain disjoint incompressible embedded tori $T_{i,1}, \ldots, T_{i,m_i}$ in S such that $\omega' = \sum_k [T_{i,k}] \in H_2(S)$.

Now the sum of the homology classes of $T''_1, \ldots, T''_q$ in $H_2(X)$ is ω_i and so $\omega_i = \sum_k [T_{i,k}]$. This finishes the proof of the claim, hence that of Lemma 5.10. □

Proof of Theorem 5.2. By Lemma 5.10, W admits an exhaustion $V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots$ by S-regular submanifolds satisfying the properties of that lemma. Let $\mathcal{S}$ be the class of Haken Seifert fibered manifolds with nonempty boundary whose Seifert fibration is unique up to isotopy. Then each V_n is either homeomorphic to $\mathbf{S}^1 \times \mathbf{D}^2$, $\mathbf{T}^2 \times \mathbf{I}$, $\mathbf{K}^2 \tilde{\times} \mathbf{I}$ (the twisted $\mathbf{I}$-bundle over the Klein bottle) or belongs to $\mathcal{S}$. Note that if $V_N \in \mathcal{S}$ for some N, then $V_n \in \mathcal{S}$ for all $n \geq N$. If no V_n belongs to $\mathcal{S}$ and $V_N \cong \mathbf{K}^2 \tilde{\times} \mathbf{I}$ for some N, then $V_n \cong \mathbf{K}^2 \tilde{\times} \mathbf{I}$ for all $n \geq N$, because neither $\mathbf{S}^1 \times \mathbf{D}^2$ nor $\mathbf{T}^2 \times \mathbf{I}$ can contain an incompressible $\mathbf{K}^2 \tilde{\times} \mathbf{I}$. Likewise, if all V_n's are solid tori or $\mathbf{T}^2 \times \mathbf{I}$'s, then either all V_n's are $\mathbf{T}^2 \times \mathbf{I}$'s for large n, or all V_n's are solid tori. This gives four cases.

Case 1 V_n belongs eventually to $\mathcal{S}$.

Then for large n, both V_n and V_{n+1} have a unique Seifert fibration up to isotopy, and V_n has incompressible boundary as a submanifold of V_{n+1}. This means that V_{n+1} admits a Seifert fibration such that every boundary component of V_n is a vertical torus. Thus any Seifert fibration on V_n extends to a Seifert fibration on V_{n+1}. Hence W has a Seifert fibration with orientable base.

Case 2 V_n is eventually $\mathbf{K}^2 \tilde{\times} \mathbf{I}$.

Since $\mathbf{K}^2 \tilde{\times} \mathbf{I}$ has only one Seifert fibration with orientable base up to isotopy, the argument for Case 1 applies. In fact, one can show that $V_{n+1} - \operatorname{Int} V_n \cong \mathbf{T}^2 \times \mathbf{I}$ for large n, so W is homeomorphic to the interior of $\mathbf{K}^2 \tilde{\times} \mathbf{I}$.

Case 3 V_n is eventually $\mathbf{T}^2 \times \mathbf{I}$.

This time, the Seifert fibration with orientable base on V_n is not unique up to isotopy, but it is still true that any Seifert fibration on V_n extends to a Seifert fibration on V_{n+1}, so W is Seifert fibered with orientable base. Again, it can be shown that in fact $W \cong \mathbf{T}^2 \times \mathbf{R}$ in this case.

Case 4 Each V_n is a solid torus.

This is the hardest case, because V_n is no longer incompressible. Without loss of generality, we assume that the tori ∂V_n are chosen sufficiently far from each other so that $d_{\mathcal{T}}(\partial V_1, V_0)$ is greater than $\operatorname{diam} H$, and there exists a point $x_0 \in V_1 - \operatorname{Int} V_0$ such that $d_{\mathcal{T}}(x_0, \partial V_1)$ and $d_{\mathcal{T}}(x_0, \partial V_2)$ are greater than $\operatorname{diam} H$.

For $n \geq 2$, define $X_n := V_n - \operatorname{Int} V_1$. Thus ∂X_n is the disjoint union of the two tori ∂V_1 and ∂V_n. The proof is based on the following lemma:

Lemma 5.12. *For all $n \geq 2$, X_n admits a Seifert fibration with orientable base, which extends to V_n.*

Proof. Let $i : X_n \to W$ and $j : X_n \to W - V_0$ be the inclusion maps. We see that ∂V_1 and ∂V_n are incompressible in $W - V_0$ (otherwise a compression would yield a sphere, which would have to bound a ball containing V_0). Thus the induced map $j_* : \pi_1 X_n \to \pi_1(W - V_0)$ is injective.

In order to apply Theorem 4.16, we are looking for a central element $A \in \pi_1 X_n$ such that $i_* A = a$. Let α be the track of x_0 and $A \in \pi_1 X_n$ be the homotopy class of α. Since $d_{\mathcal{T}}(x_0, \partial X_n) > \operatorname{diam} H$, it follows that α lies in X_n, and since $d_{\mathcal{T}}(\partial V_1, V_0) > \operatorname{diam} H$, we know that $H(X_n \times \mathbf{S}^1)$ lies in $W - V_0$. Thus

$$\begin{aligned} H_*(\pi_1(X_n \times \mathbf{S}^1)) &= H_*(\pi_1(X_n \times \{0\})) \cdot H_*(\pi_1(\{x_0\} \times \mathbf{S}^1)) \\ &= H_*(\pi_1(X_n \times \{0\})) \\ &= j_* \pi_1(X_n, x_0) \subset \pi_1(W - V_0). \end{aligned}$$

Now $j_* A$ is central in $H_*(\pi_1(X_n \times \mathbf{S}^1)) \subset \pi_1(W - V_0)$. Since j_* is injective, we conclude that A is central in $\pi_1 X_n$. Moreover, $i_* A = a$ since A is the homotopy class of a track of H. By Theorem 4.16, X_n has a Seifert fibration with orientable base, such that A is a power of the generic fiber. This condition ensures that a generic fiber cannot be contractible in W, so the Seifert fibration extends to all of V_n. $\square$

From this lemma, we can conclude that W admits a Seifert fibration with orientable base by applying Cases 1–3 above to the exhaustion of $W - \operatorname{Int} V_1$ by the X_n's, remarking that all boundary tori are incompressible in $W - \operatorname{Int} V_1$. But we can say more: for all $n \geq 2$, the fibration on X_n has base orbifold an annulus with at most one cone point, because it extends to V_n, which is a solid torus. So either all X_n's are $\mathbf{T}^2 \times \mathbf{I}$ or X_n eventually belongs to $\mathcal{S}$. In the latter case, exactly one exceptional fiber appears. In both cases, W is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^2$. $\square$

5.4 The half-way residually finite case

Definition. An infinite group is called *half-way residually finite* if it is finite or has subgroups of arbitrary large finite index.

Here is a list of easily verified facts:

i. If G is half-way residually finite, then any finite index subgroup of G is half-way residually finite.

ii. If G has a proper finite index subgroup K, then there is a normal finite index subgroup H such that $H \subset K$. (Hence the subgroups in the above definition may be chosen to be normal.)

iii. If G is residually finite, then G is half-way residually finite.

In this section, we prove the following weak version of Theorem 5.1.

Theorem 5.13. *Let $\mathcal{O}$ be a closed, irreducible, very good 3-orbifold with half-way residually finite fundamental group. If $\pi_1\mathcal{O}$ has an infinite cyclic normal subgroup, then $\mathcal{O}$ is Seifert fibered.*

We shall make extensive use of the following lemma.

Lemma 5.14. *i. If an orbifold $\mathcal{O}$ satisfies the hypotheses of Theorem 5.13, then any finite covering of $\mathcal{O}$ also satisfies these hypotheses.*

ii. Any compact, very good, irreducible orbifold $\mathcal{O}$ that is finitely covered by a Seifert fibered orbifold and whose fundamental group has an infinite cyclic normal subgroup is Seifert fibered.

Hence to prove Theorem 5.13, it suffices to prove it for a finite covering.

Proof. (i) follows from Theorem 3.23 and fact (i) above.

(ii) Suppose now that some finite covering $\mathcal{O}_1$ of $\mathcal{O}$ is Seifert fibered. Since $\mathcal{O}$ is very good, it is finitely covered by a manifold M_1. There is a common finite covering M_2 of $\mathcal{O}_1$ and M_1 that is a Seifert fibered manifold. By fact (ii) above, some finite covering M_3 of M_2 is a regular finite covering of $\mathcal{O}$. By Theorem 3.23, M_3 is irreducible. The infinite cyclic normal subgroup of $\pi_1\mathcal{O}$ induces an infinite cyclic normal subgroup of $\pi_1 M_3$, which can be represented by some Seifert fibration. Thus by the Meeks-Scott Theorem 3.25, $\mathcal{O}$ is Seifert fibered. □

We need two more elementary group-theoretic lemmas.

Lemma 5.15. *Let G be a group. Let $Z \subset G$ be an infinite cyclic normal subgroup. Then G admits a subgroup of index at most two that centralizes Z.*

Proof. The automorphism group of Z has order 2. □

Lemma 5.16. *Let G be an infinite, half-way residually finite group and $Z \subset G$ be a normal abelian subgroup. Then either G has a normal finite index subgroup G_1 that contains Z and has infinite abelianization, or Z is contained in infinitely many normal finite index subgroups.*

Proof. Assume that only finitely many normal finite index subgroups contain Z and let G_1 be their intersection. Then G_1 is normal, has finite index, contains Z and is minimal among such subgroups. We are going to show that G_1 has infinite abelianization.

By fact (i) above, G_1 is half-way residually finite. For each $i > 0$ choose a normal subgroup $N_i \subset G_1$ such that $i \leq [G_1 : N_i] < \infty$. Without loss of generality we can also assume that each N_i is normal in G. Let $\phi_i : G \to G/N_i$ be the natural projection. Then $\phi_i^{-1}(\phi_i(Z))$ is a normal subgroup G of finite index that contains Z, so it contains G_1. It follows that $G_1/N_i = \phi_i(G_1) \subset \phi_i(Z)$ is abelian. Therefore G_1 has finite abelian quotients of arbitrarily high order, hence infinite abelianization. □

Proposition 5.17. *Let M be a closed, irreducible* 3*-manifold whose fundamental group has an infinite cyclic central subgroup Z. One fixes an arbitrary triangulation on M. Then there is a cyclic homotopy H representing a generator of Z. Furthermore, if M contains embedded solid tori $V \subset V' \subset V''$ such that $d(\partial V, \partial V') > \operatorname{diam} H$, $d(\partial V', \partial V'') > \operatorname{diam} H$ then M is Seifert fibered.*

To prove this proposition, one adapts the proof of Lemma 5.12 to show that $\pi_1(M - V')$ has nontrivial center, and then one applies Theorem 4.16. Details are left to the reader.

Proof of Theorem 5.13. Using Lemma 5.14(i) and the hypothesis that $\mathcal{O}$ is very good, we may assume without loss of generality that $\mathcal{O}$ is a manifold. Let Z be an infinite cyclic normal subgroup of $\pi_1\mathcal{O}$. By Lemma 5.14(i) and Lemma 5.15, we may assume that Z is central.

By Lemma 5.16, either $\pi_1\mathcal{O}$ admits a normal finite index subgroup $G_1 \subset \pi_1 M$ containing Z and whose abelianization is infinite, or Z lies in infinitely many normal finite index subgroups. In the former case, consider the covering M_1 corresponding to G_1: it is a regular finite covering of M; furthermore, M_1 is Haken by Proposition 4.2, and $\pi_1 M_1$ has nontrivial center. We conclude by Theorem 4.16 and Lemma 5.14(ii).

Suppose now that Z lies in infinitely many normal finite index subgroups. Since $\pi_1\mathcal{O}$ is finitely generated, it has only finitely many subgroups of a given finite index. Hence there is a decreasing sequence $N_1 \supset \cdots \supset N_i \supset \cdots \supset Z$ of finite index normal subgroups such that $[\pi_1 M : N_i]$ tends to infinity.

Set $N := \bigcap N_i$ and $\Gamma := \pi_1 M/N$. It is easy to check that Γ is residually finite. By construction, N is an infinite index normal subgroup of $\pi_1\mathcal{O}$ with nontrivial center. By Corollary 5.3, the corresponding cover $\hat{M}$ is Seifert fibered.

By inspection of the possible bases, we see that either $\hat{M}$ contains an embedded incompressible torus or $\hat{M}$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^2$. We deal with either case separately.

Case 1 $\hat{M}$ contains an embedded incompressible torus T.

Since T is compact, the set of elements $g \in \Gamma$ such that $gT \cap T \neq \emptyset$ is finite. Now Γ is residually finite, so there is a finite index normal subgroup $\Gamma_1 \subset \Gamma$ such that all translates of T by elements of Γ_1 avoid T. Then $M_1 := \hat{M}/\Gamma_1$ is a finite cover of $\mathcal{O}$. Projecting T to M_1, we get an embedded incompressible torus. Thus we can conclude by applying Theorem 4.16 and Lemma 5.14(ii) as before.

Case 2 $\hat{M}$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^2$.

We are going to show that some finite cover of $\mathcal{O}$ satisfies the hypotheses of Proposition 5.17. With this goal in mind, choose large solid tori $\hat{V} \subset \hat{V}' \subset \hat{V}'' \subset \hat{M}$ whose cores generate $\pi_1\hat{M}$ and such that $d(\partial\hat{V}, \partial\hat{V}') > \operatorname{diam} H$, $d(\partial\hat{V}', \partial\hat{V}'') > \operatorname{diam} H$. Since $\hat{V}''$ is compact, the set of elements $g \in \Gamma$ such that $g\hat{V}'' \cap \hat{V}'' \neq \emptyset$ is finite. As in Case 1, the residual finiteness of Γ implies that there is a finite index normal subgroup Γ_1 of Γ such that for all $g \in \Gamma_1$, $g\hat{V}'' \cap \hat{V}'' = \emptyset$. The manifold $M_1 := \hat{M}/\Gamma_1$ is a finite cover of $\mathcal{O}$ and the projections V, V', V'' of $\hat{V}, \hat{V}', \hat{V}''$ to M_1 are embedded solid tori and satisfy the hypotheses of Proposition 5.17. Again we conclude by Lemma 5.14. □

5.5 Small Seifert orbifolds

Let us recall the statement of Theorem 5.1 from the introduction of this chapter.

Theorem 5.1 *Let $\mathcal{O}$ be a closed, irreducible 3-orbifold. Suppose that $\mathcal{O}$ is good or contains no incompressible turnover. If $\pi_1\mathcal{O}$ has an infinite cyclic normal subgroup Z, then $\mathcal{O}$ is Seifert fibered.*

An incompressible turnover is π_1-injective in a good orbifold by the Equivariant Loop Theorem (cf. Corollary 3.20). Moreover, its fundamental group has trivial center. Thus the proof of Theorem 4.16 shows that if $\mathcal{O}$ is a closed, irreducible, good orbifold which contains an incompressible turnover and such that $\pi_1\mathcal{O}$ has an infinite cyclic normal subgroup, then $\mathcal{O}$ is a bundle over $\mathbf{S}^1$ with fiber a turnover, hence Seifert fibered. The proof of Theorem 5.1 thus reduces to the small case, namely:

Theorem 5.18. *Let $\mathcal{O}$ be a closed, small 3-orbifold. If $\pi_1\mathcal{O}$ has an infinite cyclic normal subgroup Z, then $\mathcal{O}$ is Seifert fibered.*

First we sketch the proof of this theorem under the additional hypothesis that $\mathcal{O}$ is good. Set $\Gamma := \pi_1\mathcal{O}/Z$. Let $\hat{\mathcal{O}}$ be the regular covering of $\mathcal{O}$ such that $\pi_1\hat{\mathcal{O}} = Z$. The group Γ acts properly and cocompactly on $\hat{\mathcal{O}}$. By Theorem 3.24, $\hat{\mathcal{O}}$ is irreducible. Since $\pi_1\hat{\mathcal{O}}$ is torsion free and $\mathcal{O}$ is good, $\hat{\mathcal{O}}$ is a manifold.

Applying Corollary 5.3 (with $N = Z$), we see that $\hat{\mathcal{O}}$ is Seifert fibered, and in fact homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^2$.

At this point we want to use geometric group theory, more specifically quasi-isometries of groups, so the reader is referred back to Section 1.4 of Chapter 1 for an introduction to this material.

Recall that $\hat{\mathcal{O}}$ comes with a Γ-invariant quasimetric d (associated to an arbitrarily chosen Γ-invariant triangulation). By a straightforward generalization of Proposition 1.13, the quasimetric space $(\hat{\mathcal{O}}, d)$ is quasi-isometric to Γ (endowed as usual with the word metric).

The insight behind the proof is that if d was a product metric, then we could forget about the (finite diameter) $\mathbf{S}^1$ factor and get a quasi-isometry between Γ and a plane. Since d is only coarsely defined, there is no hope to prove that it is actually a product metric. However, we have the following theorem.

Theorem 5.19 ([141]). *Let $(W, \mathcal{T}, a)$ be an open, orientable, irreducible, maximal TMC. If $(W, \mathcal{T}, a)$ is uniform, then there is a compatible Seifert fibration on W with base B and projection map $p : W \to B$ and a complete Riemannian metric on B such that p is a quasi-isometry.*

This theorem is proved using PL minimal surfaces. In our case, we set $W := \hat{\mathcal{O}}$ and let $\mathcal{T}$ be any Γ-invariant triangulation and a be a generator of Z. The conclusion we get is that our group Γ is quasi-isometric to a complete Riemannian metric on $\mathbf{R}^2$.

The next step is the following result:

Theorem 5.20. *Let Γ be a finitely generated group quasi-isometric to some complete Riemannian metric on $\mathbf{R}^2$. Then Γ is virtually a surface group.*

This was first proved by G. Mess [158] in the case of a quasihomogeneous Riemannian plane, modulo a conjecture on **convergence groups** which was later proved by A. Casson and D. Jungreis [41], and D. Gabai [77] using earlier work of P. Tukia [234]. This result on convergence groups is stated in this book as Theorem 6.18. A different proof of Theorem 5.20 in full generality, also based on Theorem 6.18, was given later in [140].

From Theorem 5.20 it is not hard to prove that $\pi_1\mathcal{O}$ is isomorphic to the fundamental group of a Seifert fibered orbifold. To conclude we need the following result, see [141] for a proof:

Theorem 5.21. *Let $\mathcal{O}$ be a closed small 3-orbifold. If $\pi_1\mathcal{O}$ is infinite and isomorphic to the fundamental group of a Seifert orbifold, then $\mathcal{O}$ is Seifert fibered.*

Theorem 5.21 is an extension to small orbifolds of a theorem proved by P. Scott [200] for small manifolds. This result is in fact true for all closed irreducible orbifolds, but as of this writing, there is no known proof that does not use the Orbifold Theorem.

This completes the sketch of the proof of Theorem 5.18 when $\mathcal{O}$ is good.

Without the assumption that $\mathcal{O}$ is good, one can refine the arguments outlined above to show that $\pi_1\mathcal{O}$ is isomorphic to the fundamental group of a Seifert fibered orbifold. With this goal in mind, we look at the covering $\hat{\mathcal{O}}$. This does not *a priori* have to be a manifold. We let W be the underlying space of $\hat{\mathcal{O}}$. It is an orientable 3-manifold with infinite cyclic fundamental group and the group Γ acts properly and cocompactly on it.

Let us consider the interesting case when W is open. The difficulty is that we do not know beforehand that W is irreducible, so we need a refined version of Theorem 5.19 without the hypothesis that W is irreducible. Of course, we cannot expect to prove that W is Seifert fibered without proving the Poincaré conjecture, but we do not need as much. Here is the statement.

Theorem 5.22 ([141]). *Let $(W, \mathcal{T}, a)$ be an open, orientable, uniform, maximal TMC. Then there is an open, orientable TMC $(W', \mathcal{T}', a')$, a proper homotopy equivalence $\phi : W \to W'$ sending a to a' and which is a quasi-isometry, and a compatible Seifert fibration on W' with base B and projection map $p : W' \to B$ and a complete Riemannian metric on B such that p is a quasi-isometry.*

Hence in our setting we still get a quasi-isometry from Γ to a complete Riemannian plane, and by Theorem 5.20 Γ is virtually a surface group.

At that point, we use as above Theorem 5.21 to complete the proof of Theorem 5.18.

Chapter 6

Hyperbolic orbifolds

A *Kleinian group* is by definition a discrete subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$. Recall that we defined a *hyperbolic orbifold* to be the quotient of $\mathbf{H}^3$ by such a group. Hence the geometrization program motivates the study of these groups.

In this chapter, we first recall some basic facts of hyperbolic geometry, the ideal boundary of $\mathbf{H}^3$ and the classification of isometries in Section 6.1. In Sections 6.2 and 6.3, we review the part of Kleinian group theory relevant to this book, focusing on finite covolume and geometrically finite groups. Finally in Section 6.4 we discuss Gromov's theory of hyperbolic groups and its relevance to geometrization, in particular the so-called Weak Hyperbolization Conjecture.

6.1 Hyperbolic 3-space and its isometries

6.1.1 The ideal boundary

To understand isometries of $\mathbf{H}^3$, it is instructive to look at their action on the ideal boundary $\partial_\infty \mathbf{H}^3$, which can be defined in several ways. For practical purposes, it is useful to have an explicit description of this boundary in terms of one of the classical models for $\mathbf{H}^3$: the ball model, the upper half-space model, the Klein projective model, etc. [7, 225]; but it is also good to have an intrinsic definition, in particular when one is interested in generalizations (cf. Section 6.4).

We start with the intrinsic definition. Let $\mathcal{G}$ be the set of geodesic rays $r : [0, +\infty) \to \mathbf{H}^3$ parametrized by arclength. Two rays $r_1, r_2 \in \mathcal{G}$ are said to be equivalent if the function $t \mapsto d(r_1(t), r_2(t))$ is bounded. The *ideal boundary* $\partial_\infty \mathbf{H}^3$ is the quotient of $\mathcal{G}$ by this equivalence relation. Its elements are called ideal points. Thus for any geodesic ray, there is an associated ideal point, called its *endpoint*; to any complete geodesic are associated two ideal points, again called its *endpoints*.

For every $x_0 \in \mathbf{H}^3$, rays emanating from x_0 are in bijection with their endpoints in $\partial_\infty \mathbf{H}^3$. This identification provides a topology on $\partial_\infty \mathbf{H}^3$, which is independent of the choice of x_0 and makes it homeomorphic to the 2-sphere. One can extend this topology to a topology on $\hat{\mathbf{H}}^3 := \mathbf{H}^3 \cup \partial_\infty \mathbf{H}^3$ such that the pair $(\hat{\mathbf{H}}^3, \partial_\infty \mathbf{H}^3)$ is homeomorphic to $(\mathbf{D}^3, \mathbf{S}^2)$. In the ball model, $\mathbf{H}^3$ is represented as the open unit ball in 3-space equipped with the Poincaré metric, the ideal boundary is the unit sphere, geodesics are arcs of circles orthogonal to the boundary, and the topology on $\hat{\mathbf{H}}^3$ is the obvious one. We will sometimes work in the upper half-space model $\{(z, e^t) | z \in \mathbf{C}, t \in \mathbf{R}\}$ with the Poincaré metric; here the ideal boundary is to be thought of as $\mathbf{C} \cup \{\infty\}$, where $\mathbf{C}$ is identified with $\mathbf{C} \times \{0\}$ and ∞ lies above the picture; geodesics are either arcs of circles orthogonal to $\mathbf{C}$ or vertical half-lines; in the latter case one of the endpoints is ∞.

There is a natural conformal structure on $\partial_\infty \mathbf{H}^3$ which makes it conformally equivalent to the Riemann sphere $\mathbf{CP}^1$. This is easily seen in the upper half-space model, where $\mathbf{CP}^1$ is decomposed as $\mathbf{C} \cup \{\infty\}$. Any orientation-preserving isometry of $\mathbf{H}^3$ acts conformally on $\partial_\infty \mathbf{H}^3$ in a natural way, so that there is a canonical group homomorphism between $\mathrm{Isom}^+(\mathbf{H}^3)$ and $\mathrm{PSL}_2(\mathbf{C})$. In fact, this is an isomorphism of Lie groups, and in the sequel we shall identify them. The classification of orientation-preserving isometries of $\mathbf{H}^3$ can thus be given either geometrically or algebraically. The interaction between the two viewpoints will be crucial in the next chapter, so we explain it in some detail.

6.1.2 Classification of hyperbolic isometries

Let A be an element of $\mathrm{Isom}^+(\mathbf{H}^3) = \mathrm{PSL}_2(\mathbf{C})$ different from the identity. Then exactly one of the following occurs:

Parabolic: A is conjugate to $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Its trace is ± 2. Geometrically, A has no fixed point in $\mathbf{H}^3$ and exactly one fixed point in $\partial_\infty \mathbf{H}^3$. It leaves invariant the foliation by horospheres centered at the fixed point. The best way to view a parabolic element is in the upper-half space model, assuming the fixed point is ∞: invariant horospheres are horizontal planes, and the action on $\mathbf{C} \subset \partial_\infty \mathbf{H}^3$ is a translation $z \mapsto z + \tau$.

Semi-simple: A is conjugate to $\pm\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ with $\lambda \in \mathbf{C}^*$. One can write $\lambda = e^{(l+i\theta)/2}$ with $l, \theta \in \mathbf{R}$. Geometrically, A is a screw-motion, with translation length l and rotation angle θ; there is a unique globally invariant geodesic γ_A, called the *axis* of A. The two endpoints of the axis are the only two ideal fixed points. In the upper-half space model, if ∞ is fixed, the axis is a vertical line and the action on $\mathbf{C}$ is a similarity with one fixed point. If $|\lambda| = 1$, A is called **elliptic**, and the axis of A is fixed pointwise. (In fact the axis coincides with the set of fixed points in $\mathbf{H}^3$.) Otherwise A is called **loxodromic** and fixes no point in $\mathbf{H}^3$.

Note that a semi-simple element preserves the foliation by surfaces $\{x \in \mathbf{H}^3 \mid d(x, \gamma_A) = C\}$.

There is a convenient way of encoding geometric information about A into a complex number $u(A)$ called its *complex length:* if A is semi-simple, set $u(A) := l + i\theta$; if A is parabolic or the identity, set $u(A) := 0$. This number is defined up to $2\pi i\mathbf{Z}$ (corresponding to lifting the rotation angle to $\mathbf{R}$) and up to sign (corresponding to the choice of orientation of the axis). It is related to the trace by the formula $\operatorname{trace}(A) = \pm 2\cosh(u(A)/2)$.

6.2 Basic theory of Kleinian groups

6.2.1 Domain of discontinuity and limit set

Let Γ be a Kleinian group. By the previous section, we can view it as a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ acting on the Riemann sphere. By analogy with Fatou and Julia sets of rational fractions, we define the *domain of discontinuity* of Γ as the largest open subset $\Omega(\Gamma) \subset \mathbf{CP}^1$ on which Γ acts properly.[1] The *limit set* $\Lambda(\Gamma)$ is defined as $\mathbf{CP}^1 - \Omega(\Gamma)$. It has a number of characterizations, for instance it is the closure of the set of fixed points of non-elliptic elements of $\Gamma - \{Id\}$.

We say that Γ is *elementary* if $\Lambda(\Gamma)$ is finite. Using the dynamics of the action of an isometry on $\partial_\infty \mathbf{H}^3$, one can prove that when $\Lambda(\Gamma)$ is finite it has at most two points. In addition, one can classify elementary groups according to $|\Lambda(\Gamma)|$:

- Elliptic type: a finite subgroup of elliptic elements with a common fixed point. In this case $\Lambda(\Gamma) = \emptyset$.

[1] Recall that in this book, we call "properly" what is sometimes called "properly discontinuously".

- Parabolic type: virtually an abelian group (of rank 1 or 2) of parabolic isometries with a common fixed point at infinity. The limit set $\Lambda(\Gamma)$ consists of one single point.

- Hyperbolic type: virtually an infinite cyclic group of loxodromic isometries with a common axis, when $\Lambda(\Gamma)$ has precisely two points.

We remark that a Kleinian group is virtually abelian if and only if it is elementary. Looking at the classification of finite subgroups of O(3) we deduce:

Lemma 6.1. *Let Γ be an elementary Kleinian group. If Γ has an element of order > 5 (including ∞), then Γ has either a fixed point at $\partial_\infty \mathbf{H}^3$ or an invariant geodesic.*

If Γ is *nonelementary* (i.e. not elementary), $\Lambda(\Gamma)$ can also be characterized as the smallest nonempty closed Γ-invariant subset of $\mathbf{CP}^1$.

6.2.2 The Margulis Lemma and its consequences

Let Γ be a Kleinian group. Given a point $x \in \mathbf{H}^3$ and an $\varepsilon > 0$, we let $\Gamma_{x,\varepsilon}$ denote the subgroup of Γ generated by the set $\{\gamma \in \Gamma | d(x, \gamma x) \leq \varepsilon\}$.

Theorem 6.2 (Margulis Lemma). *There is a universal constant $\mu_0 > 0$ such that for every Kleinian group Γ, every positive number $\varepsilon \leq \mu_0$ and every $x \in \mathbf{H}^3$, the group $\Gamma_{x,\varepsilon}$ is elementary.*

Definition. A *Margulis constant* is a constant $\mu_0 > 0$ satisfying the conclusion of the theorem above.

From now on we fix a Margulis constant.

Definition. Fix a hyperbolic 3-orbifold $\mathcal{O} = \mathbf{H}^3/\Gamma$ and a real number $\varepsilon > 0$. The *ε-thin part* of $\mathcal{O}$ is the set $\mathcal{O}_{(0,\varepsilon]}$ of all points $x \in \mathcal{O}$ such that $d(\tilde{x}, \gamma\tilde{x}) \leq \varepsilon$ for some lift $\tilde{x}$ of x to $\mathbf{H}^3$ and some element $\gamma \in \Gamma$ of order $> 1/\varepsilon$ (including ∞).

The *ε-thick part* of $\mathcal{O}$ is the set $\mathcal{O}_{[\varepsilon,\infty)}$ of all points $x \in \mathcal{O}$ such that $d(\tilde{x}, \gamma\tilde{x}) \geq \varepsilon$ for all lifts $\tilde{x}$ of x to $\mathbf{H}^3$ and all elements $\gamma \in \Gamma$ of order $> 1/\varepsilon$.

Remark. With this definition, metric balls of unit radius with center in the ε-thick part have volume bounded below uniformly in ε. This is why we have made sure to include singular points with large local groups in the thin part.

Definition. A *Margulis tube* is a compact quotient of the r-neighborhood of a geodesic in $\mathbf{H}^3$ by an elementary hyperbolic group. Topologically, it can be a solid torus or a solid pillow, possibly with singular core.

A *cusp neighborhood* of rank 1 (resp. 2) is the quotient of a horoball in $\mathbf{H}^3$ by an elementary parabolic group of virtual rank 1 (resp. 2). Topologically, it is a product $[0, +\infty) \times F$, where F is a Euclidean 2-orbifold, called its *cross-section.* If the cusp has rank 2, then F is closed.

Here is an immediate consequence of the Margulis Lemma:

Corollary 6.3 (Structure of thin part). *Fix a hyperbolic 3-orbifold $\mathcal{O} = \mathbf{H}^3/\Gamma$ and a real number $\varepsilon \in (0, \mu_0]$. Then each component X of $\mathcal{O}_{(0,\varepsilon]}$ is either a Margulis tube or a cusp neighborhood.*

Furthermore, if $\mathcal{O}$ has finite volume, then $\mathcal{O}_{[\varepsilon,\infty)}$ is compact, there are no rank 1 cusps, and there are at most finitely many Margulis tubes and rank 2 cusps. As a consequence, one can choose ε small enough so that there are no Margulis tubes. Then $\mathcal{O}$ is homeomorphic to the interior of the compact orbifold $\mathcal{O}_{[\varepsilon,\infty)}$, whose boundary consists of closed Euclidean 2-orbifolds corresponding to cusps of $\mathcal{O}$.

A rank 2 cusp is called *nonrigid* if its cross-section is not a turnover. Then it is a torus or a pillow, and one can perform Dehn filling on it (cf. Chapter 2, Section 2.5). We will say more about this in Chapter 7.

Let $\mathcal{O}$ be a compact orbifold with hyperbolic interior. Let $F \subset \partial\mathcal{O}$ be a boundary component. Write U_F for the intersection of $\operatorname{Int}\mathcal{O}$ with a collar neighborhood of F in $\mathcal{O}$. Thus U_F is homeomorphic to $F \times [0, +\infty)$. When F is Euclidean, $\pi_1 F$ is parabolic and U_F is a rank 2 cusp neighborhood; in particular it has finite volume. By contrast, when F is hyperbolic, U_F has infinite volume. Thus we have:

Remark. The volume of the interior of $\mathcal{O}$ is finite iff each boundary component of $\mathcal{O}$ is Euclidean.

6.2.3 Selberg's Lemma

Selberg's lemma (Lemma 8 of [206], see also [24]) states that any finitely generated subgroup of $\mathrm{GL}_n(\mathbf{C})$ has a torsion free finite index subgroup. Since the isometry group of $\mathbf{H}^3$ is linear, we deduce:

Proposition 6.4. *Every hyperbolic orbifold with finitely generated fundamental group is very good.*

The proof of Selberg's lemma is algebraic and in general it is not easy to find an explicit manifold cover.

6.3 Existence and uniqueness of structures

The main existence result discussed here is Thurston's Hyperbolization Theorem for Haken orbifolds. Another such result is of course the Orbifold Theorem, which will be tackled in Chapter 9. Uniqueness in the finite volume case is a consequence of Mostow rigidity.

Again in this section all 2-orbifolds and 3-orbifolds are assumed to be connected and orientable unless mentioned otherwise.

6.3.1 Thurston's hyperbolization theorem

First we need a few definitions. Let Γ be a nonelementary Kleinian group and $\mathcal{O} := \mathbf{H}^3/\Gamma$ be the quotient orbifold. Consider the *convex hull* of $\Lambda(\Gamma)$, i.e. the union of geodesics in $\mathbf{H}^3$ whose endpoints are both in $\Lambda(\Gamma)$. Its image in $\mathcal{O}$ under the projection map is called the *convex core* of Γ. We say that Γ (or $\mathcal{O}$) is *geometrically finite* if the δ-neighborhood of the convex core has finite volume for some (hence in fact any) $\delta > 0$.

There are several other equivalent definitions of geometrically finite Kleinian groups, some of which can be extended to broader contexts (see e.g. [26, 27]).

A 3-orbifold $\mathcal{O}$ is called *homotopically atoroidal* if $\pi_1\mathcal{O}$ is not virtually abelian and if every subgroup of $\pi_1\mathcal{O}$ isomorphic to $\mathbf{Z}\oplus\mathbf{Z}$ is peripheral (i.e. conjugated to a subgroup of the fundamental group of a component of $\partial\mathcal{O}$).

Theorem 6.5 (Hyperbolization of Haken orbifolds). *Let $\mathcal{O}$ be a compact 3-orbifold. If $\mathcal{O}$ is Haken and homotopically atoroidal then the interior of $\mathcal{O}$ has a geometrically finite hyperbolic structure.*

Complete detailed proofs of this theorem in the manifold case can be found in [120] and [176, 177]; see [14] for an overview. The orbifold case is analogous to the manifold case, as explained in [18, Chap. 8]. Below we outline the main steps.

Recall that a Haken orbifold $\mathcal{O}$ has a hierarchy, i.e. can be built inductively, starting from a finite collection of discal 3-orbifolds and thick turnovers, and gluing at each step the pieces along incompressible 2-suborbifolds in their boundaries. Thus Theorem 6.5 is proved by induction.

The starting point is a theorem of Andreev [4, 5] which gives a combinatorial characterization of the convex polyhedra that can be embedded in $\mathbf{H}^3$ with finite volume and prescribed acute dihedral angles. (Only the right-angled case is relevant here.) The initial step of the induction is an equivariant version of this result, together with a version for products of surfaces with the interval. All this is explained in Thurston's notes [225, Chap. 13], [226, Chap. 5] using circle patterns; see also [120, Chap. 13]. Equivariance comes from the uniqueness of the hyperbolic structure.

The inductive step is reduced to a gluing theorem for hyperbolic structures, which is the heart of the proof. The reduction involves a mirror trick which enables one to consider only gluing along entire boundary components. Here is a statement of the gluing theorem. For simplicity, we give only the case without parabolic locus.

Theorem 6.6 (Final gluing). *Let $\mathcal{O}$ be a compact, possibly disconnected* 3-*orbifold with incompressible boundary whose interior has a geometrically finite hyperbolic structure. Let τ be an orientation-reversing involution of $\partial\mathcal{O}$ that exchanges the boundary components in pairs. Then the quotient orbifold $\mathcal{O}/\tau$ is hyperbolic if and only if it is homotopically atoroidal.*

The proof of Theorem 6.6 is different according to whether the 3-orbifold $\mathcal{O}$ is an **I**-bundle over a compact 2-orbifold or not.

In the first case, the quotient 3-orbifold $\mathcal{O}/\tau$ is fibered over a closed 1-orbifold, and thus finitely covered by a surface bundle over $\mathbf{S}^1$. The proof that $\mathcal{O}/\tau$ is hyperbolic can be found in [176] and [231] for the manifold case and in [18, Thm. 1] for the orbifold case.

When $\mathcal{O}$ is not an **I**-bundle, Thurston translates the gluing theorem into a fixed point theorem by using a result of Ahlfors and Bers [3, 8] and a combination theorem of Maskit [143, 144]; see also [120, Chap. 8 and 16], [177, Chap. 1 and 2]).

For a geometrically finite hyperbolic orbifold $\mathcal{O}$, the Ahlfors-Bers Theorem gives an identification between the space $\mathcal{GF}(\mathcal{O})$ of equivalence classes of geometrically finite hyperbolic structures on the interior of $\mathcal{O}$ and the Teichmüller space $\mathcal{T}(\partial\mathcal{O})$ of the boundary.

Thurston introduces the *skinning map* $\sigma : \mathcal{T}(\partial\mathcal{O}) \to \mathcal{T}(\overline{\partial\mathcal{O}})$, where $\overline{\partial\mathcal{O}}$ is $\partial\mathcal{O}$ with reversed orientation. Given a geometrically finite hyperbolic structure on the interior of $\mathcal{O}$, the covering space associated to each component $F \subset \partial\mathcal{O}$ provides a geometrically finite hyperbolic (i.e. a quasifuchsian) structure on the product $F \times \mathbf{I}$. The skinning map is defined by recording for each component $F \subset \partial\mathcal{O}$ the conformal structure on the new end which appears in this quasifuchsian covering.

The gluing involution τ induces an involution $\tau^\star\colon \mathcal{T}(\partial\mathcal{O}) \to \mathcal{T}(\overline{\partial\mathcal{O}})$. Then a fixed point for the map $\tau^\star \circ \sigma\colon \mathcal{T}(\partial\mathcal{O}) \to \mathcal{T}(\partial\mathcal{O})$ corresponds to a geometrically finite hyperbolic structure on the interior of $\mathcal{O}$ where the ends have compatible shape in the sense of Maskit's combination theorem, and thus it provides the required hyperbolic structure on the quotient $\mathcal{O}/\tau$.

Therefore Theorem 6.6 is a consequence of the following:

Theorem 6.7 (Fixed point). *Let $\mathcal{O}$ be a compact, possibly disconnected* 3-*orbifold with incompressible boundary, whose interior has a geometrically finite hyperbolic structure. Let τ be an orientation-reversing involution of $\partial\mathcal{O}$ that exchanges the boundary components in pairs. If $\mathcal{O}$ is not an* **I**-*bundle, then the map $\tau^\star\circ\sigma\colon \mathcal{T}(\partial\mathcal{O}) \to \mathcal{T}(\partial\mathcal{O})$ has a fixed point if and only if $\mathcal{O}/\tau$ is homotopically atoroidal.*

It is not difficult to check that the map $\tau^\star \circ \sigma$ decreases the Teichmüller distance strictly, and thus the fixed point, if it exists, is unique.

There are two different proofs of Theorem 6.7 in the manifold case: Thurston's and McMullen's. They both extend to the orbifold setting.

Thurston's approach uses a lot of geometric tools (e.g. measured geodesic laminations, pleated surfaces) which were introduced in Chapters 8 and 9 of his notes [225] to study ends of hyperbolic manifolds and convergence of sequences of representations of Kleinian groups in $\mathrm{PSL}_2(\mathbf{C})$ (see also [19, 37].) These results are used to find the fixed point of the map $\tau^\star \circ \sigma$ by iteration in the closure of an orbit $\{(\tau^\star \circ \sigma)^n(\rho); n \in \mathbf{Z}\} \subset \mathcal{GF}(\mathcal{O}) \cong \mathcal{T}(\partial\mathcal{O})$ of a geometrically finite hyperbolic structure $\rho \in \mathcal{GF}(\mathcal{O})$. In fact, it is sufficient to show that such an orbit is bounded in $\mathcal{GF}(\mathcal{O})$. Then Thurston's proof breaks into two steps:

- The orbit is bounded in the character variety $X(\mathcal{O})$ (see the definition in Chapter 7);
- Its closure in $X(\mathcal{O})$ is actually contained in $\mathcal{GF}(\mathcal{O})$.

In 1980, Thurston gave a precise overview of this proof in his lectures at the Bowdoin conference [227] (see also [165]). Then he wrote the proof of most of the first step in [230, 232]; using the theory of group actions on trees, J. Morgan and P. Shalen [167, 168, 169] gave an alternative and more algebraic proof of most of this step. A complete proof can be found in M. Kapovich's book [120], using Rips theory. The ingredients and most of the results needed for the second step were presented in Thurston's notes [225, Chap.8 and 9], and again a complete proof is given in [120].

In 1989, C. McMullen [151, 152, 153] gave a new and shorter proof of Theorem 6.7, based on a detailed analysis of the derivative and coderivative of the skinning map to obtain estimates on their norms. For example, if the 3-orbifold $\mathcal{O}$ is acylindrical, the quotient 3-orbifold $\mathcal{O}/\tau$ is always homotopically atoroidal; in this case, McMullen shows that the skinning map σ is strictly uniformly contracting. Since $\tau^\star$ is an isometry, the map $\tau^\star \circ \sigma$ is also strictly uniformly contracting and has a fixed point.

When the 3-orbifold $\mathcal{O}$ contains an essential annular suborbifold, some gluing involutions τ will produce homotopically non-atoroidal 3-orbifolds $\mathcal{O}/\tau$. Therefore one must take τ into account for the proof of Theorem 6.7, which is more involved in this case. For a detailed proof of Theorem 6.7 following McMullen's approach, we refer to J.-P. Otal [177] in the case of manifolds; the orbifold version is presented in [18, Chap. 8].

Unlike irreducibility and atoroidality, the Haken hypothesis in Theorem 6.5 is not necessary. For example, the Seifert-Weber dodecahedral space [244] is a small hyperbolic 3-manifold. In fact, Thurston's Hyperbolic Dehn Filling Theorem (cf. Theorem 8.4, and also [58]) provides infinitely many small hyperbolic 3-orbifolds obtained by Dehn filling on a finite volume hyperbolic 3-orbifold without essential 2-suborbifolds. This gives strong evidence for the following conjecture, which is a more precise version of a special case of th Geometrization Conjecture 3.26:

Conjecture 6.8 (Hyperbolization Conjecture). *Let $\mathcal{O}$ be a compact, irreducible 3-orbifold with infinite fundamental group. Then the interior of $\mathcal{O}$ has a geometrically finite hyperbolic structure if and only if $\mathcal{O}$ is homotopically atoroidal.*

It follows from Theorem 6.5 and the Orbifold Theorem (cf. Chap. 9) that this conjecture remains open only when $\mathcal{O}$ is a small 3-manifold.

6.3.2 Mostow rigidity

The following theorem is fundamental in the study of Kleinian groups with finite covolume:

Theorem 6.9 (Mostow rigidity). *Let Γ_1, Γ_2 be Kleinian groups such that $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ have finite volume. Every isomorphism between Γ_1 and Γ_2 is realized by conjugation in $\mathrm{Isom}^+(\mathbf{H}^3)$.*

The homotopic version of the theorem says that any homotopy equivalence between finite volume hyperbolic orbifolds is realized by an isometry. Notice that this isometry is unique in its homotopy class. In particular, it solves the so-called Borel conjecture for hyperbolic manifolds (cf. Section 4.4).

Theorem 6.9 was obtained by G. Mostow [170] for cocompact groups and extended by G. Prasad [188] to the finite covolume case. The proof relies on properties of quasi-conformal mappings of the ideal boundary of $\mathbf{H}^3$ (and can be extended to other symmetric spaces.)

For the case where the quotients are manifolds, there are now different and simpler proofs than the original one. The most geometric one is due to Gromov [94] (see also [13, Chap.C], [225, Chap.6]) and uses the notion of simplicial volume (cf. Section 9.5). The simplest one, which gives the isometry explicitly, is due to G. Besson, G. Courtois and S. Gallot [12]: it makes a more extensive use of differential geometry (e.g. Busemann functions).

Corollary 6.10. *If two finite volume hyperbolic* 3*-manifolds are homeomorphic, then they are isometric.*

The group of automorphisms of $\pi_1\mathcal{O}$ is denoted by $\mathrm{Aut}(\pi_1\mathcal{O})$, the group of inner automorphisms is denoted by $\mathrm{Inn}(\pi_1\mathcal{O})$. The quotient

$$\mathrm{Out}(\pi_1\mathcal{O}) = \mathrm{Aut}(\pi_1\mathcal{O})/\,\mathrm{Inn}(\pi_1\mathcal{O})$$

is the outer automorphism group.

Mostow rigidity gives a surjective homomorphism from $\mathrm{Isom}(\mathcal{O})$ to $\mathrm{Out}(\pi_1\mathcal{O})$. It is not hard to show that $\mathrm{Isom}(\mathcal{O})$ is finite (see [13]). Hence we get:

Corollary 6.11. *If $\mathcal{O}$ is a hyperbolic 3-orbifold of finite volume, then* $\mathrm{Out}(\pi_1\mathcal{O})$ *is finite.*

In fact, $\mathrm{Out}(\pi_1\mathcal{O})$ is isomorphic to $\mathrm{Isom}(\mathcal{O})$, since an isometry of $\mathcal{O}$ homotopic to the identity is the identity.

6.4 Hyperbolic groups, convergence groups and the Weak Hyperbolization Conjecture

Some essential results about cocompact Kleinian groups can be proved using only the large-scale properties of their actions. Hence it is reasonable to try to extend the theory to a broader class of groups having a similar behavior "at infinity". In this section we present two approaches. Following Gromov [95], we define the class of *hyperbolic spaces,* which are metric spaces satisfying a basic large-scale property of $\mathbf{H}^n$, namely the *thin triangle property.* Groups that act geometrically on these spaces are called *hyperbolic groups*. The second approach consists in looking at groups which act on a compact topological space so that a crucial property of the action of Kleinian groups on the ideal boundary of $\mathbf{H}^3$ is satisfied. These groups, called *convergence groups*, were introduced by Gehring and Martin [82].

The two notions turn out to be strongly related (cf. Theorem 6.16 below). Both points of view give useful insights and lead to a rich theory, whose exposition is well beyond the scope of this book. We will give a very brief introduction and focus on some rigidity aspects closely related to the hyperbolization conjecture.

6.4.1 Hyperbolic spaces and groups

First we define hyperbolic spaces in the sense of Gromov. Our definition is a bit more restrictive than the usual one, but sufficient for our purposes.

Let X be a metric space. A *geodesic segment* (resp. *geodesic ray*) in X is an isometric embedding of a segment (resp. half-line) into X. Thus geodesic segments are what is called minimizing geodesics in Riemannian geometry. The space X is *geodesic* if any two points can be connected by a geodesic. It is *proper* if closed metric balls in X are compact.

A *geodesic triangle* in X is a triple $(\alpha_1, \alpha_2, \alpha_3)$ of geodesic segments such that there exist points x_1, x_2, x_3 with α_i connecting x_{i+1} to x_{i+2} (with indices in $\mathbf{Z}/3$.) It is δ-*thin* if α_i lies in the δ-neighborhood of $\alpha_{i+1} \cup \alpha_{i+2}$.

Definition. A metric space is *hyperbolic* if it is geodesic, proper, and there is a constant $\delta \geq 0$ such that all geodesic triangles are δ-thin.

Recall from Chapter 1 that a group action is called geometric if it is isometric, proper, and cocompact.

Definition. A *hyperbolic group* is a group that acts geometrically on some hyperbolic space.

Example. It is well-known that any complete Riemannian manifold is geodesic and proper. This applies in particular to the eight geometries. Three of them are hyperbolic: $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^3$. The first two are to be regarded as trivial examples: any bounded space is hyperbolic, and so is any space quasi-isometric to $\mathbf{R}$. Thus the only interesting examples of hyperbolic groups that we get this way are the groups that act geometrically on $\mathbf{H}^3$, i.e. extensions of finite groups by cocompact Kleinian groups. More generally, this is true when $\mathbf{H}^3$ is replaced by $\mathbf{H}^n$.

Other important examples of hyperbolic groups are finitely generated free groups, which act geometrically on regular trees. (It is easy to check that trees are hyperbolic with $\delta = 0$.)

A key property of hyperbolicity is its quasi-isometry invariance. Refer to Section 1.4 for the definition if necessary.

Theorem 6.12. *Let A be a hyperbolic group or a hyperbolic space. Let B be a finitely generated group or a proper geodesic metric space. If B is quasi-isometric to A, then B is hyperbolic.*

The proof of Theorem 6.12 relies on the following fundamental property of hyperbolic spaces. A map $f : X_1 \to X_2$ between metric spaces is a (λ, C)-*quasi-isometric embedding* if it satisfies property (i) in the definition of a quasi-isometry (i.e. $\lambda^{-1}\, d_1(x, x') - C \leq d_2(f(x), f(x')) \leq \lambda\, d_1(x, x') + C$) but not necessarily property (ii). A (λ, C)-*quasigeodesic segment* in a space X is a quasi-isometric embedding of a segment in X.

Theorem 6.13 (Quasi-geodesic stability). *Let X be a hyperbolic space. Then for all λ, C there exists $D = D(\lambda, C) \geq 0$ such that any (λ, C)-quasigeodesic segment lies in the D-neighborhood of a geodesic segment with the same endpoints.*

6.4.2 Boundaries of hyperbolic groups and convergence groups

The intrinsic construction of the ideal boundary of $\mathbf{H}^3$ given earlier in this chapter carries over without changes to arbitrary hyperbolic spaces.

Definition. Let X be a hyperbolic space. Its *boundary* ∂X is the set of equivalence classes of geodesic rays in X, where $r_1, r_2 \in \mathcal{G}$ are equivalent if the function $t \mapsto d(r_1(t), r_2(t))$ is bounded.

The asymptotic behavior of two inequivalent rays in a hyperbolic space is qualitatively similar to that in $\mathbf{H}^n$: roughly speaking, they stay at a bounded distance for some time and then diverge exponentially. This allows to define a natural set of metrics on ∂X which induce the same topology. For precise definitions and motivation from trees, see [86]. Here we give an alternative description of the topology due to E. Swenson (cf. [73].)

Let r be a geodesic ray in X. For $C \geq 0$ define the "half-space" $H(r, C)$ to be the set of points x such that $d(x, r([C, +\infty))) > d(x, r([0, C]))$. Then a fundamental set of neighborhoods of $[r] \in \partial X$ consists of subsets $D(r,C) := \{[s], \liminf_{t\to\infty} d(s(t), X - H(r,C)) = +\infty\}$.

It is a pleasant exercise to check that this gives the correct topology for $X = \mathbf{H}^n$.

With this topology, ∂X is always compact. In fact, one can extend it to a topology on $X \cup \partial X$, which is also compact. Here it is essential to require that X is proper. Otherwise X might be bounded but noncompact; its boundary as defined above would then be empty, and there would be no hope to get a compactification in this way.

Using a version of Theorem 6.13 for rays, one sees that the boundary of a hyperbolic space is a quasi-isometry invariant. This permits to define the boundary of a hyperbolic group:

Definition. Let Γ be a hyperbolic group. Let X be a hyperbolic space on which Γ acts geometrically. Then Γ acts naturally on the set of geodesic rays in X, and therefore also on ∂X. The *boundary* $\partial\Gamma$ of Γ is by definition the boundary of X with its topology and Γ-action.

To see that $\partial\Gamma$ is well-defined, one uses the fact that any two proper geodesic spaces X, Y on which Γ acts geometrically are quasi-isometric, together with the quasi-isometry invariance of the boundary. As a consequence, we get:

Proposition 6.14. *Let A, B be hyperbolic spaces or groups. If A and B are quasi-isometric, then ∂A and ∂B are homeomorphic.*

A hyperbolic group whose boundary is finite is called *elementary*. In fact, one can show that the only elementary groups are finite groups and virtually infinite cyclic groups. Thus among cocompact groups of isometries of the eight geometries, the nonelementary hyperbolic ones are exactly those that act on $\mathbf{H}^3$. The boundary of such a group is $\mathbf{S}^2$. More generally, the boundary of a cocompact group of isometries of $\mathbf{H}^n$ is $\mathbf{S}^{n-1}$. The boundary of a finitely generated free group is a Cantor set.

In general, the boundary of a nonelementary hyperbolic group can be quite complicated, but it is always *perfect*, i.e. it has no isolated points.

6.4.3 Convergence groups

In [82], Gehring and Martin showed that some properties of Kleinian groups do not depend on the analytic structure of the sphere or the geometry of hyperbolic space, but rather on a topological condition on the dynamics of the action at infinity, which they called the *convergence property*. They studied groups of homeomorphisms of spheres satisfying this condition. Later the investigations were extended to arbitrary compact metrizable spaces by Tukia, Bowditch, Freden and others, motivated by the connections with hyperbolic groups.

Definition. Let M be an infinite compact metrizable topological space and Γ a group acting by homeomorphisms on M. Denote by ρ the associated representation of Γ in $\mathrm{Homeo}(M)$.

Then Γ is a called a *convergence group* if for each sequence $\{g_n\}$ of elements of Γ such that the $\rho(g_n)$'s are pairwise distinct, there exist points $a, b \in M$ and a subsequence $\{g_{n_k}\}$ such that $\lim \rho(g_{n_k})(x) = a$ uniformly on compact subsets that do not contain b.

Next we discuss an important characterization of convergence groups. Denote by $\Theta(M)$ the set of triples $(x, y, z) \in M \times M \times M$ such that x, y, z are pairwise distinct, topologized as a subset of $M \times M \times M$.

Proposition 6.15 ([30]). *Let M be an infinite compact metrizable topological space and Γ a group acting by homeomorphisms on M. Then Γ is a convergence group if and only if the induced action of Γ on $\Theta(M)$ is proper.*

The idea behind Proposition 6.15 is that if we think of M as some kind of boundary of a space and try to recover this space from M alone, then the space $\Theta(M)$ is a good device. For instance, take $M = \mathbf{S}^2$. Then we can construct a map $\phi : \Theta(\mathbf{S}^2) \to \mathbf{H}^3$ as follows: given $(x, y, z) \in \Theta(M)$, there is a unique geodesic ξ in $\mathbf{H}^3$ connecting x to y and a unique point $a \in \xi$ such that the ray starting at a and pointing towards z is orthogonal to ξ. Set $\phi(x, y, z) := a$. By 'naturality', ϕ is continuous. Conversely, if a is a point in $\mathbf{H}^3$, then the choice of an orthonormal frame at a gives a triple $(x, y, z) \in \Theta(\mathbf{S}^2)$ which belongs to $\phi^{-1}(a)$. Hence ϕ is surjective. It is not quite bijective, but the lack of injectivity lies in the set of orthonormal frames at a point, which is compact. In fact ϕ is proper, so $\Theta(\mathbf{S}^2)$ is as good as $\mathbf{H}^3$ as far as properness of group actions is concerned.

This suggests the following definition.

Definition. A convergence group Γ acting on M is *uniform* if the induced action on $\Theta(M)$ is cocompact.

Proposition 6.16 ([30, 73, 235]). *Let Γ be a nonelementary hyperbolic group. Then Γ acts as a uniform convergence group on $\partial\Gamma$.*

The converse is much more difficult.

Theorem 6.17 ([28]). *Let M be a perfect metrizable compact topological space. Let Γ be a uniform convergence group acting on M. Then Γ is hyperbolic, and the action is conjugate to the canonical action of Γ on its boundary.*

Remark. Another central concept is the JSJ decomposition for groups. There are various approaches, including one using convergence groups. The reader is referred to the original papers [29, 61, 62, 75, 98].

We can now state the important theorem of Tukia, Casson-Jungreis and Gabai mentioned in the previous chapter.

Theorem 6.18 ([234, 41, 77]). *A group acts as a uniform convergence group on $\mathbf{S}^1$ if and only if it is a cocompact Fuchsian group.*

Putting together Theorem 6.12, Proposition 6.14, and Proposition 6.16 we see that any group quasi-isometric to $\mathbf{H}^2$ acts as a uniform convergence group on $\mathbf{S}^1$. Hence we get:

Corollary 6.19. *A group is quasi-isometric to $\mathbf{H}^2$ if and only if it is a cocompact Fuchsian group.*

This corollary is used to complete the hyperbolic case in the proof of Theorem 5.20.

6.4.4 The Weak Hyperbolization Conjecture

The following is a weak version of the Hyperbolization Conjecture, whose conclusion is concerned with the fundamental group only.

Conjecture 6.20 (Weak Hyperbolization Conjecture). *Let M be a closed irreducible 3-manifold. If $\pi_1 M$ does not contain any $\mathbf{Z} \times \mathbf{Z}$, then $\pi_1 M$ is hyperbolic.*

Remark. In fact it would be equivalent to only assume that M is aspherical. Indeed, by Kneser's Theorem, any aspherical manifold M is a connected sum $N \# S_1 \# \cdots \# S_n$ where N is irreducible, $\pi_1 N \cong \pi_1 M$, and $S_1, \ldots, S_n$ are homotopy spheres.

We have the following implications : M is a closed hyperbolic 3-manifold $\implies$ $\pi_1 M$ is a cocompact Kleinian group $\implies$ $\pi_1 M$ is hyperbolic with boundary a 2-sphere $\implies$ $\pi_1 M$ is nonelementary hyperbolic. To deduce the full Hyperbolization Conjecture from Conjecture 6.20, it would be enough to prove the converse implications. Two of them are known to be true, and the third one is a major open problem.

Theorem 6.21 (Bestvina-Mess [10]). *Let M be a closed 3-manifold. If $\pi_1 M$ is a nonelementary hyperbolic group, then $\partial \pi_1 M$ is homeomorphic to $\mathbf{S}^2$.*

Theorem 6.22 (Gabai-Meyerhoff-N. Thurston [80]). *If M is a closed irreducible 3-manifold whose fundamental group is isomorphic to the fundamental group of a hyperbolic manifold, then M is hyperbolic.*

Conjecture 6.23 (Cannon). *Let Γ be a hyperbolic group with boundary homeomorphic to $\mathbf{S}^2$. Then Γ is a cocompact Kleinian group.*

Below are short discussions of these conjectures.

About the Weak Hyperbolization Conjecture When M is Haken, this follows of course from Thurston's Hyperbolization Theorem. However let us mention that Bestvina and Feighn [11] have given a direct proof in the case of fiber bundles over $\mathbf{S}^1$, as a corollary of a general combination theorem for hyperbolic groups. This has been extended to cover the non-fibered case by Swarup [215]. More generally, Gabai and Kazez [81] have proven the WHC for manifolds that contain a genuine essential lamination

About Cannon's Conjecture It has been proved under additional hypotheses. We already mentioned Cannon and Cooper's result [38] that if Γ is quasi-isometric to $\mathbf{H}^3$, then Γ is a cocompact Kleinian group. There is also some progress by Cannon and Swenson [39], and more recently by Bonk and Kleiner [23].

Chapter 7

Varieties of representations

In this chapter we introduce varieties of representations of 3-orbifolds. We present the Culler-Shalen theory of essential surfaces associated to ideal points via actions of the fundamental group on trees, in order to complete the proof of the existence theorem for hierarchies in Haken 3-orbifolds. In the next chapter, we use varieties of representations to sketch the proof of (an extension to orbifolds of) Thurston's hyperbolic Dehn filling theorem, due to Dunbar and Meyerhoff [58]. This is an important result in 3-dimensional topology; among other things it is an ingredient in the proof of the Orbifold Theorem.

The *variety of representations* of an orbifold is the set of all representations of the fundamental group in $\mathrm{PSL}_2(\mathbf{C})$, and has a natural structure of algebraic variety. Two representations are considered equivalent if they are conjugate. However, the space of conjugacy classes of representations is not Hausdorff. Instead of this space, one studies the *variety of characters,* which is an algebraic variety whose points are generically conjugacy classes of representations.

The variety of characters has many applications in low-dimensional geometry and topology. It can be used to analyze deformations of geometric structures, as in the proofs of the Hyperbolic Dehn Filling Theorem [224, 58] and of the Orbifold Theorem. It plays a crucial role in the proof of the Smith conjecture [166] and the study of exceptional surgeries on knots [49, 31], including the Cyclic Surgery Theorem [48].

Finally, we also mention that the variety of representations has been used to find obstructions for a group to be the fundamental group of a Kähler manifold [209, 88] or a smooth complex algebraic manifold [125].

As usual, all orbifolds are orientable.

7.1 Preliminaries

7.1.1 Varieties of representations and characters

Our aim is to associate to a compact 3-orbifold $\mathcal{O}$ two algebraic sets $R(\mathcal{O})$ and $X(\mathcal{O})$ containing information about representations of $\pi_1\mathcal{O}$ in $\mathrm{PSL}_2(\mathbf{C})$. Our basic reference is [31]. For the relevant background on algebraic geometry and invariant theory, the reader may consult [171] and [211] respectively. We will also work with representations of finitely generated groups that are not fundamental groups of orbifolds, so we develop the theory in this more general setting.

Definition. Let Γ be a finitely generated group. The set $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbf{C}))$ of all representations of Γ in $\mathrm{PSL}_2(\mathbf{C})$ is called the *variety of representations of* Γ and denoted by $R(\Gamma)$. When Γ is the fundamental group of a compact orbifold $\mathcal{O}$, we shall abbreviate $R(\pi_1\mathcal{O})$ to $R(\mathcal{O})$.

Next we show how to put an affine algebraic structure on $R(\Gamma)$. Let $\gamma_1, \ldots, \gamma_n$ be a generating system of Γ. Recall the isomorphism $\mathrm{PSL}_2(\mathbf{C}) \cong \mathrm{SO}_3(\mathbf{C})$ given by the adjoint action of $\mathrm{PSL}_2(\mathbf{C})$ on the Lie algebra $\mathrm{sl}_2(\mathbf{C}) \cong \mathbf{C}^3$. In what follows, we use this isomorphism to identify $\mathrm{PSL}_2(\mathbf{C})$ with a closed algebraic subset of $\mathbf{C}^9$ (identified with the space of 3×3-matrices of complex numbers.) We get the following embedding:

$$\begin{array}{rcl} R(\Gamma) & \hookrightarrow & \mathrm{PSL}_2(\mathbf{C}) \times \cdots \times \mathrm{PSL}_2(\mathbf{C}) \subset \mathbf{C}^{9n} \\ \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_n)). \end{array}$$

The image of $R(\Gamma)$ is a closed algebraic subset whose polynomial equations are induced by the relations of the presentation of Γ. This gives $R(\Gamma)$ the structure of a possibly reducible affine algebraic set.

Remark. This structure does not depend on the choice of the generating set.

This remark can be shown by adding or deleting a redundant generator and showing that this yields an isomorphic algebraic set. Alternatively, it follows form the results in [136].

The group $\mathrm{PSL}_2(\mathbf{C})$ acts on $R(\Gamma)$ by conjugation according to the formula $(A \cdot \rho)(\gamma) := A\rho(\gamma)A^{-1}$. Two representations are *conjugate* (or *equivalent*) if they are in the same orbit. We are interested in representations modulo this equivalence relation, so it would be natural to study the orbit space, i.e. the set-theoretic quotient $R(\Gamma)/\mathrm{PSL}_2(\mathbf{C})$. However, this space does not have a natural algebro-geometric structure. One way to see this is to notice that it does not need to be Hausdorff for the quotient topology because $\mathrm{PSL}_2(\mathbf{C})$ is not compact.

Here is a simple example. Let Γ be an infinite cyclic group. Then $R(\Gamma)$ can be identified with $\mathrm{PSL}_2(\mathbf{C})$. The terms of the sequence $A_n := \pm \begin{pmatrix} 1 & 1/n^2 \\ 0 & 1 \end{pmatrix}$ all lie in the same orbit (A_n is conjugate to A_1 by $\pm \begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix}$), but the limit is $\pm I$, which is not in the same orbit. Hence the orbit of A_1 is not closed and the quotient is not Hausdorff.

So instead, we shall work with the quotient of $R(\Gamma)$ by $\mathrm{PSL}_2(\mathbf{C})$ in the category of affine algebraic sets. For this we consider the algebra of invariant functions $\mathbf{C}[R(\Gamma)]^{\mathrm{PSL}_2(\mathbf{C})}$, i.e. algebraic functions on $R(\Gamma)$ which are invariant by conjugation. By invariant theory (see e.g. [211]), this algebra is finitely generated. Therefore there exists an affine algebraic set $X(\Gamma)$ whose function algebra is isomorphic to $\mathbf{C}[R(\Gamma)]^{\mathrm{PSL}_2(\mathbf{C})}$. In addition, the inclusion $\mathbf{C}[X(\Gamma)] \subset \mathbf{C}[R(\Gamma)]$ induces a surjective morphism $t\colon R(\Gamma) \to X(\Gamma)$.

Definition. The algebraic set $X(\Gamma)$ is called the *variety of characters* of Γ.

$X(\Gamma)$ has the properties that a "quotient in the category of affine algebraic sets" should have. For instance, if A is any affine algebraic set, then any morphism $f : R(\Gamma) \to A$ invariant by conjugation descends to a unique morphism $X(\Gamma) \to A$. It follows that the construction is functorial in the sense that if Γ' is another finitely generated group, then any group homomorphism $\phi : \Gamma \to \Gamma'$ induces a morphism of algebraic sets $\phi^* : X(\Gamma') \to X(\Gamma)$. Moreover, if ϕ is surjective, then ϕ^* is injective. We will apply this for instance to the epimorphism $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}}) \to \pi_1\mathcal{O}$ for a 3-orbifold $\mathcal{O}$ and identify $X(\mathcal{O})$ with a subset of $X(\mathcal{O} - \Sigma_{\mathcal{O}})$.

Notice that the word "component" used in connection with $R(\Gamma)$ or $X(\Gamma)$ refers to irreducible components of an algebraic set, which may not coincide with the topological (i.e. connected) components.

Given a representation $\rho \in R(\Gamma)$, its *character* is the map:

$$\begin{array}{rcl} \chi_\rho : \Gamma & \to & \mathbf{C} \\ \gamma & \mapsto & \mathrm{trace}(\rho(\gamma))^2 \end{array}$$

Notice that $\mathrm{trace}(\rho(\gamma))$ is defined up to sign, so $\mathrm{trace}(\rho(\gamma))^2$ is well-defined.

In Corollary 7.2 below, we construct a natural bijection between $X(\Gamma)$ and the set of characters of Γ, which justifies the name of $X(\Gamma)$. This bijection maps $t(\rho)$ to χ_ρ, for every $\rho \in R(\Gamma)$.

Given $\gamma \in \Gamma$, the map $\rho \mapsto \mathrm{trace}(\rho(\gamma))^2$ is an algebraic function invariant by conjugation. This leads us to the following definition.

Definition. Given $\gamma \in \Gamma$, we define the regular function:

$$\begin{array}{rcl} J_\gamma \colon X(\Gamma) & \to & \mathbf{C} \\ t(\rho) & \mapsto & \chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))^2. \end{array}$$

Proposition 7.1. *The algebra $\mathbf{C}[R(\Gamma)]^{\mathrm{PSL}_2(\mathbf{C})}$ is finitely generated by the functions $\{J_\gamma\}_{\gamma \in \Gamma}$.*

See [107] for a proof of this proposition. The main consequences are:

Corollary 7.2. *There is a natural bijection between $X(\Gamma)$ and the set of characters of Γ.*

Proof. Consider the mapping that sends $t(\rho) \in X(\Gamma)$ to the character χ_ρ. This is a well-defined surjection from $X(\Gamma)$ to the set of characters of Γ, because the projection $t \colon R(\Gamma) \to X(\Gamma)$ is surjective. Proposition 7.1 implies that it is injective. □

Choosing a finite set of generators $J_{\gamma_1}, \ldots, J_{\gamma_N}$, we have:

Corollary 7.3. *There exist $\gamma_1, \ldots, \gamma_N \in \Gamma$ such that $(J_{\gamma_1}, \ldots, J_{\gamma_N}) : X(\Gamma) \to \mathbf{C}^N$ is an embedding.*

Remark. There are other natural functions to be considered. For instance, if $[\gamma_1, \gamma_2] = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$ denotes the commutator of $\gamma_1, \gamma_2 \in \Gamma$, then the trace

$$\mathrm{trace}(\rho([\gamma_1, \gamma_2])) = \mathrm{trace}([\rho(\gamma_1), \rho(\gamma_2)])$$

is well-defined for $\rho \in R(\Gamma)$, without sign ambiguity, and therefore it induces a function on $X(\Gamma)$. By Proposition 7.1, this function is a polynomial in the functions J_γ.

In the sequel, a special role will be played by a particular kind of representations which are called *irreducible*.

Definition. One can see elements of $\mathrm{PSL}_2(\mathbf{C})$ as acting on the set of lines in $\mathbf{C}^2$. A representation $\rho \colon \pi_1 \mathcal{O} \to \mathrm{PSL}_2(\mathbf{C})$ is *reducible* if all elements of the image of ρ have a common invariant line. This is equivalent to saying that ρ is equivalent to a representation by upper triangular matrices. A representation that is not reducible is called *irreducible*. The set of all irreducible representations in $R(\mathcal{O})$ is denoted by $R(\mathcal{O})^{irr}$.

Remark. Let $\mathcal{O}$ be a hyperbolic 3-orbifold. There is an isomorphism of $\pi_1\mathcal{O}$ onto a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ called the *holonomy representation* (cf. Section 8.2). This representation is irreducible unless $\pi_1\mathcal{O}$ is elementary. To show this, notice that the action of $\mathrm{PSL}_2(\mathbf{C})$ on $\mathbf{P}^1\mathbf{C}$ is equivalent to the action on the sphere at infinity $\partial_\infty\mathbf{H}^3$. Hence the image of a reducible representation fixes a point in $\partial_\infty\mathbf{H}^3$ and it is an elementary group.

Remark. More precisely, one can show that a representation in $\mathrm{PSL}_2(\mathbf{C})$ has elementary image if and only if either it is reducible or it fixes a point of $\mathbf{H}^3$ (i.e. it is elliptic).

Here is a useful irreducibility criterion.

Lemma 7.4. *A representation $\rho \in R(\Gamma)$ is reducible iff* $\mathrm{trace}(\rho(\gamma)) = 2$ *for every γ in the commutator subgroup of Γ. (Remember that traces of commutators are well-defined.)*

The same proof as in [49, Lemma 1.2.1] applies here. As a direct consequence, we get:

Corollary 7.5. *The image $t(R(\Gamma)^{irr})$ is a Zariski open subset of $X(\Gamma)$.*

The image $t(R(\Gamma)^{irr})$ is denoted by $X(\Gamma)^{irr}$. Its elements are called *irreducible* characters. Note that the word "irreducible" in this chapter can have four (!) different meanings, according to whether it refers to a representation, a character, an algebraic set, or a 3-orbifold. No confusion should result from this.

Lemma 7.6. *Two irreducible representations in $R(\Gamma)$ are conjugate iff they have the same character. Moreover, the* $\mathrm{PSL}_2(\mathbf{C})$*-action on $R(\Gamma)^{irr}$ has finite stabilizers.*

Proposition 7.7. *The projection $t\colon R(\Gamma) \to X(\Gamma)$ induces a bijection between the components R_0 of $R(\Gamma)$ that have irreducible representations and components $X_0 = t(R_0)$ of $X(\Gamma)$ that have irreducible characters.*

Lemma 7.6 and Proposition 7.7 are proved in [107].

Corollary 7.8. *Let R_0 be a component of $R(\Gamma)$ that contains an irreducible representation and $X_0 = t(R_0)$ the corresponding component. Then*

$$\dim R_0 = \dim X_0 + 3.$$

Proof. By [171, Cor. 3.5], $\dim R_0 - \dim X_0$ equals the dimension of the generic fibers of the projection $t\colon R_0 \to X_0$. By Lemma 7.6, these fibers have dimension $\dim \mathrm{PSL}_2(\mathbf{C}) = 3$, because the stabilizer of the action by conjugation is finite. □

7.1.2 Examples

We start with the turnover $S^2(n_1, n_2, n_3)$. It is proved in [91] that the number of conjugacy classes of $R(S^2(n_1, n_2, n_3))$ is finite. Thus:

Proposition 7.9. *$X(S^2(n_1, n_2, n_3))$ is a finite set.*

Before the next example, we need the basic identity:

$$\operatorname{trace} AB + \operatorname{trace} A^{-1}B = \operatorname{trace} A \operatorname{trace} B \tag{7.1}$$

$\forall A, B \in \mathrm{SL}_2(\mathbf{C})$. This identity can be deduced from $A^2 - (\operatorname{trace} A)A + Id = 0$ (i.e. the Cayley-Hamilton identity) by multiplying by $A^{-1}B$ and taking traces.

Example. We compute $X(\mathbf{Z})$. For any $n \in \mathbf{Z}$, J_n is a polynomial in J_1 (using (7.1)). Hence $\mathbf{C}[X(\mathbf{Z})]$ is a polynomial algebra freely generated by J_1, and $X(\mathbf{Z}) \cong \mathbf{C}$ is the complex line parametrized by J_1.

Notice that $R(\mathbf{Z}) \cong \mathrm{PSL}_2(\mathbf{C})$ has dimension three. This does not contradict Corollary 7.8, because $R^{irr}(\mathbf{Z}) = \emptyset$.

Example. Our next example is the free group F_n of rank $n \geq 2$. We have an obvious isomorphism:

$$R(F_n) \cong \mathrm{PSL}_2(\mathbf{C}) \times \overset{(n)}{\cdots} \times \mathrm{PSL}_2(\mathbf{C})$$

Thus, $X(F_n)$ has only one irreducible component. By Corollary 7.8,

$$\dim X(F_n) = 3n - 3.$$

However an explicit computation of $X(F_n)$ would be more involved.

Example. Let F be a compact 2-orbifold with c cone points and with underlying space a surface of genus g and b boundary components. We shall estimate the dimension of some components of $X(F)$.

First consider the case $b = c = 0$ and $g > 1$. The group $\pi_1 F$ admits a presentation with $2g$ generators and one relation. Hence $R(F)$ is an algebraic subset of $\mathrm{PSL}_2(\mathbf{C})^{2g}$ (with the obvious embedding by generators) and the relation induces 3 equations, because $\dim \mathrm{PSL}_2(\mathbf{C}) = 3$. Hence $\dim R(F) \geq 6g - 3$ and $\dim X_j(F) \geq 6g - 6$, for each component $X_j(F)$ that contains irreducible characters.

Assume now that $b > 0$ or $c > 0$. We use the surface S obtained by removing discs centered at the cone points of F. To compute $R(F)$ from $R(S)$, we consider the curves $\mu_i \in \pi_1 F$ around cone points that are boundary curves of S. A representation $\rho \in R(S)$ induces a representation in $R(F)$ if and only if for each curve μ_i, $\rho(\mu_i)^{n_i} = \pm\text{id}$, where n_i is the order of the corresponding cone point.

We deal first with the case where $\rho(\mu_i)$ is non-trivial, i.e. a rotation of angle $2\pi m/n_i$ for some integer $0 < m < n_i$. This condition is given by the equation:

$$J_{\mu_i}(\chi_\rho) = 4\cos^2(\pi m/n_i).$$

Thus, some components $X_j(F)$ of $X(F)$ are obtained from $X(S)$ by c algebraic equations (one for each cone point). Now $\pi_1 S$ is a free group of rank $2g-1+b+c$ and therefore $\dim X(S) = 3(2g-2+b+c)$. Hence

$$\dim X_j(F) \geq 6g - 6 + 3b + 2c.$$

There are other components that correspond to representations that map some μ_i to $\pm\text{id}$, i.e. that factor through another orbifold with fewer cone points. Those components can have lower dimension.

Example. Let $\mathcal{O}$ be the exterior of the figure eight knot in $\mathbf{S}^3$. We take the presentation $\pi_1\mathcal{O} = \langle \alpha, \beta, \mu \mid \mu\alpha\mu^{-1} = \alpha\beta, \mu\beta\mu^{-1} = \beta\alpha\beta \rangle$, coming from the fibration of $\mathcal{O}$ over $\mathbf{S}^1$ with fiber a punctured torus. Here α and β generate the group of the fiber and μ is a meridian that commutes with the longitude $\alpha\beta\alpha^{-1}\beta^{-1}$.

It is easy to write α as a word in α, β, μ such that each generator appears with even total exponent. This means that there is no sign indeterminacy on $\text{trace}(\rho(\alpha))$, and it induces a function in $\mathbf{C}[X(\mathcal{O})]$. Write x for this function and let $y = J_\mu$. In [107], it is proved that $\mathbf{C}[X(\mathcal{O})] \cong \mathbf{C}[x, y]/p(x, y)$, where

$$p(x, y) = (2 - x)((1 - x)y + x^2 + x - 1).$$

Thus $X(\mathcal{O})$ is a plane curve with two components. One of them, the straight line $x = 2$, corresponds to abelian representations. The other component is the most interesting one. We will see how it is related to deformations of hyperbolic structures on $\mathcal{O}$ and Dehn fillings.

Let $\mathcal{O}(n)$ denote the orbifold with underlying space $\mathbf{S}^3$, singular locus the figure eight knot, and cyclic local group of order n. We can compute $X(\mathcal{O}(n))$ from $X(\mathcal{O})$ by writing

$$y = 4\cos^2(\pi m/n)$$

for some integer $0 < m < n$. In particular $X(\mathcal{O}(n))$ is finite.

7.1.3 Dimension and smoothness of $X(\mathcal{O})$

We will need the following proposition, which is due to Thurston [225, Thm 5.6].

Proposition 7.10. *Let M be a compact 3-manifold. Let $\rho_0 \in R(M)^{irr}$ be an irreducible representation such that for each torus component T of ∂M, $\rho_0(\pi_1(T)) \neq \{\pm \mathrm{id}\}$. Then every irreducible component of $X(M)$ that contains the character of ρ_0 has dimension $\geq s - \frac{3}{2}\chi(\partial M)$, where s is the number of torus components of ∂M.*

The proof for representations in $\mathrm{SL}_2(\mathbf{C})$ can be found in [49, Prop. 3.2.1] and is easily adapted to our case.

When $\mathcal{O}$ is a compact 3-orbifold, $\chi(|\partial \mathcal{O}|)$ denotes the Euler characteristic of the underlying surface of $\partial \mathcal{O}$. The singular set of the boundary $\partial \mathcal{O}$ is finite and its cardinality is denoted by $|\Sigma_{\partial \mathcal{O}}|$. Recall that a *meridian* is a finite order element $\mu \in \pi_1 \mathcal{O}$ represented by a loop that bounds a small disc meeting an edge or a circle of $\Sigma_{\mathcal{O}}$ in exactly one point.

Corollary 7.11. *Let $\mathcal{O}$ be a compact 3-orbifold. Let $\rho_0 \in R(\mathcal{O})^{irr}$ be an irreducible representation such that:*

i. for each nonsingular torus component T of $\partial \mathcal{O}$, $\rho_0(\pi_1(T)) \neq \pm \mathrm{id}$,

ii. for each meridian μ, $\rho_0(\mu) \neq \pm \mathrm{id}$.

Let X_0 be an irreducible component of $X(\mathcal{O})$ that contains $t(\rho_0)$. Then

$$\dim X_0 \geq s - \frac{3}{2}\chi(|\partial \mathcal{O}|) + |\Sigma_{\partial \mathcal{O}}|,$$

where s is the number of nonsingular torus components of $\partial \mathcal{O}$.

Proof. We set $M = \mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})$, where $\mathcal{N}(\Sigma_{\mathcal{O}})$ is an open tubular neighborhood of $\Sigma_{\mathcal{O}}$. Since $\pi_1 M$ surjects onto $\pi_1 \mathcal{O}$, we have an inclusion $X(\mathcal{O}) \subset X(M)$. Let Y_0 be the component of $X(M)$ that contains X_0. By Proposition 7.10, $\dim Y_0 \geq s + c - \frac{3}{2}\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})))$, where c is the number of circles of $\Sigma_{\mathcal{O}}$.

Let e be the number of edges of $\Sigma_{\mathcal{O}}$, so that $X(\mathcal{O}) \cap Y_0$ is the zero set of $c + e$ equations in Y_0. Since X_0 is an irreducible component of $X(\mathcal{O}) \cap Y_0$, we have

$$\dim X_0 \geq \dim Y_0 - c - e \geq s + c - \frac{3}{2}\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}}))) - c - e.$$

We get the corollary from this inequality and the formulae

$$\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}}))) = \chi(|\partial \mathcal{O}|) - v - |\Sigma_{\partial \mathcal{O}}| \quad \text{and} \quad 2e = 3v + |\Sigma_{\partial \mathcal{O}}|,$$

where v denotes the number of vertices of $\Sigma_{\mathcal{O}}$. $\square$

Remark. The contribution of a component of $\partial\mathcal{O}$ in the formula of Corollary 7.11 is positive if and only if it is different from a sphere with at most three branching points, i.e. different from a nonsingular sphere, a spindle, a football, a teardrop or a turnover (for the turnover the contribution is zero, and for the other ones it is negative). It follows that the right hand side is positive if (i) each boundary component of $\mathcal{O}$ has nonpositive Euler characteristic, and (ii) at least one of them is not a turnover.

Note that each pillow and each nonsingular torus in $\partial\mathcal{O}$ counts for one in the lower bound of $\dim X_0$. In fact, we will see that when $\operatorname{Int}\mathcal{O}$ carries a hyperbolic metric of finite volume and ρ_0 is the holonomy representation of this structure (cf. Section 8.2), then the bound is sharp, i.e. $\dim X_0$ equals the number of boundary components that are not turnovers.

Now assume that $\operatorname{Int}(\mathcal{O}-\Sigma_{\mathcal{O}})$ has a geometrically finite hyperbolic structure whose cusp ends correspond to the circles and edges of $\Sigma_{\mathcal{O}}$ and to the tori of $\partial\mathcal{O}$. We have seen that the holonomy representation ρ_0 is irreducible. Define $\chi_0 := t(\rho_0)$. Let X_1 be an irreducible component of the set

$$\{\chi \in X(\mathcal{O}-\Sigma_{\mathcal{O}}) \mid \chi(\mu) = \chi_0(\mu) \text{ for each meridian } \mu\}$$

that contains χ_0. Let Γ be the product of the fundamental groups of the components of $\partial\mathcal{O}-\Sigma_{\partial\mathcal{O}}$. The natural group homomorphism $\Gamma \to \pi_1(\mathcal{O}-\Sigma_{\mathcal{O}})$ induces a morphism $r\colon X_1 \to X(\Gamma)$.

Corollary 7.12. *Assume that $\mathcal{O}$ has at least one boundary component that is not a turnover. Then the image of r has positive dimension.*

Proof. According to [120, Lemma 9.37], χ_0 is a smooth point of X_1 and the tangent map at χ_0

$$r_*\colon T_{\chi_0}X_1 \to T_{r(\chi_0)}X(\partial\mathcal{O}-\Sigma_{\partial\mathcal{O}})$$

is injective. In addition, Corollary 7.11 applies to X_1 (with the same proof), thus we get:

$$\dim(\operatorname{Im}(r)) = \dim X_1 \geq s - \frac{3}{2}\chi(|\partial\mathcal{O}|) + |\Sigma_{\partial\mathcal{O}}|$$

and we have just remarked that this is positive if $\mathcal{O}$ has at least one boundary component different from a turnover. □

7.2 Ideal points and essential surfaces

7.2.1 Ideal Points

Let $\mathcal{C} \subset \mathbf{C}^n$ be an *affine curve*, i.e. a closed algebraic subset of dimension 1. We can associate to $\mathcal{C}$ a *projective completion* $\hat{\mathcal{C}}$ and its *smooth projective model* $\overline{\mathcal{C}}$ (see for instance [171]). There is a canonical embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ and a canonical birational map $\overline{\mathcal{C}} \to \hat{\mathcal{C}}$, which is generically one-to-one. Points of $\overline{\mathcal{C}}$ mapped to $\mathcal{C}$ are called *ordinary*, and the other points of $\overline{\mathcal{C}}$ are called *ideal*. Every regular function $\mathcal{C} \to \mathbf{C}$ induces a rational function $\overline{\mathcal{C}} \to \mathbf{CP}^1$.

Let Γ be a finitely generated group. Recall that we defined for each $\gamma \in \Gamma$ a regular function J_γ on $X(\Gamma)$. Whenever $\mathcal{C}$ is a curve in $X(\Gamma)$, we shall still denote by J_γ the induced function $\overline{\mathcal{C}} \to \mathbf{CP}^1$.

Let us now fix some notation for this whole section: let $\mathcal{O}$ be a compact, irreducible 3-orbifold; set $M := \mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})$, where $\mathcal{N}(\Sigma_{\mathcal{O}})$ is an open tubular neighborhood of $\Sigma_{\mathcal{O}}$.

We will be interested in ideals points of curves $\mathcal{C}$ in $X(M)$ such that for each meridian μ, J_μ evaluated at this ideal point is bounded. It will be of special relevance to know whether some function J_γ takes a finite value or the value ∞ at ideal points of $\mathcal{C}$. Note that for a given ideal point x, there is at least one element $\gamma \in \pi_1 M$ such that $J_\gamma(x) = \infty$, by Corollary 7.3.

Theorem 7.13. *Let $\mathcal{O}, M$ be as above. Let $\mathcal{C}$ be a curve in $X(M)$ and x an ideal point of $\mathcal{C}$ such that for each meridian μ, $|J_\mu(x)| < \infty$. Then there exists an essential 2-suborbifold $F \subset \mathcal{O}$ with the following property: for every component Q of $\mathcal{O} - F$ and every element $\gamma \in \operatorname{Im}(\pi_1(Q - \Sigma_Q) \to \pi_1 M)$, the function J_γ takes a finite value at x.*

Corollary 7.14. *Let $\mathcal{O}$ be a compact irreducible 3-orbifold such that the interior of the manifold $\mathcal{O} - \Sigma_{\mathcal{O}}$ has a geometrically finite hyperbolic structure and at least one component of $\partial\mathcal{O}$ is not a turnover. Then $\mathcal{O}$ contains an essential 2-suborbifold with non-empty boundary.*

Proof of the corollary. As in the previous section, we consider the character χ_0 of the holonomy of the hyperbolic structure on $\mathcal{O} - \Sigma_{\mathcal{O}}$ and an irreducible component X_1 of the set

$$\{\chi \in X(\mathcal{O} - \Sigma_{\mathcal{O}}) \mid \chi(\mu) = \chi_0(\mu) \text{ for each meridian } \mu\}$$

that contains χ_0. Let Γ be the product of the fundamental groups of the components of $\partial\mathcal{O} - \Sigma_{\partial\mathcal{O}}$ and $r : X_1 \to X(\Gamma)$ the natural morphism. Images of elements of Γ in $\pi_1(\mathcal{O} - \Sigma_{\mathcal{O}})$ are called *peripheral*.

By Corollary 7.12, we can choose a curve $\mathcal{C}$ in the image of r and a curve $\mathcal{C}' \subset r^{-1}\mathcal{C} \subset X_1$. Pick an ideal point x of $\mathcal{C}'$ induced by an ideal point of $\mathcal{C} \subset X(\Gamma)$, and consider a suborbifold $F \subset \mathcal{O}$ given by Theorem 7.13. Since x is induced by an ideal point of $\mathcal{C}$, there exists a peripheral element $\gamma \in \pi_1(\mathcal{O} - \Sigma_{\mathcal{O}})$ such that J_γ is infinite at x. Hence $\partial F = F \cap \partial\mathcal{O} \neq \emptyset$. □

Theorem 7.13 follows from the next two propositions.

Recall that a group acts on a tree *without edge inversions* if whenever an edge e is fixed by some element γ, then γ also fixes the endpoints of e.

Proposition 7.15. *Let $\mathcal{O}, M, \mathcal{C}, x$ be as in Theorem 7.13. Then $\pi_1 M$ acts on a simplicial tree T without edge inversions so that an element $\gamma \in \pi_1 M$ stabilizes a vertex if and only if $J_\gamma(x) \in \mathbf{C}$. In particular, each meridian stabilizes a vertex.*

We have remarked earlier that there is at least one element $\gamma \in \pi_1 M$ such that $J_\gamma(x) = \infty$. Hence the above action is *nontrivial*, i.e. admits no global fixed point. This is crucial to ensure that the next construction gives a nonempty essential suborbifold.

Proposition 7.16. *Assume that $\pi_1 M$ acts nontrivially on a simplicial tree T without edge inversions. If each meridian stabilizes a vertex, then there exists an essential 2-suborbifold $F \subset \mathcal{O}$ such that for every component Q of $\mathcal{O} - F$, $\pi_1(Q - \Sigma_Q)$ stabilizes a vertex.*

The proofs of both propositions are sketched in the next two paragraphs. For details (except for the part concerning orbifolds), we recommend the original paper of Culler and Shalen [49], as well as the survey [208].

7.2.2 From ideal points to actions on trees

Sketch of proof of Proposition 7.15. First of all, we lift $\mathcal{C} \subset X(M)$ to a curve $\widetilde{\mathcal{C}} \subset R(M)$. We choose an ideal point $\tilde{x}$ of $\widetilde{\mathcal{C}}$ that projects to x. We notice that $J_\gamma(x) \in \mathbf{C}$ if and only if $(J_\gamma \circ t)(\tilde{x}) \in \mathbf{C}$.

We consider the function field $\mathbf{C}(\mathcal{C})$. To the ideal point x there corresponds a *discrete valuation*

$$\nu_x : \mathbf{C}(\mathcal{C}) \to \mathbf{Z}$$

that measures the order of the zeros of the function at x (counted negatively for poles). In particular,

$$J_\gamma(x) \in \mathbf{C} \text{ if and only if } \nu_x(J_\gamma) \geq 0.$$

The field $\mathbf{C}(\widetilde{\mathcal{C}})$ is an extension of $\mathbf{C}(\mathcal{C})$ and the valuation $\nu_{\tilde{x}}$ is an extension of a positive multiple of ν_x [49, 208]. We shall not work with the field $\mathbf{C}(\widetilde{\mathcal{C}})$, but with a finite extension F of it, to be described later. A positive multiple of $\nu_{\tilde{x}}$ extends to a valuation $\nu\colon F \to \mathbf{Z}$ [49, 208] .

The idea is to take the tree T as the Bruhat-Tits building for GL_2 of the field F with the discrete valuation ν. The construction of the tree is explained in the book of Serre [204] as well as in [53], where it is proved that $\mathrm{PSL}_2(F)$ acts on T with the following properties:

(1) the action is without edge inversions,

(2) the stabilizers of vertices are conjugate to $\mathrm{PSL}_2(R)$, where R is the ring of integers associated to ν (i.e. $R = \{a \in F \mid \nu(a) \geq 0\}$).

The action of $\pi_1 M$ on T is induced by the action of $\mathrm{PSL}_2(F)$ and a representation $\mathcal{P}\colon \pi_1 M \to \mathrm{PSL}_2(F)$, defined as follows. For every $\gamma \in \pi_1 M$ and every representation $\rho \in \widetilde{\mathcal{C}}$,

$$\rho(\gamma) = \pm \begin{pmatrix} a_\gamma(\rho) & b_\gamma(\rho) \\ c_\gamma(\rho) & d_\gamma(\rho) \end{pmatrix}.$$

The coefficients $a_\gamma, b_\gamma, c_\gamma, d_\gamma$ are not well-defined functions on ρ. However, monomials of degree two (a_γ^2, $a_\gamma b_\gamma$ and so on) are well-defined. So we take F to be the finite extension of $\mathbf{C}(\tilde{\mathcal{C}})$ that contains the coefficient functions $a_\gamma, b_\gamma, c_\gamma, d_\gamma$ for γ in a generating system of $\pi_1 M$ and define the *tautological representation*

$$\mathcal{P}(\gamma) = \pm \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

Lemma 7.17. *For an element $\gamma \in \pi_1 M = \pi_1(\mathcal{O} - \Sigma_{\mathcal{O}})$, γ stabilizes a vertex if and only if $\nu(J_\gamma) \geq 0$.*

Proof. Since the stabilizers of vertices are conjugate to $\mathrm{PSL}_2(R)$, one has to prove that $\nu(J_\gamma) \geq 0$ if and only if $\mathcal{P}(\gamma)$ is conjugate to a matrix in $\mathrm{PSL}_2(R)$. One implication is easy, because $\mathrm{trace}(\mathcal{P}(\gamma))^2 = J_\gamma$. To show the converse, just write $\mathcal{P}(\gamma)$ in the so-called normal form: if $\mathcal{P}(\gamma)$ is not $\pm\mathrm{id}$, then choose a basis $\{v_1, v_2\}$ of $\mathbf{C}(\mathcal{C})^2$ such that $\mathcal{P}(\gamma)(v_1) = v_2$. Changing to the basis $\{v_1, v_2\}$ corresponds to conjugating $\mathcal{P}(\gamma)$ to

$$\pm \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}.$$

By looking at the determinant and the trace, we get $b = \pm 1$ and $d^2 = J_\gamma$. Therefore, if $v_{\tilde{x}}(J_\gamma) \geq 0$, then $\mathcal{P}(\gamma)$ is conjugate to a matrix in $\mathrm{PSL}_2(R)$. ☐

This finishes the sketch of the proof of Proposition 7.15.

Remark. This construction can be generalized: instead of considering a curve in a representation space, one can work with an algebraic subset of any dimension, for instance the representation space itself. In this case, the valuations involved may have rank > 1 and/or be nondiscrete, and the associated objects are affine buildings [181].

7.2.3 From actions on trees to essential suborbifolds

The following proposition can be found in [49]. Recall that in the situation of Proposition 7.16, the fundamental group of $M = \mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})$ acts non-trivially on T without edge inversions, so that each meridian fixes a vertex.

Proposition 7.18. *Let $\mathcal{O}$ and M be as in Proposition 7.16. In particular assume $\pi_1 M$ acts nontrivially on a simplicial tree T without edge inversions. Then M contains an essential surface $S \subset M$ such that for each meridian $\mu \subset \partial M$, $\mu \cap S = \emptyset$, and for every component Q of $M - S$, $\mathrm{im}(\pi_1 Q \to \pi_1 M)$ is contained in a vertex stabilizer.*

Remark. Notice that the conclusion of Proposition 7.18 holds for the manifold $M = \mathcal{O} - \mathcal{N}(\Sigma_{\mathcal{O}})$, but not for the orbifold $\mathcal{O}$. This is a step in the proof of Proposition 7.16, which gives a suborbifold in $\mathcal{O}$.

Sketch of proof. The first step consists in building a $\pi_1 M$-equivariant map $f : \widetilde{M} \to T$. We may assume that f is simplicial with respect to some fixed $\pi_1 M$-invariant triangulation, and that for every lift $\tilde{\mu}$ of a meridian, $f(\tilde{\mu})$ is contained in a vertex.

The set of midpoints of edges is denoted by E. Since f is simplicial and equivariant, $f^{-1}(E)$ is a $\pi_1 M$-invariant bicollared surface in $\widetilde{M}$. Since the action on T is without inversions, the projection F of this surface to M is orientable. Moreover, F does not intersect any meridian $\mu \subset \partial M$.

To complete the proof, one makes F incompressible by repeated surgeries on f using the Loop Theorem. This surgery process is classical in 3-manifold topology and goes back to Stallings [212]. The fact that the resulting surface is *nonempty* (and not boundary-parallel) comes from the nontriviality of the action of $\pi_1 M$ on T and from the fact that at each step, the surface is *dual* to this action. For details, see [208]. □

Proof of Proposition 7.16. Let S be the surface of Proposition 7.18. Since $\mu \cap S = \emptyset$, every component of ∂S either belongs to $\partial \mathcal{O}$ or is parallel to some meridian. This follows from the fact that the meridians cut $\partial M - \partial \mathcal{O}$ into a family of pairs of pants and annuli. Hence we glue to S the corresponding meridian disks (with a cone point) to obtain a properly embedded 2-suborbifold $F_0 \subset \mathcal{O}$, with $F_0 \cap M = S$.

The suborbifold F_0 can be compressible, and we must apply a new surgery process. Let D be a compressing disk for F_0. Since S is essential in M, D has a singular cone point. The compression surgery on F_0 along D induces a surgery on $S \subset M$ along the annulus $D \cap M$. The map f can be easily modified so that the new surface resulting from surgery on S is again the inverse image of a middle point of an edge.

To show that this process stops, we use a notion of complexity of the 2-suborbifold F_0 as described in [106], based on the Euler characteristic of the components of F_0 (the Euler characteristic of S does not change under this surgery). Before defining the complexity, we remark that the Euler characteristic of every 2-suborbifold of $\mathcal{O}$, while not necessarily an integer, lies in $\frac{1}{m}\mathbf{Z}$, where m is the smallest common multiple of the orders of the local groups of $\mathcal{O}$. We define the complexity as:

$$(\ldots, n_{\frac{-1}{m}}, n_0, n_{\frac{1}{m}}, n_{\frac{2}{m}}, \ldots, n_2)$$

where n_r is the number of components of F_0 with Euler characteristic $r \in \frac{1}{m}\mathbf{Z}$, and we order complexities lexicographically. Notice that 2 is an upper bound for the Euler characteristic r. It is clear that this surgery process decreases the complexity of F_0 .

A similar surgery process allows to make F_0 ∂-incompressible. We can also eliminate spherical and boundary-parallel components of F_0. Since the action on the tree is not trivial, we end up with a non-empty suborbifold $F \subset \mathcal{O}$ with the required properties. □

Remark. This construction has several useful generalizations. If $\pi_1 M$ is replaced by a finitely presented group Γ, then M can be replaced by a presentation complex K. Inverse images of midpoints of edges by a general position equivariant map $\tilde{K} \to T$ are then embedded 1-complexes called *tracks* (cf. [53]). This can be further generalized to settings where the tree T is not simplicial. Then tracks are replaced by laminations and the process to put them in normal forms is called *Rips' machine.* For those results and some of their applications, including compactness results for representation varieties, see [9, 182] or [120, Chap.12].

Chapter 8

Volumes and hyperbolic Dehn filling

Section 8.1 deals with topics connected to volumes of hyperbolic 3-orbifolds, including the extension to orbifolds of the Thurston-Jørgensen theory. In 8.2 we extend our definition of hyperbolic orbifolds to include incomplete structures; we introduce the notions of developing map and holonomy, which will be used in 8.3, where we prove the hyperbolic Dehn filling for orbifolds.

All orbifolds considered in this chapter are orientable.

8.1 The set of volumes of hyperbolic 3-orbifolds

All orbifolds considered in this section have **finite volume**.

Let $\mathcal{V} \subset [0, +\infty)$ be the set of all volumes of hyperbolic 3-orbifolds. We would like to know how $\mathcal{V}$ looks like a subset of the real line with its canonical order and topology. For manifolds, there is a beautiful theorem due to Jørgensen and Thurston [225, 92]. We will present its generalization to orbifolds by Dunbar and Meyerhoff [58]. While this result is important in itself and leads to fascinating and difficult open questions, we mainly use it as an opportunity to discuss geometric convergence, compactness theorems and hyperbolic Dehn surgery, which will be used in the next chapter.

By Corollary 6.3, every (finite volume) hyperbolic 3-orbifold is homeomorphic to the interior of a compact 3-orbifold. By triangulating, one sees that there are countably many compact 3-orbifolds up to homeomorphism. Hence the set $\mathcal{H}$ of hyperbolic 3-orbifolds modulo homeomorphism is countable. By

Mostow rigidity, $\mathcal{H}$ is also the set of hyperbolic 3-orbifolds modulo isometry. Thus we can consider the volume map vol : $\mathcal{H} \to \mathbf{R}$. Our set $\mathcal{V}$ is the image of this map, so it is countable. However, it is not clear whether it is closed or not, or whether there are isolated points or accumulation points.

To study $\mathcal{V}$, we introduce a topology on $\mathcal{H}$.

Definition. We say that $\mathcal{O}_n$ converges to $\mathcal{O}_\infty$ in the *geometric topology* if $\forall \varepsilon > 0$, the ε-thick part of $\mathcal{O}_\infty$ is $(1+\delta_n)$-bi-Lipschitz homeomorphic to the ε-thick part of $\mathcal{O}_n$, with $\delta_n \to 0$, provided that n is large enough.

To start off, we state two facts which enable us to recover the topology of $\mathcal{V}$ from that of $\mathcal{H}$.

Fact 1 (Continuity) If $\lim \mathcal{O}_n = \mathcal{O}_\infty$, then $\lim \operatorname{vol}(\mathcal{O}_n) = \operatorname{vol}(\mathcal{O}_\infty)$.

This fact is not hard to prove with our definition of convergence. All we have to show is that the volume of the ε-thin part of $\mathcal{O}_n$ and $\mathcal{O}_\infty$ goes to zero as $\varepsilon \to 0$, uniformly on n. This is a consequence of the following lemma, which is an elementary computation in hyperbolic geometry:

Lemma 8.1. *Given a Margulis tube V and a cusp neighborhood N such that ∂V and ∂N are isometric Euclidean 2-orbifolds. Then* $\operatorname{vol} N \geq \operatorname{vol} V$.

Notice that we cannot find a universal upper bound for the volume of an ε-thin neighborhood independently of the orbifold. For instance the boundary torus of an ε-thin cusp neighborhood can have arbitrarily large diameter, hence arbitrarily large volume. Thus we need Lemma 8.1 in order to bound the volume of the ε-thin part of $\mathcal{O}_n$ in terms of ε-thin part of the *fixed* orbifold $\mathcal{O}_\infty$, which goes to zero as $\varepsilon \to 0$.

Fact 2 (Properness) For each $v > 0$, the sets $\operatorname{vol}^{-1}(\{v\})$ and $\operatorname{vol}^{-1}([0, v])$ are compact.

Sketch of proof. One has to show that every sequence $\mathcal{O}_n$ of hyperbolic 3-orbifolds in $\operatorname{vol}^{-1}([0, v])$ has a convergent subsequence. We use Hausdorff-Gromov convergence and the Gromov pre-compactness criterion (see the discussion in Section 9.2). By Margulis' Theorem, the μ_0-thick part of $\mathcal{O}_n$ is nonempty and we choose a thick point $x_n \in (\mathcal{O}_n)_{[\mu_0,\infty)}$. Now we consider the sequence of pointed metric spaces $(\mathcal{O}_n, x_n)$. By Gromov's pre-compactness criterion, this sequence subconverges to a length space for the Hausdorff-Gromov topology. To prove that the limit is an orbifold and that the convergence is geometric, we need to prove the following assertion: for every $R > 0$ there exists $\varepsilon = \varepsilon(R) > 0$ such that $B_R(x_n) \subset (\mathcal{O}_n)_{[\varepsilon,\infty)}$. This assertion follows from volume estimates: first $\mathrm{vol}(B_{\mu_0}(x_n))$ is uniformly bounded below, because $x_n \in (\mathcal{O}_n)_{[\mu_0,\infty)}$; second, if $y \in (\mathcal{O}_n)_{(0,\varepsilon]}$, then $\mathrm{vol}(B_R(y))$ is bounded above by $C\varepsilon$ for some uniform constant $C > 0$. □

To analyze $\mathrm{vol}^{-1}([0, v])$, we need to analyze accumulation points of $\mathcal{H}$. We first give an informal discussion.

Suppose an orbifold $\mathcal{O}_\infty$ is the limit of a sequence of distinct orbifolds $\mathcal{O}_n$. Then for large n, $\mathcal{O}_n$ and $\mathcal{O}_\infty$ have homeomorphic thick parts. Hence the only way they can differ topologically is that some Margulis tubes become cusps in the limit or vice versa, that is $\mathcal{O}_n$ and $\mathcal{O}_\infty$ are related by **Dehn filling**. Recall that this can happen only if the cusps involved are *nonrigid*, i.e. the cross-section is not a turnover.

Intuitively, it is conceivable that a sequence of closed orbifolds $\mathcal{O}_n$ could develop a thinner and thinner Margulis tube that becomes a cusp in the limit. This is called a *cusp opening*. Topologically, the $\mathcal{O}_n$'s should be obtained from $\mathcal{O}_1$ by removing a solid torus or solid pillow and gluing back another one with different surgery coefficients. The convergence of the sequence to a one-cusped orbifold should correspond to a "blowing-up" of the surgery coefficients.

Fact 3 (Cusp opening) If $\lim \mathcal{O}_n = \mathcal{O}_\infty$ and the $\mathcal{O}_n$'s are pairwise distinct, then for large n, $\mathcal{O}_n$ is obtained by Dehn filling on $\mathcal{O}_\infty$. In particular, $\mathcal{O}_\infty$ has more nonrigid cusps than $\mathcal{O}_n$.

Furthermore, this is relevant to the order structure on $\mathcal{V}$ by

Fact 4 (Limit volume bigger) In the previous situation, $\mathrm{vol}\,\mathcal{O}_n < \mathrm{vol}\,\mathcal{O}_\infty$.

This fact is proved in [225, Chap.6], as an extension of the proof of Mostow rigidity. For large values of n, this estimate can also be obtained from hyperbolic Dehn filling and Schläfli's formula (see [162] and Lemma 9.18 in Section 9.4 of Chapter 9).

Corollary 8.2. $\mathrm{vol}^{-1}(v)$ *is finite.*

At this point, it is not clear whether $\mathcal{H}$ has accumulation points at all. To prove their existence, we need two more facts.

Fact 5 For any integer $p \geq 0$, there exists an orbifold in $\mathcal{H}$ with exactly p nonrigid cusps.

This is a consequence of the Hyperbolization Theorem for Haken orbifolds (Thm. 6.5).

Fact 6 If $\mathcal{O} \in \mathcal{H}$ has p nonrigid cusps, then for any $0 \leq k \leq p$ there is a sequence of orbifolds with k nonrigid cusps converging to $\mathcal{O}$.

This follows from the Hyperbolic Dehn Filling Theorem (Theorem 8.4), which will be discussed later in this chapter. We now come to the main theorem of this section:

Theorem 8.3 (Structure of the volume set, [225], [58]). *The set $\mathcal{V}$ of volumes of all hyperbolic 3-orbifolds is a closed, well-ordered subset of $\mathbf{R}$ of order type ω^ω. Furthermore, there are at most finitely many orbifolds with the same volume.*

This theorem easily follows from Facts 1–6.

Theorem 8.3 raises lots of extremely interesting questions, such as: what is the hyperbolic 3-orbifold (resp. 3-manifold) of least volume with n nonrigid cusps (where $n = 0, 1, \ldots$). These questions can be asked for orientable manifolds/orbifolds (as in this book) or allow nonorientable objects. (The theory goes through without major difficulties.)

For $n = 0$ (i.e. in the closed case) the hyperbolic manifold of minimal volume is conjectured to be the Fomenko-Matveev-Weeks manifold. This manifold was found independently by Fomenko and Matveev in [72], by using spines of manifolds, and by J. Weeks, using the program SnapPea.

For $n = 1$, the smallest volume cusped hyperbolic 3-orbifold and 3-manifold were determined respectively by R. Meyerhoff [147] and C. Adams [1]. Those examples happen to be non-orientable, and C. Cao and R. Meyerhoff [40] proved that the figure eight knot exterior and its sister are the orientable cusped manifolds of least volume.

For the study of volumes of arithmetic hyperbolic manifolds we refer to Borel's article [25].

8.2 Complete vs incomplete hyperbolic structures

In the proof of the Hyperbolic Dehn Filling Theorem, we will need to consider 3-orbifolds that carry *incomplete* hyperbolic metrics. They will be studied through their holonomy representations and developing maps, which we define in this section.

Let $\mathcal{O}$ be an n-orbifold. A *hyperbolic structure* on $\mathcal{O}$ is a set of Riemannian metrics of constant sectional curvature -1 on a covering of $\mathcal{O}$ by uniformizing charts such that the group actions on these charts are isometric and transition maps are isometries. This allows to measure lengths of paths in $\mathcal{O}$, hence induces a distance function. A hyperbolic structure is *complete* if the induced metric space is complete (i.e. Cauchy sequences converge.)

Examples of complete hyperbolic structures on orbifolds are obtained by taking polyhedra in $\mathbf{H}^n$ and gluing their faces pairwise, respecting certain geometric conditions (see [190]). For instance, one can obtain a closed hyperbolic surface F of genus 2 from a right-angled octahedron H in $\mathbf{H}^2$. The natural map $H \to F$ lifts to an isometric embedding in the universal covering $\tilde{F}$ (with the pullback metric). In fact, one can construct a $\pi_1 F$-invariant tiling of $\tilde{F}$ by copies of H, and it follows that $\tilde{F}$ is isometric to $\mathbf{H}^2$.

This leads to the more general notion of *developing map* (cf. [225, 145, 43]): let $\mathcal{O}$ be an n-orbifold with a possibly incomplete hyperbolic structure. Let $\phi_0 : \tilde{U}_0 \to U_0$ be a chart for the hyperbolic structure and $*$ be a regular point of U_0. We are going to construct a map D from the universal cover $\tilde{\mathcal{O}}$ to $\mathbf{H}^n$. Recall from Chapter 2 that $\tilde{\mathcal{O}}$ can be defined as the set of homotopy classes of paths in $\mathcal{O}$ with initial point $*$. We begin by fixing a lift $\tilde{*}$ of $*$ to $\tilde{U}_0$ and an isometric embedding $f_0 : \tilde{U}_0 \to \mathbf{H}^n$.

Let α be a path in $\mathcal{O}$ with initial point $*$. Cover the image of α by a chain of open subsets $U_0, \ldots, U_n$ which are images of charts $\phi_i : \tilde{U}_i \to U_i$ for the hyperbolic structure. There is a unique way of choosing isometric embeddings $f_i : \tilde{U}_i \to \mathbf{H}^n$ for $i = 1, \ldots, n$ such that the path α can be locally lifted to the charts and then sent continuously to $\mathbf{H}^n$, starting at $f_0(\tilde{*})$. The endpoint of the image path in $\mathbf{H}^n$ depends only on the homotopy class of α; hence this defines a map $D : \tilde{\mathcal{O}} \to \mathbf{H}^n$.

The map D is well-defined up to postcomposition with an isometry of $\mathbf{H}^n$. It is called the *developing map* of the hyperbolic structure on M. There exists a representation $\rho : \pi_1\mathcal{O} \to \mathrm{Isom}(\mathbf{H}^n)$ such that for each $x \in \tilde{\mathcal{O}}$ and each $\gamma \in \pi_1\mathcal{O}$ we have $D(\gamma \cdot x) = \rho(\gamma)(D(x))$. This representation is unique up to equivalence and called the *holonomy representation*.

If the structure is complete, then D is a global isometry and ρ is discrete and faithful. Hence we can identify $\tilde{\mathcal{O}}$ with $\mathbf{H}^n$, $\pi_1\mathcal{O}$ with $\rho(\pi_1\mathcal{O})$, and $\mathcal{O}$ with $\mathbf{H}^n/\rho(\pi_1\mathcal{O})$. This shows that studying complete hyperbolic structures on 3-orbifolds is equivalent to studying Kleinian groups.

In general, D is a local isometry, which shows that $\mathcal{O}$ is good, but it doesn't need to be one-to-one nor onto, as shown in the following example.

Example. Let $\mathcal{O}$ be an open metric ball in $\mathbf{H}^3$ minus its center. The developing map of the natural (incomplete) hyperbolic structure on $\mathcal{O}$ is again a ball minus its center, isometric to the initial one. In particular we can recover the completion of $\mathcal{O}$ from the image of the developing map.

In our second example, $\mathcal{O}$ is a Margulis tube minus its core. Then the image of the developing map is a tubular neighborhood of a geodesic minus the geodesic itself, and again we can recover the completion of $\mathcal{O}$ from the image of the developing map. Notice that now D is not one-to-one.

In the next section, we will need to extend these notions to affine structures. They come up only as a technical tool, so the rest of this section can be skipped on first reading.

An *affine structure* on an orbifold is defined similarly as a set of compatible invariant affine structures on domains of charts. The definitions of the developing map and the holonomy are the same. The key point to ensure well-definedness of the developing map is that two affine maps $f_1 : U_1 \to \mathbf{R}^n$ and $f_2 : U_2 \to \mathbf{R}^n$ that coincide on an open subset of $U_1 \cap U_2$ coincide on all of $U_1 \cap U_2$. There is no metric associated to an affine structure, hence we use the developing map to *define* completeness.

Definition. An affine structure on an orbifold is *complete* if its developing map is a covering map.

Example. Consider an affine structure on the 2-torus whose holonomy sends a basis of $\mathbf{Z}^2$ to two similarities with a unique common fixed point. If we view the universal covering of the torus as the Euclidean plane, and the action of $\mathbf{Z}^2$ as an action by translations, then the developing map is viewed as the complex exponential (the fixed point of the holonomy being $0 \in \mathbf{C}$), cf. [224, Ex. 3.3.4].

8.3 Hyperbolic Dehn filling for orbifolds

8.3.1 The Hyperbolic Dehn Filling Theorem

Let $\mathcal{O}$ be a compact 3-orbifold such that $\mathrm{Int}(\mathcal{O})$ has a complete hyperbolic structure with finite volume. We saw in Chapter 6 that $\partial\mathcal{O}$ is a union of Euclidean 2-orbifolds. In this section we explore the connections between three sets of ideas:

Geometric convergence (cf. Section 8.1). The picture to keep in mind is that of *cusp openings:* $\mathcal{O}$ is approximated in the geometric topology by sequences of orbifolds with thinner and thinner Margulis tubes turning into cusps.

Dehn filling (cf. Section 2.5). Recall that it is a topological operation consisting in gluing one or more solid tori (resp. solid pillows) to some torus (resp. pillow) components of $\partial\mathcal{O}$.

Deformation of holonomy. We denote by $\rho_0 \in R(\mathcal{O})$ the holonomy of a finite volume hyperbolic structure and set $\chi_0 := t(\rho_0) \in X(\mathcal{O})$. Mostow rigidity implies that once an orientation on $\mathcal{O}$ has been chosen, ρ_0 is unique up to equivalence, hence χ_0 is unique. The local structure of $X(\mathcal{O})$ carries information about deformations of the hyperbolic structure.

We have already explained a connection between geometric convergence and Dehn filling: sequences of orbifolds converging to $\mathcal{O}$ are topologically obtained by Dehn filling on $\mathcal{O}$. This raises an obvious question:

Question. Which Dehn fillings on $\mathcal{O}$ are hyperbolic? In particular, is there a sequence of hyperbolic fillings converging to $\mathcal{O}$?

Thurston's Hyperbolic Dehn Filling Theorem deals with this question. Let us briefly explain the connection with the character variety. If one removes standard neighborhoods of the cusps obtained by quotienting horoballs and tries to glue in Margulis tubes, one obtains a singular hyperbolic metric. To get a smooth metric, one needs to deform the metric on the thick part in order to make it fit the tubes. Hence it is of interest to understand deformations of hyperbolic structures, and this can be studied through deformations of their holonomies in the character variety. In other words, Mostow rigidity tells us that ρ_0 is the only discrete faithful representation of $\pi_1\mathcal{O}$ into $\mathrm{PSL}_2(\mathbf{C})$ up to equivalence. Hence nearby representations will be either nonfaithful or nondiscrete. The hope is that some suitably chosen discrete nonfaithful ones will factor through faithful representations of fundamental groups of some Dehn fillings on $\mathcal{O}$ which are holonomies of hyperbolic structures on them.

To state the theorem, we need to fix some notation first. Let s, p, t be the number of components of $\partial\mathcal{O}$ that are respectively tori, pillows, and turnovers. For each torus T_j in $\partial\mathcal{O}$, we fix two generators m_j and l_j of $\pi_1 T_j$, that are represented by two simple loops in T_j. For each pillow P_j in $\partial\mathcal{O}$, we also fix two generators m_j and l_j of the torsion free subgroup of $\pi_1 P_j$, that represent two simple closed curves in P_j (each curve bounds a disc with two cone points). Then to an $(s+p)$-tuple $((p_1, q_1), \ldots (p_{s+p}, q_{s+p})) \in (\mathbf{Z}^2 \cup \{\infty\})^{s+p}$ we associate an orbifold obtained by (p_j, q_j)-Dehn filling on the j-th boundary component, defined as follows: if $(p_j, q_j) = \infty$, we just remove the j-th boundary component; otherwise we glue something to the j-th component of $\partial\mathcal{O}$. If it is a torus, then we write $p = rd$ and $q = sd$, with $r, s \in \mathbf{Z}$ coprime and $d = \text{g.c.d.}(p, q)$, and glue the singular solid torus $S^1 \times D^2(|d|)$, where $D^2(|d|)$ denotes the quotient of the 2-disk by a rotation of order $|d|$. The surgery meridian is the curve $rm + sl$. For pillows we glue a solid pillow, possibly with a singular core, in a similar way (cf. Section 2.5).

Theorem 8.4 (Thurston's Hyperbolic Dehn Filling). *Let $\mathcal{O}$ be a compact 3-orbifold with boundary such that $\text{Int}(\mathcal{O})$ has a complete hyperbolic structure with finite volume. Then there exists a neighborhood $\mathcal{U}$ of $(\infty, \ldots, \infty)$ in $(\mathbf{Z}^2 \cup \{\infty\})^{s+p}$ such that for every $x \in \mathcal{U}$, the Dehn filling associated to x is hyperbolic.*

The topology on $\mathbf{Z}^2 \cup \{\infty\}$ is given by Alexandrov compactification. Hence elements of a neighborhood $\mathcal{U}$ need to have sufficiently large coefficients for *all* cusps.

In the manifold case, the proof is given in Thurston's notes [225, Chap.5], and it has been generalized to orbifolds by Dunbar and Meyerhoff [58]; see also [18, Appendix B].

Remark. To establish the connection with convergence of orbifolds, there remains to prove that sequences of hyperbolic Dehn fillings on $\mathcal{O}$ with coefficients going to $(\infty, \ldots, \infty)$ do converge to the finite volume hyperbolic structure on $\mathcal{O}$. This involves connecting algebraic convergence and geometric convergence and we will not do this here. See [58].

We turn to a more detailed outline of the proof. The first thing we need is a local study of $X(\mathcal{O})$ near χ_0. We already have a lower bound of the dimension. Recall that a cusp is called nonrigid when the corresponding component of $\partial\mathcal{O}$ is not a turnover. Since χ_0 is irreducible, Corollary 7.11 implies that every component of $X(\mathcal{O})$ that contains χ_0 has dimension at least $s + p$. As soon as there is a nonrigid cusp, this dimension is positive. By contrast, we have:

Proposition 8.5. *If $s = p = 0$ then χ_0 is an isolated point.*

Proof. Let $\rho \in R(\mathcal{O})$ be a representation in a neighborhood of ρ_0. We may choose the neighborhood sufficiently small so that ρ is still the holonomy representation of a structure on $\mathcal{O}$ (see [37]). If $\mathcal{O}$ is compact, then this structure is complete, and by Mostow rigidity the hyperbolic structure on $\mathcal{O}$ must be the same. Therefore ρ is conjugate to ρ_0. If $\mathcal{O}$ is not compact, by hypothesis the ends of $\mathcal{O}$ are parabolic cusps with horospherical section a turnover T. By Proposition 7.9, the restriction to the turnover $\rho|_{\pi_1 T}$ is conjugate to $\rho_0|_{\pi_1 T}$. In particular, the new structure on $\mathcal{O}$ with holonomy ρ is complete and therefore

Mostow rigidity applies. Thus ρ is again conjugate to ρ_0. We have proved that the orbit by conjugation of ρ_0 is isolated; therefore χ_0 is also isolated. $\square$

In the course of the proof of the Hyperbolic Dehn Filling Theorem, we will generalize this: χ_0 is always a smooth point of $X(\mathcal{O})$ and the dimension of the unique component of $X(\mathcal{O})$ through χ_0 is *exactly* $s+p$.

The first step of the proof consists in producing a branched covering from an open subset W of $\mathbf{C}^{s+p}$ to a neighborhood V of χ_0 in $X(\mathcal{O})$, using the maps J_γ defined earlier. This branched covering maps the origin to χ_0.

In the second step, we construct a homeomorphism from W to a neighborhood U of $(\infty,\dots,\infty)$ in $(\mathbf{R}^2 \cup \{\infty\})^{s+p}$ that maps the origin to $(\infty,\dots,\infty)$. This gives a correspondence between points near $(\infty,\dots,\infty)$ in generalized Dehn filling space and characters near χ_0. Two points near $(\infty,\dots,\infty)$ correspond to the same character if and only if they differ by changes of sign of some coordinates.

In the third step, we prove that any point of V corresponding to an element of $\mathcal{U} := U \cap (\mathbf{Z}^2)^{s+p}$ is the character of the holonomy of an incomplete structure on $\mathrm{Int}(\mathcal{O})$ whose completion is precisely the orbifold with the corresponding Dehn filling coefficients. This is achieved by deforming the developing map of ρ_0.

Each step of the proof is discussed in one of the next subsections.

Remark. We thus establish a correspondence between certain points of $X(\mathcal{O})$ near χ_0 and hyperbolic Dehn fillings. This correspondence can be enlarged to give geometric interpretations of other points of $X(\mathcal{O})$. This uses cone manifolds and will be discussed in the next chapter.

8.3.2 Algebraic deformation of holonomies

Theorem 8.6. *The map*

$$J_m = (J_{m_1}, \dots, J_{m_{s+p}}) : X(\mathcal{O}) \to \mathbf{C}^{s+p}$$

is locally bianalytic at χ_0.

Sketch of proof. We follow the proofs of [225] and [249, 250]. Let X_0 be an irreducible component of $X(\mathcal{O})$ that contains χ_0. As we saw in subsection 8.3.1, $\dim X_0 \geq s+p$. By Mostow Rigidity, χ_0 is an isolated point of $J_m^{-1}(J_m(\chi_0))$. Using those properties, the openness principle and other standard results in complex algebraic geometry [171], one shows that J_m is open at χ_0, and that $\dim X_0 \leq s+p$. It follows that J_m is locally either bianalytic or a branched cover.

The latter possibility is eliminated by using Mostow rigidity on the hyperbolic orbifolds obtained by Dehn filling (see [249, 250, 18] for details). $\square$

8.3.3 Generalized Dehn filling coefficients

Recall that an orientation-preserving isometry of $\mathbf{H}^3$ has a *complex length*, which is well-defined up to sign and addition of an integer multiple of $2i\pi$. We use this to parametrize a deformation of the holonomy of the complete structure.

Proposition 8.7. *There exists a neighborhood W of the origin in $\mathbf{C}^{s+p}$ and an analytic map*

$$\begin{array}{ccc} W & \to & R(\mathcal{O}) \\ u & \mapsto & \rho_u \end{array}$$

such that for every $u = (u_1, \ldots, u_{s+p}) \in W$ and every $j = 1, \ldots, s+p$, the isometry $\rho_u(m_j)$ has complex length u_j.

Proof. Recall from Chapter 6 that the trace of a matrix in $\mathrm{PSL}_2(\mathbf{C})$ equals ± 2 times the hyperbolic cosine of half the complex length. Thus we consider the map

$$\begin{array}{ccc} W & \to & \mathbf{C}^{s+p} \\ u & \mapsto & (4\cosh^2(u_1/2), \ldots, 4\cosh^2(u_{s+p}/2)). \end{array}$$

By Theorem 8.6, there exists a neighborhood $V \subset X(\mathcal{O})$ of χ_0 such that $J_m(V) \subset \mathbf{C}^{s+p}$ is a neighborhood of $(4, \ldots, 4)$ and the restriction $J_m : V \to J_m(V)$ is bianalytic. Set

$$\chi_u := J_m^{-1}(4\cosh^2(u_1/2), \ldots, 4\cosh^2(u_{s+p}/2)).$$

It is proved in [185] that there exists an analytic section $s\colon V \to R(\mathcal{O})$ to the canonical projection. We define $\rho_u = s(\chi_u)$. It follows from the construction that the complex length of $\rho_u(m_j)$ is u_j. $\square$

Recall that for $j = 1, \ldots, s$, the j-th boundary component of $\mathcal{O}$ is a torus T_j and m_j and l_j generate $\pi_1 T_j$, whereas for $j = s+1, \ldots, s+p$, the j-th boundary component of $\mathcal{O}$ is a pillow P_j and m_j and l_j generate the maximal torsion free subgroup of $\pi_1 P_j$.

Lemma 8.8. *For $j = 1, \ldots, s+p$, there is an analytic map $A_j\colon W \to PSL_2(\mathbf{C})$ such that for every $u \in W$:*

$$\rho_u(m_j) = \pm A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1}.$$

Proof. Let $\widetilde{\rho_u(m_j)}$ be a lift of $\rho_u(m_j)$ to $\mathrm{SL}_2(\mathbf{C})$. Then $\widetilde{\rho_0(m_j)}$ has a unique eigenvalue, which is equal to 1 or -1. Call it $\epsilon_j = \pm 1$. Pick an element $w_2 \in \mathbf{C}^2$ that is not an eigenvector of $\widetilde{\rho_0(m_j)}$. Set $w_1(u) = (\epsilon_j \widetilde{\rho_u(m_j)} - e^{-u_j/2}) w_2$. Take $A_j(u)$ to be the transition matrix between the canonical basis and $\{w_1(u), w_2\}$.
□

The next corollary shows that one can define a function $v_j(u)$ as the complex length of $\rho_u(l_j)$. The sign of $v_j(u)$ is well-defined, because the sign of u_j fixes an orientation of the geodesic invariant by $\rho_u(m_j)$ (which is also invariant by $\rho_u(l_j)$). The indeterminacy of $2\pi i\,\mathbf{Z}$ is eliminated by analyticity.

Corollary 8.9. *For $j = 1, \ldots, s+p$, there exists a unique pair of analytic functions $v_j, \tau_j : W \to \mathbf{C}$ such that $v_j(0) = 0$ and for every $u \in W$:*

$$\rho_u(l_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

i. $\tau_j(0) \in \mathbf{C} - \mathbf{R}$*;*

ii. $\sinh(v_j(u)/2) = \tau_j(u) \sinh(u_j/2)$*;*

iii. v_j *is odd in* u_j *and even in* u_r*, for* $r \neq j$*;*

iv. $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$.

Proof. Use the commutativity relation $m_j l_j = l_j m_j$ and Lemma 8.8. Property (i) comes from the fact that ρ_0 is faithful and discrete. Commutativity implies (ii), which in turn implies (iv). To prove (iii), notice that complex length is defined up to sign, and use the isomorphism of Theorem 8.6 and property (iv).
□

Proposition 8.10. *For each $j = 1, \ldots, s+p$, there is a unique function $(p_j, q_j) : W \to \mathbf{R}^2 \cup \{\infty\}$ such that:*

$$\begin{cases} (p_j(u), q_j(u)) = \infty & \text{if } u_j = 0; \\ p_j(u) u_j + q_j(u) v_j(u) = 2\pi i & \text{if } u_j \neq 0. \end{cases}$$

Furthermore,

$$\begin{aligned} W &\to (\mathbf{R}^2 \cup \{\infty\})^{s+p} \\ u &\mapsto (p_1, q_1), \ldots, (p_{s+p}, q_{s+p}) \end{aligned}$$

is a homeomorphism between W and a neighborhood U of $\{\infty, \ldots, \infty\}$.

Proof. The defining equation is equivalent to:

$$\begin{cases} p_j \operatorname{Re}(u_j) + q_j \operatorname{Re}(v_j) & = & 0 \\ p_j \operatorname{Im}(u_j) + q_j \operatorname{Im}(v_j) & = & 2\pi. \end{cases}$$

Thus

$$(p_j, q_j) = (-2\pi \operatorname{Re}(v_j), 2\pi \operatorname{Re}(u_j)) / \operatorname{Im}(\overline{u}_j v_j).$$

By Corollary 8.9, $\operatorname{Re}(v_j) = \operatorname{Re}(u_j(\tau_j(u) + O(|u|^2))$ and

$$\operatorname{Im}(\overline{u}_j v_j) = |u_j|^2 \operatorname{Im}(\tau_j(u) + O(|u|^2)).$$

Since $\operatorname{Im}(\tau_j(0)) \neq 0$, those computations prove the proposition. □

8.3.4 Deformation of developing maps

Let $D_0 \colon \widetilde{\operatorname{Int}\mathcal{O}} \to \mathbf{H}^3$ be the developing map for the complete structure on $\operatorname{Int}\mathcal{O}$. The following proposition completes the proof of Theorem 8.4.

Proposition 8.11. *For each $u \in W$ there is a map $D_u : \widetilde{\operatorname{Int}\mathcal{O}} \to \mathbf{H}^3$ that is a developing map of a hyperbolic structure on $\operatorname{Int}(\mathcal{O})$ whose holonomy is ρ_u, and such that when $(p_j(u), q_j(u)) \in \mathbf{Z}^2 \cup \{\infty\}$, the completion of $\operatorname{Int}\mathcal{O}$ is the Dehn filling on $\mathcal{O}$ with coefficients $(p_j(u), q_j(u))$.*

Proof. We write $\operatorname{Int}\mathcal{O} = N \cup C_1 \cup \cdots \cup C_{s+p+t}$, where $N \cong \mathcal{O}$ is a compact retract of $\operatorname{Int}\mathcal{O}$ and $C_1, \ldots, C_{s+p+t}$ are standard neighborhoods of the cusps. The idea is to deform the developing maps on N and on the C_i's.

Lemma 8.12. *There exists a family of local diffeomorphisms $D_u^0 : \tilde{N} \to \mathbf{H}^3$, which depends on $u \in W$ continuously for the compact $\mathcal{C}^1$-topology, such that D_u^0 is ρ_u-equivariant and $D_0^0 = D_{0|\tilde{N}}$.*

Lemma 8.13. *There exists a family of local diffeomorphisms $D_u^j : \tilde{C}_j \to \mathbf{H}^3$ which is continuous in $u \in W$ for the compact $\mathcal{C}^1$-topology, such that D_u^j is ρ_u-equivariant, $D_0^j = D_{0|\tilde{C}_j}$ and the structure on C_j can be completed to give the Dehn filling with the coefficients $p_j(u), q_j(u)$.*

Assuming Lemmas 8.12 and 8.13, the maps D_u of Proposition 8.11 are obtained by using bump functions and the fact that we have uniform convergence for compact subsets in the $\mathcal{C}^1$-topology, as in [37] or [18, Appendix B]. □

Proof of Lemma 8.12. This is an orbifold version of [37, Lemma 1.7.2]. We start with a finite covering $\{U_1, \ldots, U_n\}$ of a neighborhood of N. We take each U_i to be either simply connected or the quotient of a ball by a finite orthogonal group. In particular, $U_i \cap \Sigma$ is either empty, an unknotted arc or a graph with three edges meeting at a vertex.

Let $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ denote the universal covering projection and let V_1 be a connected component of $p^{-1}U_1 = \sqcup_{\gamma\in\pi_1\mathcal{O}} \gamma V_1$. If $U_1 \cap \Sigma_{\mathcal{O}} = \emptyset$, then we define $\Delta_1 : V_1 \to \mathbf{H}^3$ to be the restriction of D_0^0 and we extend it ρ_t-equivariantly to $p^{-1}U_1 = \sqcup_{\gamma\in\pi_1\mathcal{O}} \gamma V_1$.

If $U_1 \cap \Sigma_{\mathcal{O}} \neq \emptyset$, then we must be more careful. Here we need to use a family of isometries $\{I_u \in \mathrm{Isom}(\mathbf{H}^3)\}_{u\in U}$ such that:

(i) I_u conjugates $\rho_0(\pi_1 U_1)$ and $\rho_u(\pi_1 U_1)$;

(ii) $I_0 = \mathrm{id}$;

(iii) I_u depends analytically on u.

The construction of such a family is elementary, using the fact that the fixed point set of $\rho_u(\pi_1 U_1)$ depends analytically on u and that $\rho_u(\pi_1 U_1)$ is a finite orthogonal group.

In this case we define $\Delta_1 : V_1 \to \mathbf{H}^3$ to be the restriction of $I_u \circ D_0^0$ and we extend it ρ_t-equivariantly to $p^{-1}U_1 = \sqcup_{\gamma\in\pi_1\mathcal{O}} \gamma V_1$.

Now we make the same construction for each open set U_i and we glue the maps by using refinements of the covering and bump functions, as explained in [37] or [18, Appendix B]. The result of gluing D_u^0 is ρ_u-equivariant, and converges to D_0^0 as $u \to 0$ uniformly on compact subsets for the $\mathcal{C}^1$-topology. In particular, D_u^0 is a local embedding. □

Proof of Lemma 8.13. The proof is different according to whether C_j is a torus, a pillow or a turnover.

The turnover case is the easiest, because the restrictions of the representations to $\pi_1 C_j$ are rigid by Proposition 7.9: $\rho_0|_{\pi_1 C_j}$ and $\rho_u|_{\pi_1 C_j}$ are conjugated by an isometry. Therefore D_u^j is obtained by composing D_0^j with the conjugating isometry.

We discuss now the case where C_j is a torus, following Thurston's notes [225]. By Lemma 8.8 and Corollary 8.9, we may assume:

$$\rho_u(m_j) = \pm \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix}, \qquad \rho_u(l_j) = \pm \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix}.$$

Thus the holonomy of C_j preserves $\infty \in \partial\mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$ and acts on $\mathbf{C} = \partial\mathbf{H}^3 - \{\infty\}$ by affine transformations.

Let $T_j = \partial C_j$ be the horospherical torus. The complete structure on C_j induces a Euclidean structure on T_j, that we may view as a complete affine structure. The developing map D_0 of the complete structure restricted to $\tilde{T}_j$ gives a diffeomorphism $\Delta_0 = D_0|_{\tilde{T}_j} : \tilde{T}_j \to \mathbf{C}$. Since T_j is compact, we may deform Δ_0 to a family of local diffeomorphisms $\Delta_u : \tilde{T}_j \to \mathbf{C}$ which are ρ_u-equivariant, as in Lemma 8.12. This gives a family of affine structures on T^2. One can describe explicitly the maps Δ_u by deforming the image of a fundamental domain (i.e. by deforming a square in $\mathbf{C}$ to a quadrilateral, as in [224, Sec. 3.3 and 3.4]). When $u_j = 0$, Δ_u is still a covering of $\mathbf{C}$. When $u_j \neq 0$, the holonomy $\rho_u|_{(\pi_1 T_j)}$ fixes a point $x_{u_j} \in \mathbf{C}$, and $\Delta_u : \tilde{T}_j \to \mathbf{C} - \{x_{u_j}\}$ lifts through the universal covering projection $\pi : \mathbf{C} \to \mathbf{C} - \{x_{u_j}\}$, where $\pi(z) = \exp(z) + x_{u_j}$. The lift $\tilde{\Delta}_u$ is a covering of $\mathbf{C}$ invariant by the lifted action of the holonomy (see [224] for details).

We want now to pass from the map $\Delta_u : \tilde{T}_j \to \mathbf{C}$ to a developing map $D^j_u : \tilde{C}_j \to \mathbf{H}^3$. Technically, it is useful to fix a point $p \in D^j_0(\tilde{T}_j) \subset \mathbf{H}^3$ and to construct D^j_u so that $p \in D^j_u(\tilde{T}_j)$ for every $u \in U$.

When $u_j = 0$, we define $S(u_j)$ to be the horosphere centered at ∞ that contains the point $p \in \mathbf{H}^3$. The horosphere $S(u_j)$ can be identified to $\mathbf{C} = \partial\mathbf{H}^3 - \{\infty\}$ by means of the geodesics that have ∞ as one of the limit points (i.e. geodesics orthogonal to $S(u_j)$). Thus we define $D^j_0|_{\tilde{T}_j} : \tilde{T}_j \to S(u)$ as the composition of Δ_u with this identification. We use the product structure of $\tilde{C}_j \cong \tilde{T}_j \times [0, +\infty)$ to extend $D^j_0|_{\tilde{T}_j}$, by means of the geodesics having ∞ as limit point.

When $u_j \neq 0$, the holonomy $\langle \rho_u(m_j), \rho_u(l_j) \rangle$ preserves a geodesic γ_{u_j}. The ends of γ_{u_j} are ∞ and $x_{u_j} \in \mathbf{C}$, which are the points of $\mathbf{C} \cup \{\infty\}$ fixed by $\langle \rho_u(m_j), \rho_u(l_j) \rangle$. We define $S(u_j)$ to be the set of points whose distance to γ_{u_j} equals the distance from the fixed point p to γ_{u_j}. In the half-space model, $S(u_j)$ is a Euclidean cone, having x_{u_j} as vertex. In the ball model, it is a banana.

We identify the complement of the fixed point $\mathbf{C} - \{x_{u_j}\}$ with $S(u_j)$ by using the geodesics orthogonal to γ_{u_j}, and define again $D^j_u|_{\tilde{T}} : \tilde{T} \to S(u)$ as the composition of Δ_u with this identification. The family of geodesics orthogonal to γ_{u_j} and the product structure of $\tilde{C}_j$ can be used again to define $D^j_u : \tilde{C}_j \to \mathbf{H}^3 - \gamma_{u_j}$.

It is easy to check that D^j_u depends continuously on u for the compact $\mathcal{C}^1$-topology. What we want to check now is that the completion is the one determined by the generalized Dehn filling coefficients (p_j, q_j).

The metric completion of the image $D^j_u(\tilde{C}_j)$ consists in adding the invariant geodesic γ_{u_j}.

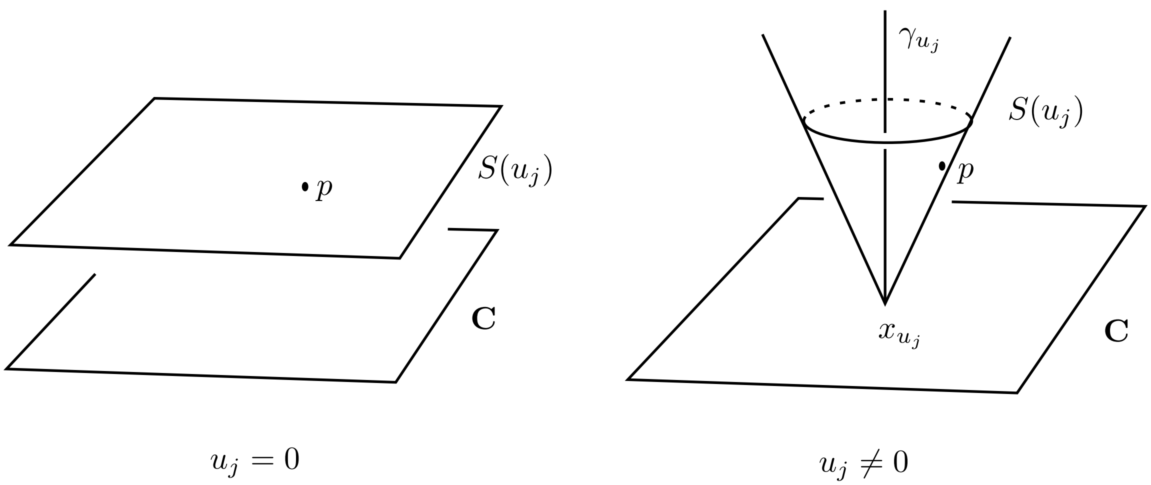

Figure 8.1: The hypersurface $S(u_j)$.

To understand the completion of C_j we use the following lemma, whose proof is left to the careful reader.

Lemma 8.14. *Assume $u_j \neq 0$. Let $\phi\colon \pi_1 T_j \to \mathbf{R}$ be the representation induced by the action of $\langle \rho_u(m_j), \rho_u(l_j)\rangle$ by translations along the invariant geodesic γ_{u_j}.*

(i) If $p_j/q_j \in \mathbf{R} - \mathbf{Q}$, then $\ker\phi$ is trivial and the image of ϕ is non-discrete.

(ii) If $p_j/q_j = r_j/s_j$ for some coprime integers $r_j, s_j \in \mathbf{Z}$ then $\ker\phi \cong \mathbf{Z}$ is generated by $p_j m_j + q_j l_j$ and the image of ϕ is discrete. In addition $r_j m_j + s_j l_j$ acts as a rotation around the invariant geodesic of angle $2\pi r_j/p_j$.

As a particular case of (ii), if $q_j = 0$, then $\ker\phi \cong \mathbf{Z}$ is generated by m_j, the image of ϕ is discrete and m_j acts as a rotation of angle $2\pi/p_j$.

When $p_j/q_j \in \mathbf{R} - \mathbf{Q}$, since the action on γ_{u_j} is not discrete, the completion of C_j is the one-point compactification. When $p_j/q_j \in \mathbf{Q}$, the completion of C_j consists in adding the quotient of γ_{u_j} by $\mathbf{Z}$ acting by translation, i.e. adding a closed geodesic. The fact that this geodesic is singular or not depends only on the angle $2\pi r_j/p_j$.

Finally, in the case where C_j is a pillow, we just use the fact that $\pi_1 C_j$ is an extension of $\mathbf{Z} \oplus \mathbf{Z}$ by an involution, and that all the previous constructions can be made invariant by this involution. □

Historical Remark The first example of a deformation of the holonomy representation of a complete hyperbolic manifold was constructed by Jørgensen in

[119]. He considered the exterior of the figure eight knot, which is fibered over the circle with fiber a punctured torus. The longitude λ of the knot is represented by a peripheral curve in this fiber. Jørgensen constructed a one real parameter deformation of the holomomy representation such that the image of λ is a rotation of angle $\alpha \in [0, 2\pi)$ (here $\alpha = 0$ corresponds to the complete structure). In particular for $\alpha = \frac{2\pi}{n}$, this is an orbifold fibered over the circle.

In his notes [225] Thurston constructed a one complex parameter deformation of the holonomy representation of the figure eight knot exterior, that lead to a constructive proof of the Hyperbolic Dehn Filling Theorem in this case. He even gave an explicit list of the exceptional slopes (i.e. whose fillings give non-hyperbolic manifolds). The problem of determining the exceptional slopes in general is a subject in itself, where many difficult questions are still open (cf. the survey paper [90]). Recently C. Hodgson and S. Kerckhoff [111] obtained a universal bound on the number of exceptional slopes for Dehn fillings on an orientable finite volume hyperbolic 3-manifold with one cusp.

Chapter 9

The Orbifold Theorem

In this chapter we present the Orbifold Theorem.

It follows from Chapter 3 that any compact, connected, orientable 3-orbifold that does not contain a bad 2-suborbifold, can be split along a finite collection of disjoint embedded spherical and toric 2-suborbifolds into irreducible, Seifert fibered or atoroidal 3-orbifolds. The Orbifold Theorem states that the atoroidal pieces are also geometric provided that they have a non-empty singular locus. Here is the full statement.

Theorem 9.1 (Orbifold Theorem). *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible* 3*-orbifold with non-empty singular locus. If $\mathcal{O}$ is atoroidal, then $\mathcal{O}$ is geometric.*

There is also an important application of Theorem 9.1 to non-free finite group actions on the 3-sphere. Some previous partial results were obtained in [51, 68], as well as in [155, 132] for finite group actions on the 3-ball.

Corollary 9.2. *An orientation preserving, smooth, non-free, finite group action on $\mathbf{S}^3$ is conjugate to an orthogonal action.*

The Orbifold Theorem was announced by Thurston in 1981 [228, 229]. Unfortunately, he never published his proof. Recently in 2000, two different proofs were worked out, see [16, 17] and [43]. A proof of the case where the singular locus is not empty, with cyclic local groups already appeared in [18].

By Theorem 3.3 $\mathcal{O}$ can be cut along turnovers into Haken and small pieces. The geometrization of Haken orbifolds can be proved by adapting the proof for Haken manifolds, as explained in Theorem 6.5. In addition, hyperbolic structures can be glued along turnovers, because a turnover in a hyperbolic orbifold is always totally geodesic. So the proof of Theorem 9.1 can be reduced to the case of small orbifolds.

We sketch here the proof of the Orbifold Theorem in the small case. To avoid some technicalities while still conveying the main ideas of the proof, we make further assumptions of closedness and cyclic local groups.

Theorem 9.3. *Let $\mathcal{O}$ be a closed, small, orientable 3-orbifold with nonempty singular locus. Assume that all local groups are cyclic. Then $\mathcal{O}$ is geometric.*

Corollary 9.4. *Let M be a small orientable 3-manifold. Let $\phi\colon M \to M$ be a non-trivial orientation-preserving diffeomorphism of finite order. If ϕ has fixed points, then M admits a $\langle\phi\rangle$-invariant geometric structure.*

Proof. By hypothesis, the orbifold $\mathcal{O} = M/\phi$ is closed with nonempty singular locus and its singular points have cyclic local groups. To apply Theorem 9.3, we only have to show that $\mathcal{O}$ is small.

Suppose that $\mathcal{O}$ is reducible and consider a spherical decomposition (given by Theorem 3.2.) The decomposing system of spherical 2-orbifolds lifts to a system of 2-spheres in M. Each one of them bounds a ball in M. Consider an innermost 2-sphere; it bounds a ball $B \subset M$ disjoint from the other spheres.

Let Q be the quotient of B by its stabilizer Γ in $\langle\phi\rangle$. Then ∂Q is a spherical 2-orbifold. We cap it off by attaching a discal 3-orbifold. The resulting closed 3-orbifold $\hat{Q}$ is irreducible and has fundamental group Γ. If Γ were trivial, then Q would be a ball, contradicting the minimality of the spherical decomposition. It follows that Γ is a nontrivial, finite cyclic group and that Q has nonempty singular locus. Moreover, the Equivariant Loop Theorem (Theorem 3.19) implies that $\hat{Q}$ is small.

Hence we can apply Theorem 9.3 to $\hat{Q}$. The conclusion is that $\hat{Q}$ is spherical, which means that Q is discal, again contradicting the minimality of the spherical decomposition.

Therefore $\mathcal{O}$ is irreducible. As above, the Equivariant Loop Theorem tells us that $\mathcal{O}$ is small. □

In the remainder of this chapter we will assume that $\mathcal{O}$ is an orbifold satisfying the hypothesis of Theorem 9.3. In particular, it is closed and orientable.

Lemma 9.5. *Either $\mathcal{O} - \Sigma_{\mathcal{O}}$ has a complete hyperbolic structure with finite volume or $\mathcal{O} - \Sigma_{\mathcal{O}}$ is Seifert fibered. In the latter case, $\mathcal{O}$ is Seifert fibered and $\Sigma_{\mathcal{O}}$ is a union of fibers.*

Proof. Since $\mathcal{O}$ is small, $\mathcal{O} - \Sigma_{\mathcal{O}}$ is irreducible and atoroidal:

- If a 2-sphere $S^2 \hookrightarrow \mathcal{O} - \Sigma_{\mathcal{O}}$ bounds a discal 3-orbifold in $\mathcal{O}$, then it must bound a ball in $\mathcal{O} - \Sigma_{\mathcal{O}}$, because $S^2 \cap \Sigma_{\mathcal{O}} = \emptyset$.

- If a 2-torus $T^2 \hookrightarrow \mathcal{O} - \Sigma_{\mathcal{O}}$ is compressible in $\mathcal{O}$, then either it is compressible in $\mathcal{O} - \Sigma_{\mathcal{O}}$ or there is a compression disk Δ such that $\Delta \cap \Sigma_{\mathcal{O}}$ is one point. By cutting T^2 along Δ and gluing back two copies of Δ, we get a spherical 2-suborbifold that bounds a discal 3-orbifold in $\mathcal{O}$, hence T^2 is parallel to the boundary of a tubular neighborhood of $\Sigma_{\mathcal{O}}$.

By the Hyperbolization Theorem for Haken 3-manifolds, $\mathcal{O} - \Sigma_{\mathcal{O}}$ is either hyperbolic or Seifert fibered. When $\mathcal{O} - \Sigma_{\mathcal{O}}$ is Seifert fibered, the fibers of the fibration are never isotopic to a meridian of $\Sigma_{\mathcal{O}}$, otherwise $\mathcal{O}$ would be reducible (gluing a vertical annulus with a discal orbifold bounded by the meridian would give an essential spherical or bad 2-suborbifold). Thus we can extend the Seifert fibration to $\mathcal{O}$ by adding the components of $\Sigma_{\mathcal{O}}$ as fibers. □

Remark. If $\mathcal{O}$ is closed, small and has only cyclic local groups, then $\mathcal{O}$ is very good. This is proved in two steps. First notice that the underlying space of $\mathcal{O}$ is a rational homology sphere, by smallness. Then prove that the meridians are linearly independent in $H_1(\mathcal{O} - \Sigma_{\mathcal{O}}, \mathbf{Q})$, by a Mayer-Vietoris argument with a neighborhood of $\Sigma_{\mathcal{O}}$ and $\mathcal{O} - \Sigma_{\mathcal{O}}$. Hence some finite covering of $\mathcal{O} - \Sigma_{\mathcal{O}}$ extends to a manifold covering of $\mathcal{O}$. See [18, Chap.7].

From now on we assume that $\mathcal{O} - \Sigma_{\mathcal{O}}$ is hyperbolic.

In Section 9.1 we define cone manifolds, which are the basic tool, and set the stage for the proof. This leads in Sections 9.2 and 9.3 to some considerations about sequences of cone manifolds and to a proof of the Orbifold Theorem modulo some results to be proved later. Those are tackled in the remaining sections.

9.1 Cone manifolds

Before defining cone manifolds we recall briefly the notion of a *path metric space.* In a metric space X one defines the *length* of a path ξ as the supremum of the lengths of piecewise geodesic paths inscribed in ξ. Then X is a *path metric space* if for all $x, y \in X$, the distance between x and y is the infimum of the length of paths joining x to y. For instance, a Riemannian manifold is (by definition) a path metric space.[1]

[1] Obviously a geodesic space as defined in Section 6.4 is a path metric space, but the converse is not true. (Consider for instance Euclidean space minus a point.) However, a

If a topological space X results from an isometric gluing construction on one or more path metric spaces, then there is an obvious way to measure lengths of paths in X, and one can *define* a metric on X by taking the infimum of lengths of paths joining two points. We call this *the* path metric space obtained by the gluing construction.

Definition. A 3-dimensional *cone manifold* C of constant curvature $K \leq 0$ is a complete path metric space whose underlying space is a smooth 3-manifold $|C|$ and such that every $x \in C$ has a neighborhood U_x that embeds isometrically in one of the model spaces $\mathbf{H}^3_K(\alpha)$ defined below.

To define $\mathbf{H}^3_K(\alpha)$, we first let $\mathbf{H}^3_K$ denote the complete, simply-connected Riemannian manifold of constant sectional curvature $K \leq 0$. Thus $\mathbf{H}^3_{-1} \cong \mathbf{H}^3$ and $\mathbf{H}^3_0 \cong \mathbf{E}^3$. For $\alpha \in (0, 2\pi)$, consider in $\mathbf{H}^3_K$ a solid angular sector S_α obtained by taking the intersection of two half-spaces, such that the dihedral angle at its infinite edge is α. Then $\mathbf{H}^3_K(\alpha)$ is the path metric space obtained by gluing together the faces of S_α by an isometric rotation around the edge. Let Σ be the image of the edge in $\mathbf{H}^3_K(\alpha)$. The induced metric on $\mathbf{H}^3_K(\alpha) - \Sigma$ is an incomplete Riemannian metric of constant curvature K, whose completion is precisely $\mathbf{H}^3_K(\alpha)$. Points of Σ have no neighborhood isometric to a ball in a Riemannian manifold.

In cylindrical or Fermi coordinates, the metric on $\mathbf{H}^3_K(\alpha) - \Sigma$ is:

$$ds^2_K = \begin{cases} dr^2 + \left(\frac{\alpha}{2\pi}\frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}\right)^2 d\theta^2 + \cosh^2(\sqrt{-K}r)dh^2 & \text{for } K < 0 \\ dr^2 + \left(\frac{\alpha}{2\pi}r\right)^2 d\theta^2 + dh^2 & \text{for } K = 0 \end{cases}$$

where $r \in (0, +\infty)$ is the distance from Σ, $\theta \in [0, 2\pi)$ is the rescaled angle parameter around Σ and $h \in \mathbf{R}$ is the distance along Σ.

More generally, if C is a cone manifold and $x \in C$, we say that x is *regular* if it has a neighborhood isometric to a subset of $\mathbf{H}^3_K$. Otherwise it is *singular*. The set of singular points is denoted by Σ_C and called the *singular locus*.[2] To every singular point is associated a *cone angle*, which is the only α such that x has a neighborhood isometric to a subset of $\mathbf{H}^3_K(\alpha)$. The induced metric on $|C| - \Sigma_C$ is a Riemannian metric of constant curvature whose completion is precisely the cone manifold C. It is easy to see that the cone angle is constant

complete, locally compact path metric space is geodesic. This applies to cone manifolds. For more on these notions, see [97].

[2]These definitions are very similar to those for orbifolds, but there is a fundamental difference between the two concepts: orbifolds are topological objects which may carry metrics, whereas cone manifolds are *by definition* metric spaces.

on components of Σ_C, so we can talk about the cone angle *along* a component of Σ_C.

We leave to the reader the definitions of 2-dimensional cone manifolds. Note that with our definition, the singular locus of a 3-dimensional cone manifold is a 1-submanifold. One can give more general definitions where the cone manifold may have arbitrary dimension, the singular locus may have a more complex topology, or cone angles may be greater than 2π. (Compare [17, 43].)

Here are some useful definitions:

- The developing map of C is the developing map of the induced metric on $C - \Sigma_C$:

$$D\colon \widetilde{C - \Sigma_C} \to \mathbf{H}^3_K\,.$$

  It is not a covering map because the metric is incomplete. ($\widetilde{C - \Sigma_C}$ is the universal covering of $C - \Sigma_C$).

- The associated holonomy representation $\rho : \pi_1(C - \Sigma_C) \to \mathrm{Isom}^+(\mathbf{H}^3_K)$ is called the holonomy of C. It is defined by :

$$D \circ \gamma = \rho(\gamma) \circ D,$$

  where γ acts as a covering translation of the universal covering. The image $\rho(\pi_1(C - \Sigma_C))$ need not be discrete.

- When $\mu \in \pi_1(C - \Sigma_C)$ is a meridian around a component of Σ, then $\rho(\mu)$ is an *elliptic element,* more precisely a rotation of angle α around a geodesic. We have the equality:

$$Tr(\rho(\mu)) = \pm 2\cos\frac{\alpha}{2}\,.$$

Remark. The orbifold $\mathcal{O}$ has a Riemannian metric of constant curvature iff there exists a cone manifold C with $(|C|, \Sigma_C) \cong (|\mathcal{O}|, \Sigma_{\mathcal{O}})$ and with cone angles $\frac{2\pi}{m_1}, \dots, \frac{2\pi}{m_k}$, where $m_1, \dots, m_k$ are the orders of the local groups of the edges of $\Sigma_{\mathcal{O}}$. This motivates the next definition.

Definition. The angles $\frac{2\pi}{m_1}, \dots, \frac{2\pi}{m_k}$ are called *the orbifold angles.*

We now give some examples to be kept in mind while reading the proof of the Orbifold Theorem.

Example. We start with a 2-dimensional example. Consider an equilateral triangle in $\mathbf{H}^2$, $\mathbf{E}^2$ or $\mathbf{S}^2$ with angle $\alpha \in (0, \pi]$. In the hyperbolic case (resp. Euclidean, resp. spherical), one has $\alpha < \pi/3$ (resp. $\alpha = \pi/3$, resp. $\alpha > \pi/3$). Let $S(\alpha, \alpha, \alpha)$ be the double of this triangle, i.e the length space obtained by gluing two copies of the triangle by an isometry. Then $S(\alpha, \alpha, \alpha)$ is a cone 2-manifold with underlying space $\mathbf{S}^2$ and three cone points of cone angle 2α. When $n = \pi/\alpha$ is an integer, it is a hyperbolic (resp. Euclidean, resp. spherical) structure on an orbifold. We see on this example how cone manifolds can be seen as "interpolating continuously between geometric orbifolds".

It is worth noting that when α goes to $\pi/3$ from below, the diameter of the hyperbolic cone manifold $S(\alpha, \alpha, \alpha)$ goes to 0. Thus there is a limit angle which corresponds to a degeneration of the hyperbolic structure.

Example. This kind of behavior happens in dimension 3. In [108], it is proved that for every $\alpha \in (0, \pi)$ there is a cone manifold of constant curvature with underlying space $\mathbf{S}^3$, singular locus the figure eight knot and angle α. The structure is explicitly constructed; it is hyperbolic for $\alpha < 2\pi/3$, Euclidean for $\alpha = 2\pi/3$ and spherical for $2\pi/3 < \alpha \leq \pi$. Again by looking at angles of the form $2\pi/n$, one gets geometric structures on certain orbifolds. Since orbifold coverings are branched coverings, one also gets geometric structures on branched coverings of the figure eight knot. It is also natural to consider the finite volume hyperbolic structure on the complement of the figure eight knot as a cone manifold structure with cone angle 0, because hyperbolic cone manifolds with small angles are obtained by deforming this structure.

9.1.1 Deforming cone manifolds

Before proceeding, we have to define formally hyperbolic cone manifold structures on orbifolds.

Definition. Let $\mathcal{O}$ be a closed, orientable 3-orbifold with cyclic local groups. Let $\Sigma_1, \ldots, \Sigma_k$ be the components of $\Sigma_{\mathcal{O}}$. Let $(\alpha_1, \ldots, \alpha_k)$ be a k-tuple of real numbers belonging to the interval $(0, \pi)$.

A *hyperbolic cone manifold structure* on $\mathcal{O}$ with *cone angles* $(\alpha_1, \ldots, \alpha_k)$ is a pair (C, ϕ) where C is a hyperbolic cone manifold and ϕ is a homeomorphism of pairs $(|\mathcal{O}|, \Sigma_{\mathcal{O}}) \to (|C|, \Sigma_C)$ such that for all i, the cone angle along $\phi(\Sigma_i)$ is α_i.

By convention, we define a hyperbolic cone manifold structure on $\mathcal{O}$ with angles $(0, \ldots, 0)$ to be a complete hyperbolic structure of finite volume on the 3-manifold $\mathcal{O} - \Sigma_{\mathcal{O}}$.

Throughout this chapter, we fix an order on the components of the singular locus of our orbifold $\mathcal{O}$, let m_i denote the order of the local group of the i-th component, and set:

$$\mathcal{I} := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{There exists a hyperbolic cone manifold structure} \\ \text{on } \mathcal{O} \text{ with cone angles } (\frac{2\pi t}{m_1}, \dots, \frac{2\pi t}{m_k}). \end{array} \right\}$$

Our hypothesis that $\mathcal{O} - \Sigma_{\mathcal{O}}$ is hyperbolic translates into the fact that $0 \in \mathcal{I}$. The first step is to deform this structure to get hyperbolic cone manifold structures with small angles.

Theorem 9.6. *There exists $\varepsilon > 0$ such that for every k-tuple $\alpha_1, \dots, \alpha_k$ with $0 < \alpha_i < \varepsilon$, there exists a hyperbolic cone manifold structure on $\mathcal{O}$ with cone angles $(\alpha_1, \dots, \alpha_k)$.*

This is the cone manifold version of Thuston's Hyperbolic Dehn Filling Theorem. The proof of Theorem 8.4 that we have sketched can be adapted to prove this version. Using the notation of Section 8.3.1, i.e. letting $\mu_1, \dots, \mu_k$ be the meridians of Σ_C, we are interested in the generalized coefficients $(p_1, q_1), \dots, (p_k, q_k)$, where $p_i = 2\pi/\alpha_i$ and $q_i = 0$. Notice that in the proof of Theorem 8.4 a continuous family of deformations is constructed, so the generalized version of the Hyperbolic Dehn Filling Theorem provides a non-complete structure on $\mathcal{O} - \Sigma_{\mathcal{O}}$ with those coefficients. The completion of this structure is precisely the cone manifold structure with cone angles $2\pi/\alpha_1, \dots, 2\pi/\alpha_k$ (cf. Lemmas 8.13 and 8.14).

To further deform cone angles, we will use the following theorem of Hodgson and Kerckhoff:

Theorem 9.7 ([110]). *The space of hyperbolic cone manifold structures on $\mathcal{O}$ with cone angles $< 2\pi$ is open, and it is locally parametrized by the cone angles $(\alpha_1, \dots, \alpha_k)$.*

From Theorems 9.6 and 9.7 we deduce:

Corollary 9.8. *$\mathcal{I}$ is open.*

Theorem 9.7 contains a local rigidity statement: there are no deformations of hyperbolic cone structures with cone angles fixed. The global rigidity has been obtained by Kojima:

Theorem 9.9 ([129]). *Two hyperbolic cone manifold structures on $\mathcal{O}$ with the same cone angles are isometric, provided that their cone angles are $< \pi$.*

For a generalization of these results to hyperbolic or spherical cone structures with cone angles $\geq \pi$ on a 3-orbifold with noncyclic local groups, see [245].

Theorem 9.9 implies that for each $t \in \mathcal{I}$, the hyperbolic cone manifold structure on $\mathcal{O}$ with angles $(\frac{2\pi t}{m_1}, \ldots, \frac{2\pi t}{m_k})$ is unique. We shall denote it by $C(t)$.

We can now explain the basic idea of the proof of the Orbifold Theorem. If $1 \in \mathcal{I}$, then $\mathcal{O}$ is hyperbolic. Otherwise we must analyze the accidents that can occur at the *limit of hyperbolicity* $t_\infty = \sup \mathcal{I}$. This is done by looking at sequences $C(t_n)$ with $t_n \to t_\infty$, so we need some preliminaries about convergence of sequences of cone manifolds.

9.2 Limits of cone manifolds

We will consider two notions of convergence: Gromov-Hausdorff and pointed bi-Lipschitz.

9.2.1 Gromov-Hausdorff convergence

We begin by recalling the classical notion of Hausdorff distance. Given a metric space Z and two non-empty subsets $X, Y \subset Z$, the *Hausdorff distance* between X and Y is the infimum of all $\varepsilon > 0$ such that X lies in the ε-neighborhood of Y and vice-versa. If one makes no assumption on X, Y, it can happen than $d(X,Y) = 0$ and $X \neq Y$ (e.g. $Z = \mathbf{R}$, $X = [0,1]$, $Y = [0,1] \cap \mathbf{Q}$), but if both X, Y are closed, then $d(X,Y) = 0$ if and only if $X = Y$. In fact, the Hausdorff distance turns the set of non-empty closed subsets of Z into a metric space.

If we are given two metric spaces X, Y abstractly, then we may look at all possible isometric embeddings of X, Y into a space Z and consider the infimum of Hausdorff distances of the images. This leads to a useful notion for *compact* metric spaces.

Definition. [97] Given two compact metric spaces X and Y, the *Gromov-Hausdorff distance* between X and Y, denoted by $d_{GH}(X,Y)$, is the infimum of the Hausdorff distance between $i(X)$ and $j(Y)$, for any metric space Z and any pair of isometric embeddings $i : X \hookrightarrow Z$ and $j : Y \hookrightarrow Z$.

A sequence of compact metric spaces X_n *converges for the Gromov-Hausdorff topology* to a compact metric space X_∞ if $\lim_{n\to\infty} d_{GH}(X_n, X_\infty) = 0$.

Remark. The Gromov-Hausdorff distance is symmetric, satisfies the triangle inequality, and two compact metric spaces X and Y are isometric if and only if $d_{GH}(X,Y) = 0$ (see [97] for details). Notice that this last point implies uniqueness up to isometry of the limit of a convergent sequence.

Example. A sequence of compact metric spaces converges to a point for the Gromov-Hausdorff distance if and only if the diameter converges to zero.

Example. For $\alpha \in (0, \pi)$, consider the hyperbolic quadrilateral of angle $\alpha/2$ such that two of its opposite edges have length 1. Let $S^2(\alpha, \alpha, \alpha, \alpha)$ be the hyperbolic cone 2-manifold obtained by doubling this quadrilateral. It is easily seen that for any sequence $\alpha_n \to \pi$, $S^2(\alpha_n, \alpha_n, \alpha_n, \alpha_n)$ converges to a segment of length one for the Gromov-Hausdorff distance.

We notice that in the previous examples the topology of the limits is different from the topology of the approximating sequences. We will see later that, for cone manifolds, such degenerations corresponds to special behaviors called collapses.

Gromov-Hausdorff convergence is too restrictive for our purposes, because we are interested in sequences X_n with $\operatorname{diam} X_n \to \infty$. Such a sequence cannot converge to a compact space in any reasonable sense. For instance, intuitively, a sequence of round 2-spheres of radius n should converge to $\mathbf{E}^2$. But if X_n is obtained by gluing a round 2-sphere of radius n to a round 3-sphere of radius n (the union occurring at a single point), then what should $\lim X_n$ be: $\mathbf{E}^2$ or $\mathbf{E}^3$?

This problem is solved by considering sequences of *pointed spaces*, i.e. pairs (X, x) where X is a metric space and x is a point of X. This works well when the spaces considered are *proper* (recall that it means that metric balls are compact.)

Definition. Let (X_n, x_n) be a sequence of pointed proper metric spaces and (X_∞, x_∞) be a pointed proper metric space. Then (X_n, x_n) converges to (X_∞, x_∞) for the *pointed Gromov-Hausdorff topology* if for every $R > 0$

$$\lim_{n \to \infty} d_{GH}(B_R(x_n), B_R(x_\infty)) = 0.$$

If the limit exists, it is unique up to isometry [33]. The next example illustrates the importance of the choice of basepoint in a hyperbolic context.

Example. Let M be a noncompact hyperbolic manifold. Set $X_n = M$ and choose $x_n \in M$.

- When the sequence x_n stays in a compact subset of M, (X_n, x_n) subconverges to some (X_∞, x_∞) with X_∞ isometric to M.
- When x_n goes to infinity in a cusp of maximal rank ($\dim M - 1$), (X_n, x_n) converges to a line. The cusp is a warped product of a compact Euclidean manifold with a line, and the diameter of the Euclidean manifold containing x_n converges to zero as x_n goes to infinity.

- When x_n goes to infinity in a geometrically finite end of infinite volume, (X_n, x_n) converges to a hyperbolic space of dimension $\dim M$. This holds true because one can find metric round balls $B_{R_n}(x_n)$ with $R_n \to \infty$.

Proposition 9.10 ([97]). *A pointed Gromov-Hausdorff limit of geodesic spaces is geodesic.*

9.2.2 Gromov's pre-compactness criterion

For a metric space X and for constants $R > \varepsilon > 0$, let $N(R, \varepsilon, X)$ denote the maximal number of disjoints balls of radius ε in a ball of radius R in X.

Theorem 9.11 (Pre-compactness criterion [97]). *A sequence of pointed metric geodesic spaces (X_n, x_n) is pre-compact for the pointed Hausdorff-Gromov topology (i.e. every subsequence subconverges to a pointed metric space) if and only if, for every $\varepsilon > 0$ and $R > 0$, $N(R, \varepsilon, X_n)$ is bounded on n.*

To apply this criterion to cone manifolds we need:

Proposition 9.12 (Bishop-Gromov inequality). *Let C be a cone manifold of curvature $k \leq 0$ and cone angles $\leq 2\pi$ and $p \in C$. Let q be a point in the model space $\mathbf{H}^3_k$. Then, for $0 < r < R$:*

$$\frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}(B_r(q))} \geq \frac{\mathrm{vol}(B_R(p))}{\mathrm{vol}(B_R(q))}.$$

Remark. As for non-singular manifolds, the Bishop-Gromov inequality implies that a sequence of cone manifolds with cone angles $\leq 2\pi$ and curvature in $[-1, 0]$ satisfies Gromov's pre-compactness criterion.

Proof of the Bishop-Gromov inequality. The proof uses the so-called Dirichlet Domain. Let p be a point in a hyperbolic cone manifold. Consider all the segments σ starting at p that are length minimizing. The *Dirichlet domain* D_p is the set of points that are interior to such a minimizing segment.

Notice that singular points of C are never contained in D_p, unless p itself is singular.

Lemma 9.13. *Let C be a cone manifold of curvature $k \leq 0$ and $p \in C$.*

i. If the cone angles are $\leq 2\pi$, then D_p is a star-shaped polyhedron in the model space $\mathbf{H}^3_k$ or $\mathbf{H}^3_k(\alpha)$, according to whether p is smooth or singular.

ii. If the cone angles are $\leq \pi$, then D_p is convex.

This lemma is proved for instance in [18, Chap.3]. We deduce now the Bishop-Gromov inequality from this lemma. The volume of $B_r(p)$ is the same as the volume of the corresponding ball in the Dirichlet domain. Since the Dirichlet domain is star-shaped, the inequality follows from the classical Bishop-Gromov inequality for Riemannian manifolds. □

9.2.3 Bi-Lipschitz convergence of cone manifolds

We have seen that there is no control on the topology of a Gromov-Hausdorff limit. For this reason, we need another notion of convergence.

Definition. A sequence of pointed cone manifolds (C_n, x_x) *converges* to a cone-manifold (C_∞, x_∞) for the *pointed bi-Lipschitz topology* if for every $R > 0$ and $\varepsilon > 0$, there exists an integer n_0 such that, for $n > n_0$, there is a $(1+\varepsilon)$-bi-Lipschitz map $f_n\colon B_R(x_\infty) \to C_n$ satisfying:

i. $d(f_n(x_\infty), x_n) < \varepsilon$,

ii. $B_{R-\varepsilon}(x_n) \subset f_n(B_R(x_\infty))$, and

iii. $f_n(B_R(x_\infty) \cap \Sigma_\infty) = \Sigma_n \cap f_n(B_R(x_\infty))$.

Remark. When $(C_n, x_n) \to (C_\infty, x_\infty)$ for the pointed bi-Lipschitz topology, if the limit C_∞ is compact, then for n large enough the pairs $(|C_n|, \Sigma_n)$ and $(|C_\infty|, \Sigma_\infty)$ are homeomorphic.

A *standard ball* is a metric ball in a model space $\mathbf{H}^3_k(\alpha)$ which either does not intersect the singular axis or is centered on it.

Definition. The *cone injectivity radius* of a point p in a cone manifold is

$$\text{cone-inj}(p) = \sup\{r > 0 \mid B_r(p) \text{ is contained in a standard ball }\}.$$

Definition. A cone manifold C is called *δ-thin* if

$$\sup_{x\in C} \text{cone-inj}(x) \leq \delta.$$

A sequence of cone manifolds C_n *collapses* if C_n is δ_n-thin for some sequence $\delta_n \to 0$.

If $\operatorname{diam} C_n$ goes to 0, then obviously the sequence collapses. For instance, our first 2-dimensional example (page 150) $S(\alpha_n, \alpha_n, \alpha_n)$ collapses when $\alpha_n \to \pi/3$.

The converse is false. For instance, one obtains a collapsing sequence of flat metrics on the 3-torus by starting with a product metric and pinching one factor to a point. In this example the diameter is eventually constant. Notice also that in the example $S^2(\alpha, \alpha, \alpha, \alpha)$ in 9.2.1, the limit is an interval of length 1.

If a sequence C_n does not collapse, then by definition there is a sequence $x_n \in C_n$ such that for some subsequence, the numbers cone-inj(x_n) are uniformly bounded away from zero. Thus the following theorem is relevant to non-collapsing sequences [18, Chap. 3]:

Theorem 9.14 (Compactness Theorem). *Let (C_n, x_n) be a sequence of pointed cone manifolds. Suppose that there exist constants $a, \omega > 0$ such that for all n we have:*

i. cone-inj$(x_n) > a$;

ii. C_n has constant curvature $K_n \in [-1, 0]$;

iii. all cone angles of C_n lie in $[\omega, \pi]$.

Then a subsequence of (C_n, x_n) converges for the pointed bi-Lipschitz topology to a cone 3-manifold (C_∞, x_∞) with curvature $K_\infty = \lim_{n\to+\infty} K_n$ and cone angles that are limits of the cone angles of C_n.

When the sequence does collapse, it does not have a limit in the pointed bi-Lipschitz sense, but the examples above suggest that it may converge to a lower dimensional cone-manifold in the Gromov-Hausdorff topology.

Theorem 9.14 is proved by using Gromov's compactness criterion to say that there is a subsequence that converges to a metric space for the Gromov-Hausdorff topology. To show that the limit is a cone manifold, one proves a uniform lower bound for the injectivity radius of points in C_n that are at bounded distance from x_n. This is where the upper bound on the cone angles is used, via convexity of the Dirichlet domain (Lemma 9.13).

9.3 Analyzing limits of cone manifolds

We now come back to our outline of proof of the Orbifold Theorem. Let (t_n) be an increasing sequence in $\mathcal{I}$ with limit t_∞. Assume that $C(t_n)$ does not collapse. Then Theorem 9.14 implies that $C(t_n)$ subconverges to a hyperbolic cone manifold C_∞ for the pointed bi-Lipschitz topology.

Theorems 9.15 and 9.16 below are proved in [18] (see [17] when the singular locus is any trivalent graph).

Theorem 9.15 (Stability). *If $C(t_n)$ does not collapse, then C_∞ is compact.*

We prove this theorem in Section 9.4. Assuming it, bi-Lipschitz convergence implies that the limit C_∞ is a hyperbolic cone structure on $\mathcal{O}$. Since $\mathcal{I}$ is open, it follows that $t_\infty = 1$ and we are done.

So from now on, we consider the harder case where $C(t_n)$ collapses. Our main tool is the following theorem, whose proof is sketched in Section 9.6. Recall that δ-thin means that $\delta > 0$ is an upper bound for the cone injectivity radius.

Theorem 9.16 (Fibration). *Let C be a cone manifold structure on $\mathcal{O}$ of constant curvature in $[-1, 0]$ and with cone angles less than or equal to the orbifold angles of $\mathcal{O}$. For $\omega > 0$ there exists $\delta > 0$ such that if C has cone angles $\geq \omega$, $\operatorname{diam}(C) \geq 1$ and C is δ-thin, then $\mathcal{O}$ is Seifert fibered.*

Now the proof of the Orbifold Theorem goes as follows:

Case 1 $\operatorname{diam}(C(t_n))$ is bounded away from zero.

In this case Theorem 9.16 proves that $\mathcal{O}$ is Seifert fibered, hence geometric.

Case 2 $\operatorname{diam}(C(t_n)) \to 0$.

Consider the **rescaled** sequence

$$\overline{C}(t_n) = \frac{1}{\operatorname{diam}(C(t_n))} C(t_n)$$

of cone 3-manifolds with constant curvature $K_n = -\operatorname{diam}(C(t_n))^2 \in [-1, 0)$ and diameter equal to 1. If $\overline{C}(t_n)$ collapses, then for n sufficiently large Theorem 9.16 applies again. Otherwise by Theorem 9.14, a subsequence converges to a compact Euclidean cone 3-manifold C_∞ with diameter one. Hence C_∞ corresponds to a closed Euclidean cone structure on $\mathcal{O}$. If $t_\infty = 1$, this proves that $\mathcal{O}$ is Euclidean. Hence we assume that $t_\infty < 1$. Our goal is to show that $\mathcal{O}$ is spherical.

Recall that $\mathcal{O}$ is very good, hence it has a finite regular covering which is a manifold M. The Euclidean cone metric C_∞ lifts to a Euclidean cone metric on M. The singular locus $\widetilde{\Sigma}_{\mathcal{O}}$ of this metric is a link in M and every cone angle equals $2\pi t_\infty < 2\pi$. Moreover, this metric is invariant by the group of deck transformations of the covering. By a radial deformation in a tubular neighborhood of the singular locus $\widetilde{\Sigma}_{\mathcal{O}}$, this cone metric can be desingularized to a smooth metric of non-negative curvature which is still invariant by the group of deck transformations. Then we apply Hamilton's theorem [101] to this metric as in [18] to conclude that M has a metric of constant curvature $+1$ invariant by the group of deck transformations. Hence $\mathcal{O}$ is spherical.

This finishes the sketch of the proof of the Orbifold Theorem.

9.4 Proof of the stability theorem

We have an increasing sequence t_n in $\mathcal{I}$ that converges to t_∞ and the corresponding cone manifolds $C(t_n)$ converge to a hyperbolic cone manifold C_∞ for the pointed bi-Lipschitz topology. We have to prove that C_∞ is compact.

Set $M := \mathcal{O} - \Sigma_{\mathcal{O}}$, $C_n^{smooth} := C(t_n) - \Sigma_{C(t_n)}$, and $C_\infty^{smooth} = C_\infty - \Sigma_{C_\infty}$. By definition of a hyperbolic cone manifold structure, there is for each n a homeomorphism $\phi_n : (|\mathcal{O}|, \Sigma_{\mathcal{O}}) \to (|C(t_n)|, \Sigma_{C(t_n)})$ such that for all i, the cone angle along $\phi_n(\Sigma_i)$ is $\frac{2\pi t_n}{n_i}$.

Using these homeomorphisms, we can identify $\pi_1 M$ with $\pi_1 C_n^{smooth}$, and thus consider the holonomy ρ_n of C_n^{smooth} as a representation of $\pi_1 M$ in $\mathrm{PSL}_2(\mathbf{C})$. We write $\chi_n \in X(M)$ for the character of this holonomy.

Lemma 9.17. *The singular locus of C_∞ is compact.*

Sketch of proof. Seeking a contradiction, assume that C_∞ has non-compact singular locus. Using bi-Lipschitz convergence, it is easy to see that the number of singular components of C_∞ is not greater than that of $C(t_n)$; in particular it is finite. Thus C_∞ has a non-compact singular component, and again by bi-Lipschitz convergence, there exists a connected component Σ_i of $\Sigma_{\mathcal{O}}$ such that the length of $\phi_n(\Sigma_i)$ goes to infinity.

According to [18, Section 2.2], there exists an algebraic curve $\mathcal{C} \subset X(M)$ containing χ_n for all n. By passing to a subsequence, we assume that (χ_n) converges in $\overline{\mathcal{C}}$. The limit x must be an ideal point, because if $\gamma \in \pi_1 M$ is a peripheral element represented by a curve parallel to the arbitrarily long singular component Σ_i, then the real part of the complex length of $\rho_n(\gamma)$ goes to infinity.

Now for each meridian μ of Σ, $J_\mu(\chi_n) = 4\cos\frac{\pi t_n}{m}$ with $m \in \{m_1, \ldots, m_k\}$, so by continuity, $J_\mu(x)$ is finite. Hence Theorem 7.13 gives an incompressible suborbifold in $\mathcal{O}$. This contradicts our hypothesis that $\mathcal{O}$ is small. $\square$

Lemma 9.18. *For every $t \in \mathcal{I}$, vol$(C(t)) \leq \mathrm{vol}(C(0))$.*

Proof. The proof uses Schläfli's formula [162]. For a smooth deformation of a polyhedron P_t in hyperbolic space, the variation of volume is:

$$d\mathrm{vol}(P_t) = -\frac{1}{2}\sum_e \mathrm{length}(e) d\alpha_e$$

where the sum is taken over all edges e of P_t and α_e denotes the dihedral angle of e. For our family of cone manifolds $C(t)$, we can take a totally geodesic triangulation that varies smoothly with t (see [186]). By adding up all contributions of volume, we realize that only edges corresponding to singularities are relevant, and we have:

$$d\mathrm{vol}(C(t)) = -\frac{1}{2}\sum_{i=1}^{k} \ \mathrm{length}(\Sigma_i) \ d(2\pi/m_i t) = -\pi \sum_{i=1}^{k} \ \mathrm{length}(\Sigma_i)\frac{1}{m_i} \ d\,t,$$

where $\Sigma_1, \ldots, \Sigma_k$ denote the components of Σ_C. Hence the volume of $C(t)$ decreases with t. □

Assume that C_∞ is not compact. By Lemma 9.18, the volume of $C(t_n)$ is bounded above, thus C_∞ has finite volume. Since its singular locus is compact, the ends of C_∞ are smooth and we can apply a local version of the Margulis Lemma (see [17]). In particular one can prove easily:

Proposition 9.19. *The manifold C_∞ has a finite number of ends, which are smooth cusps.*

Lemma 9.20. *The manifold C_∞^{smooth} is hyperbolic.*

Sketch of proof. The incomplete metric can be deformed around the singularity to a metric of strictly negative curvature [129]. The metric is unchanged along the complete smooth cusps of C_∞^{smooth}. This implies that C_∞^{smooth} is irreducible and atoroidal, since strictly negative curvature forbids essential spheres or essential tori. Then the result follows from Thurston's Hyperbolization Theorem. □

Let Y be a compact core of C_∞^{smooth}. By convergence, there exists a $(1+\varepsilon_n)$-bi-Lipschitz embedding $f_n : Y \to C_n^{smooth}$ with $\varepsilon_n \to 0$.

Lemma 9.21. *$C(t_n) - f_n(Y)$ is a union of smooth or singular solid tori.*

Sketch of proof. The boundary ∂Y is a union of tori $T_1, \ldots, T_r$. Since $C(t_n)^{smooth}$ is hyperbolic, each $f_n(T_i)$ is either compressible or end-parallel in $C(t_n)^{smooth}$. Assume first that $f_n(T_i)$ is end-parallel, i.e. $f_n(T_i)$ bounds an end-neighborhood U of C_n^{smooth} homeomorphic to $\mathbf{T}^2 \times [0, +\infty)$. If $f_n(T_i)$ separates $f_n(Y)$ from U, then the component of $C(t_n) - f_n(Y)$ corresponding to T_i is a singular solid torus. Now it is impossible that $f_n(Y) \subset U$ for infinitely many n, because by [18, 3.5.4], $\rho_n \circ f_{n*}$ converges to the holonomy of the incomplete structure on C_∞^{smooth}, which is non-abelian.

When $f_n(T_i)$ is compressible, a standard topological argument already explained in this book shows that one of the following occurs:

(1) $f_n(T_i)$ bounds a solid torus disjoint from $f_n(Y)$.

(2) $f_n(T_i)$ bounds a solid torus that contains $f_n(Y)$.

(3) $f_n(T_i)$ is contained in a ball.

As before, an argument with convergence of holonomy representations eliminates cases (2) and (3), because the holonomy of C_∞^{smooth} is non-abelian and the holonomy of T_i is nontrivial. $\square$

For each n, let $\lambda_1^n, \ldots, \lambda_s^n$ ($s \leq k$) be curves on ∂Y such that:

i. there is one curve λ_i^n for each component of ∂Y corresponding to a cusp of C_∞.

ii. $f_n(\lambda_i^n)$ is a meridian of a possibly singular solid torus lying in $C(t_n) - f_n(Y)$.

Lemma 9.22. *For each i, the sequence of closed curves λ_i^n represents infinitely many distinct homotopy classes on ∂Y. Hence, after passing to a subsequence, the length of λ_n^i goes to infinity with n.*

Proof. Otherwise, for some i the curve $\lambda_i^n = \lambda_i$ is independent of n. Thus $\rho_n(f_{n*}(\lambda_i))$ converges to the holonomy of λ_i in C_∞^{smooth}, which is parabolic. This gives a contradiction with the fact that $\rho_n(f_{n*}(\lambda_i))$ is either trivial or a rotation of angle $\frac{2\pi}{m_i} t_n$ (that converges to $\frac{2\pi}{m_i} t_\infty$). $\square$

For each n we consider the Dehn filling of Y along $\lambda_1^n, \ldots, \lambda_s^n$. This manifold is the underlying space of $C(t_n)$ minus open regular neighborhoods of the components of $\Sigma_{C(t_n)}$ that correspond to the components of Σ_{C_∞}. Thus we may assume that topologically this Dehn filling is independent of n. Now using Lemma 9.22, the Hyperbolic Dehn Filling Theorem, and volume estimates (Schläfli's formula), one can show that those Dehn fillings are different. This contradiction finishes the proof of Theorem 9.15. $\square$

9.5 Gromov's simplicial volume

A crucial ingredient in the proof of the fibration theorem is the notion of simplicial volume, introduced by M. Gromov [94].

Let M be a topological space. Our first goal is to define a semi-norm on $H_k(M, \mathbf{R})$. A real k-chain on M is a linear combination $c = \Sigma_i a_i \sigma_i$, where the a_i's are real numbers and the σ_i's are continuous maps from the standard k-simplex to M. We set $\|c\| := \Sigma_i |a_i|$. The semi-norm $\|z\|$ of an element $z \in H_k(M, \mathbf{R})$ is defined as the infimum of the norms of cycles representing z.

If M is a closed n-manifold, then it has a fundamental class $[M] \in H_n(M, \mathbf{R})$. We define the *simplicial volume* of M, sometimes also called *Gromov norm*, by $\|M\| := \|[M]\|$. More generally:

Definition. Let M be a compact orientable n-manifold,

$$\|M\| := \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \;\middle|\; \begin{array}{l} \sum_{i=1}^{n} \lambda_i \sigma_i \text{ is a cycle representing a fundamental} \\ \text{class in } H_3(M, \partial M; \mathbf{R}), \text{ where } \sigma_i : \Delta^3 \to M \\ \text{is a singular simplex and } \lambda_i \in \mathbf{R},\ i = 1, \ldots, n. \end{array} \right\}$$

A basic idea we will exploit is that nonvanishing of simplicial volume is associated to some kind of "hyperbolic" behavior. Let us illustrate this on examples.

Proposition 9.23. *Let M be a closed orientable manifold. If there exists a self-map $f\colon M \to M$ with $|\deg(f)| \geq 2$, then $\|M\| = 0$.*

This follows from the fact that the degree of a map $f : M \to N$ can be defined by $f_*[M] = \deg(f) \cdot [N]$.

Corollary 9.24. *Spheres and tori of dimension ≥ 1 have zero simplicial volume.*

By contrast, hyperbolic manifolds have nonzero simplicial volume. More precisely their simplicial volume is equal to the hyperbolic volume up to a constant factor:

Theorem 9.25. *For $n \geq 2$, let v_n be the supremum of volumes of geodesic simplices in $\mathbf{H}^n$. Then for any complete hyperbolic n-manifold M with finite volume, we have:*

$$\|M\| = \frac{\operatorname{vol} M}{v_n}.$$

Remark. Proving the inequality $\|M\| \geq \operatorname{vol} M / v_n$ is not too hard. It uses the idea of "straightening" cycles, together with the fact that for any "straight" cycle c representing $[M]$, the volume of M is equal to the weighted sum of the volumes of the simplices of c provided that they are counted "algebraically", i.e. taking into account orientations and multiplicities. The other direction is more involved. See [13, C4] for a detailed proof.

Here are some important properties of the simplicial volume:

Properties

- $||M_1 \sharp M_2|| = ||M_1|| + ||M_2||$.

- For 3-dimensional manifolds, $||M_1 \bigcup_{T^2} M_2|| \leq ||M_1|| + ||M_2||$ with equality if the boundary torus T^2 is incompressible in both M_1 and M_2 (cf. [94], see also [131]).

Thus we obtain:

Corollary 9.26. *Let M be a compact, orientable, Haken 3-manifold. Then $\|M\| \neq 0$ if and only if M has at least one hyperbolic piece in its toric splitting.*

Simplicial volume will be used in the next section to analyze collapses. We shall use Corollary 9.26 and Gromov's Vanishing Theorem below (see [94] and [112]). We say that a covering of a manifold has dimension k if its nerve has dimension k (i.e. each point of the manifold is contained in at most $k+1$ sets of the covering).

We say that a subset S in a manifold M is *amenable* if the image of $\pi_1 S \to \pi_1 M$ is amenable. Notice that virtually abelian groups are amenable.

Theorem 9.27 (Vanishing Theorem). *If M is a closed, orientable n-manifold with an $(n-1)$-dimensional covering by amenable sets, then $\|M\| = 0$.*

9.6 The fibration theorem

Throughout this section, we assume that C is a cone manifold structure on $\mathcal{O}$ of constant curvature in $[-1, 0]$, with cone angles between ω and the orbifold angles of $\mathcal{O}$. We also assume that C is δ-thin (i.e. each point has cone injectivity radius $< \delta$). We shall show the existence of a constant $\delta_0(\omega) > 0$ such that if $\delta < \delta_0(\omega)$ then $\mathcal{O}$ is Seifert fibered.

The strategy of the proof is the following. We choose first a Seifert fibered suborbifold $W_0 \subset \mathcal{O}$ such that $\mathcal{O}_0 := \mathcal{O} - \mathrm{Int}(W_0)$ is Haken. Since $\mathcal{O}_0$ is Haken, it has a toric splitting into geometric pieces by Thurston's Hyperbolization Theorem (Theorem 6.5) and it is very good by [149]. Using Theorem 9.27 and Corollary 9.26, we show that any finite regular manifold covering of $\mathcal{O}_0$ has only Seifert pieces in its toric splitting (i.e. it is a *graph manifold* in the sense of [238]). Hence no piece of the toric splitting of $\mathcal{O}_0$ is hyperbolic, and there is a collection of essential toric 2-suborbifolds that decompose $\mathcal{O}$ into Seifert fibered 3-suborbifolds (i.e. $\mathcal{O}$ itself is a "graph orbifold"). Since $\mathcal{O}$ is small, it is in fact Seifert fibered.

9.6.1 Local Euclidean structures

A cone manifold of constant curvature 0 is called *Euclidean*. To understand the local geometry of thin cone manifolds of nonpositive curvature, we need some facts about non-compact Euclidean cone 3-manifolds. We first give some examples.

Example. The following are non-compact Euclidean cone 3-manifolds.

i. The model spaces $\mathbf{H}^3_0$ and $\mathbf{H}^3_0(\alpha)$.

ii. Quotients of $\mathbf{H}^3_0$ (resp. $\mathbf{H}^3_0(\alpha)$) by an infinite cyclic group generated by a screw motion (resp. a screw motion respecting the singular axis.) The underlying space is $\mathbf{S}^1 \times \mathbf{R}^2$, and the singular locus is empty (resp. a core circle).

iii. The product of $\mathbf{R}$ with a closed Euclidean cone 2-manifold.

iv. A slightly more complicated example is obtained by taking the quotient of the previous one by a metric involution τ that reverses the orientation of both factors. For instance $\mathbf{T}^2$ admits an involution such that the quotient is topologically an annulus. This gives $\mathbf{S}^1 \times \mathbf{R}^2$ with singular locus two circles of angle π.

Definition. A *soul* S of a non-compact Euclidean cone 3-manifold E is a totally geodesic compact submanifold with boundary either empty or singular with cone angle π, such that E is isometric to the normal bundle over S (with infinite radius).

In Example (i) above the soul is a point. In Example (ii) it is a circle. We leave it as an exercise to determine the soul in Examples (iii) and (iv).

Proposition 9.28. *Every non-compact Euclidean 3-cone manifold with cone angles $\leq \pi$ has a soul.*

This proposition can be used to classify Euclidean cone 3-manifolds. See [17, 43] for a complete list.

Next lemma is the cone manifold analogue of [42, part 2, Proposition 3.4] in the case of manifolds, which gives a local description of collapsing manifolds.

Lemma 9.29. *For every $\varepsilon > 0$ and $D > 1$, there exists $\delta_0 = \delta_0(\varepsilon, D, \omega) > 0$ such that, if C is a cone 3-manifold satisfying all hypotheses of Theorem 9.16, in particular is δ-thin with $\delta < \delta_0$, then for each $x \in C$ there is a neighborhood $U_x \subset C$ of x, a number $\nu_x \in (0,1)$ and a $(1+\varepsilon)$-bi-Lipschitz homeomorphism f*

between U_x and the ν_x-neighborhood of the soul S of a non-compact Euclidean cone 3-manifold. In addition $\dim S = 1$ *or* 2*, and*

$$\max(d(f(x), S), \mathrm{diam}(S)) \leq \nu_x / D.$$

Sketch of proof. The proof is by contradiction. If the assertion were false, then there would exist $\varepsilon > 0$, $D > 1$ and a sequence of cone manifolds C_n with diameter ≥ 1, curvature in $[-1, 0]$ such that C_n is $\frac{1}{n}$-thin, and there would exist points $x_n \in C_n$ for which the conclusion of the lemma does not hold with the constants ε, D.

Set $\lambda_n = \text{cone-inj}(x_n)$. By the compactness theorem (Thm. 9.14), a subsequence of $(\frac{1}{\lambda_n} C_n, x_n)$ converges to a non-compact Euclidean 3-manifold (E, x_∞). Since cone-inj$(x_\infty) = 1$, the soul of E has dimension one or two. Using the properties of pointed bi-Lipschitz convergence, one can prove that the conclusion of the lemma holds for x_n provided that n is large enough (see [17] and [18, Chap.4] for details). □

The neighborhoods U_x in this lemma are called (ε, D)*-Margulis' neighborhoods*, and the Euclidean cone 3-manifolds with soul S are called *local models.* The local models E are described according to the dimension of their soul S:

- When S is 2-dimensional and orientable, then E is isometric to the product $S \times \mathbf{R}$. The possible 2-dimensional cone manifolds S are the torus $\mathbf{T}^2$, the pillow $S^2(\pi, \pi, \pi, \pi)$ and the turnovers $S^2(\alpha, \beta, \gamma)$, with $\alpha + \beta + \gamma = 2\pi$.
- When S is 2-dimensional but non-orientable (possibly with mirror boundary), then $E = \tilde{S} \times \mathbb{R}/\tau$, where $\tilde{S}$ is the orientation covering of S and τ is an involution that preserves the product structure and reverses the orientation of each factor. It is a twisted line bundle over S.
- When $\dim(S) = 1$, then either $S = \mathbf{S}^1$ or $S = \mathbf{S}^1/\mathbf{Z}_2$. In the former case, E is a solid torus, possibly with a singular core. In the latter case, E is a solid pillow.

We apply this lemma to each point of C with some constants $D > 1$, $\varepsilon > 0$ to be specified later. Consider the thickening

$$W_x := f^{-1}(\overline{\mathcal{N}_{\lambda \nu_x}(S)})$$

of the soul of U_x where $0 < \lambda < \frac{1}{D}$. We will also view W_x as a suborbifold of $\mathcal{O}$.

The topology of W_x is easily described from the classification of non-compact Euclidean cone 3-manifolds. Moreover, not all the models can occur: W_x contains no turnover, because $\mathcal{O}$ is small and the components of $\Sigma_{\mathcal{O}}$ are circles. One can deduce:

Lemma 9.30. *Each W_x is Seifert fibered. In particular, ∂W_x is a union of tori and pillows.*

Proof. Notice that $\mathcal{N}_{\lambda\nu_x}(S)$ is a Euclidean structure on the interior of W_x, with cone angles less than or equal to the orbifold angles. Then the lemma is proved by looking at all the possible cases for S, using the fact that S is never a turnover nor a quotient of one. □

The proof of the following lemma is technical and uses metric properties of the W_x's; we refer to [17] for the proof.

Lemma 9.31. *If $\varepsilon = \varepsilon(\omega) > 0$ is small enough, $D = D(\omega)$ is large enough and if $W_x \cap \Sigma \neq \emptyset$, then $\mathcal{O} - \mathrm{Int}(W_x)$ is Haken.*

9.6.2 Covering by virtually abelian subsets

We assign a special role to one of the subsets W_x along which we will cut $\mathcal{O}$ later on. Namely, we choose $x_0 \in C$ such that $W_{x_0} \cap \Sigma_C \neq \emptyset$ and its radius ν_{x_0} is almost maximal:

$$\nu_{x_0} \geq \frac{1}{1+\varepsilon} \sup\{\nu_x | W_x \cap \Sigma \neq \emptyset\}.$$

We set $W_0 := W_{x_0}$, $\mathcal{O}_0 := \mathcal{O} - \mathrm{Int}(W_0)$, $\nu_0 := \nu_{x_0}$. In view of Lemma 9.31, $\mathcal{O}_0$ is Haken.

Definition. We say that a subset $S \subset \mathcal{O}$ is *virtually abelian in $\mathcal{O}_0$* if for each connected component Z of $S \cap \mathcal{O}_0$, the image of $\pi_1(Z) \to \pi_1(\mathcal{O}_0)$ in the fundamental group of the corresponding component of $\mathcal{O}_0$ is virtually abelian.

Proposition 9.32 ([17, 18]). *If D is large enough and $\varepsilon > 0$ is sufficiently small, then there exists a 2-dimensional complex $K^{(2)}$ and a continuous map $f: C \to K^{(2)}$ such that:*

i. $f(W_0)$ is a vertex of $K^{(2)}$.

ii. The inverse image of the open star of each vertex is virtually abelian in $\mathcal{O}_0$.

Notice that by taking the inverse images of the open stars of vertices of $K^{(2)}$, this proposition tells us that $\mathcal{O}_0$ has a covering by virtually abelian subsets. This covering has dimension 2 (i.e. each point belongs to at most 3 open sets).

Sketch of proof. The proof is divided in several steps.

Step 1: Construction of a covering by virtually abelian subsets.

For $x \in C$ we define:

$$va(x) = \sup\{r > 0 \mid B_r(x) \text{ is virtually abelian in } \mathcal{O}_0\}$$

and

$$r(x) = \inf(\frac{va(x)}{8}, 1).$$

By construction, if $B_{r(x)}(x) \cap B_{r(y)}(y) \neq \emptyset$, then

$$3/4 \leq r(x)/r(y) \leq 4/3 \tag{9.1}$$

and $B_{r(x)}(x) \subset B_{4r(y)}(y)$. In addition, for D sufficiently large we have

$$W_0 \subset B_{\frac{r(x_0)}{9}}(x_0), \tag{9.2}$$

because $va(x_0) \geq \frac{1}{1+\varepsilon}\nu_{x_0}(1 - 1/D)$ and W_0 is contained in the ball of radius $2(1+\varepsilon)\nu_{x_0}/D$ centered at x_0.

To construct the covering we consider sequences $\{x_0, x_1, \ldots\}$ starting with the distinguished point $x_0 \in W_0$, such that:

$$\text{The balls } B_{\frac{1}{4}r(x_0)}(x_0), B_{\frac{1}{4}r(x_1)}(x_1), \ldots \text{ are pairwise disjoint.} \tag{9.3}$$

Such a sequence must be finite. We choose a maximal sequence satisfying (9.3). By maximality and (9.1), the balls $B_{\frac{2}{3}r(x_0)}(x_0), \ldots, B_{\frac{2}{3}r(x_p)}(x_p)$ cover C. Consider the covering of C by the open sets

- $V_0 = B_{r(x_0)}(x_0)$ and
- $V_i = B_{r(x_i)}(x_i) - W_0$ for $i = 1, \ldots, p$.

Define $r_i := r(x_i)$ and $B_i := B_{r(x_i)}(x_i)$.

Using (9.1) and (9.3) we easily get:

Lemma 9.33. *There is a uniform bound N on the number of balls B_i that intersect a fixed ball B_k.*

Step 2: Constructing the Lipschitz map to the nerve of the covering.

Let K denote the nerve of the covering. By Lemma 9.33, $\dim K \leq N$.

Lemma 9.34. *Each $x \in C$ belongs to an open set V_k such that $d(x, \partial V_k) \geq r_k/3$.*

Proof. Since the balls $B_{\frac{2}{3}r_0}(x_0),\dots,B_{\frac{2}{3}r_p}(x_p)$ cover C, we may assume that $x \notin B_{\frac{2}{3}r_0}(x_0)$, $x \in B_{\frac{2}{3}r_k}(x_k)$ for some $k > 0$, and $B_k \cap B_0 \neq \emptyset$, otherwise the lemma is clear. Using inclusion (9.2) we get:

$$d(x, W_0) \geq d(x, x_0) - \frac{1}{9}r_0 \geq \frac{2}{3}r_0 - \frac{1}{9}r_0 \geq \frac{3}{4}\cdot\frac{5}{9}r_k > \frac{1}{3}r_k$$

Hence $d(x, \partial V_k) \geq \frac{1}{3}r_k$. □

On V_k we define

$$\psi_k = \tau(\frac{1}{r_k}d(\partial V_k, \cdot))$$

where $\tau\colon [0,1] \to [0,1]$ is an auxiliary function that vanishes in a neighborhood of the origin, satisfies $\tau|_{[\frac{1}{3},1]} \equiv 1$ and is 4-Lipschitz. Thus ψ_k is $\frac{4}{r_k}$-Lipschitz and we extend it trivially to C.

Now set $\phi_k = \psi_k / \sum_0^p \psi_i$. A consequence of Lemma 9.34 is that

$$\sum_{i=0}^{p} \psi_i(x) \geq 1, \qquad \forall x \in C,$$

hence the application ϕ_k is well-defined on C. Since at most N open sets V_i intersect a given V_k, ϕ_k is $\frac{L}{r_k}$-Lipschitz on each V_k for some uniform constant $L > 0$.

We define $f_N : C \to K$ as $f_N = (\phi_0, \dots, \phi_p)$, which is $\frac{L}{r_k}$-Lipschitz on V_k.

Step 3: Pushing f_N to the 2-skeleton.

We first show that $f_N : C \to K^{(N)}$ can be homotoped to $f_{N-1} : C \to K^{(N-1)}$ which is still $\frac{L'}{r_k}$-Lipschitz on V_k. This homotopy is obtained by composing with radial projection on each N-simplex Δ^N of K. Thus it suffices to prove the next lemma, which guaranties that radial projection from z has bounded diameter:

Lemma 9.35. *If $N > 3$, then there exists $z \in \Delta^N$ at distance $> \theta_0$ from both $\partial\Delta^N$ and the image of f_N, for $\theta_0 > 0$ uniform.*

Proof. Let $\theta > 0$ be a constant for which such a $z \in \Delta^N$ does not exists. Then $f_N(C) \cap \operatorname{Int}\Delta^N$ contains a subset of at least $c_1(N)\cdot\frac{1}{\theta^N}$ points with pairwise distance $\geq \theta$. Let A be the set of inverse images. Notice that $A \subset V_k \subset B_k$, where v_k is any vertex of Δ^N. Since f_N is $\frac{L}{r_k}$-Lipschitz, points in A are separated by distance $\frac{1}{L}r_k\theta$. Volume comparison (using the Bishop-Gromov inequality, Prop. 9.12) implies that A contains at most $c_2 \cdot \left(\frac{L}{\theta}\right)^3$ points. Then inequality

$$c_1(N)\cdot\frac{1}{\theta^N} \leq c_2 \cdot \left(\frac{L}{\theta}\right)^3$$

provides the uniform lower bound for θ. □

Notice that this lemma works as long as $N > 3$, thus applying it several times we can homotope f_N to a map $f_3 : C \to K^{(3)}$ which is $\frac{L}{r_k}$-Lipschitz on each V_k. To push it further to $K^{(2)}$ by a radial projection in each 3-simplex, we show that no 3-simplex Δ^3 is contained in the image of f_3 using a volume argument.

Notice that $\mathrm{vol}(f_3(C) \cap \Delta^3) = \mathrm{vol}(f_3(f_3^{-1}\,\mathrm{Int}(\Delta^3)))$. Since $f_3^{-1}\,\mathrm{Int}(\Delta^3) \subset f_3^{-1}(v_k) \subseteq V_k$, we get:

$$\mathrm{vol}(f_3(C) \cap \Delta^3) \leq \mathrm{vol}\, f_3(V_k) \leq \left(\frac{L}{r_k}\right)^3 \mathrm{vol}\, V_k$$

because f_3 is $\frac{L}{r_k}$-Lipschitz on V_k. Using the Bishop-Gromov inequality (Prop. 9.12) and the description of the local models, we prove that

$$\mathrm{vol}\, V_k \leq \mathrm{vol}\, B_k \leq b\frac{1}{D} r_k^3$$

for some uniform $b > 0$ (see [18, Chap.6], [17]). Thus

$$\mathrm{vol}(f_3(C) \cap \Delta^3) \leq b\frac{1}{D}$$

and it suffices to choose $D > b/\,\mathrm{vol}\,\Delta^3$, so that $\mathrm{vol}(f_3(C) \cap \Delta^3) < \mathrm{vol}(\Delta^3)$. □

9.6.3 Vanishing of simplicial volume

The orbifold $\mathcal{O}_0$ is Haken and therefore has a toric splitting into Seifert fibered and hyperbolic suborbifolds. In particular it is very good [149].

Proposition 9.36. *All components in the toric splitting of $\mathcal{O}_0$ are Seifert fibered.*

Proof. Since the orbifold $\mathcal{O}_0$ is very good, there is a finite covering $p\colon M \to \mathcal{O}_0$ by a manifold M whose boundary ∂M is a union of tori. A hyperbolic piece in the toric splitting of $\mathcal{O}_0$ lifts to a hyperbolic piece in that of M. We have to show that there are no hyperbolic components in the toric splitting of M.

We may assume that the boundary of M is incompressible because otherwise M is a solid torus and the assertion holds. We construct a closed manifold $\overline{M}$ by Dehn filling on M as follows. Let $Y \subset M$ be a component of the toric splitting such that $Y \cap \partial M \neq \emptyset$. When Y is hyperbolic we choose, using the Hyperbolic Dehn Filling Theorem (Thm. 8.4) , the Dehn fillings at the tori of $Y \cap \partial M$ in such a way that the resulting manifold $\overline{Y}$ remains hyperbolic. When Y is Seifert fibered, we fill so that $\overline{Y}$ is Seifert fibered and the components of $\partial Y - \partial M$ remain incompressible (i.e. the surgery slope meets the fiber in at least two points). This can be done because the base of the Seifert fibration of Y is neither an annulus nor a disk with zero or one cone point. The manifold $\overline{M}$ has a toric splitting along the same tori as M and with the same number of hyperbolic (and also Seifert fibered) components.

It suffices to show that $\overline{M}$ has zero simplicial volume, because then Corollary 9.26 implies that $\overline{M}$ contains no hyperbolic component in its toric splitting. To this purpose we will apply Gromov's Vanishing Theorem (Thm. 9.27).

We compose the map of Proposition 9.32 $f\colon \mathcal{O} \to K^{(2)}$ with the projection p and extend the resulting map $M \to K^{(2)}$ to a map $h\colon \overline{M} \to K^{(2)}$ by mapping the filling solid tori to the vertex v_{V_0}. Notice that h is continuous because $f(\partial\mathcal{O}_0) = \{v_{V_0}\}$. The inverse images under h of open stars of vertices are virtually abelian as subsets of $\overline{M}$. These subsets yield an open covering of $\overline{M}$ with covering dimension ≤ 2. By Gromov's Vanishing Theorem, the simplicial volume of $\overline{M}$ vanishes. □

Conclusion of the proof of Theorem 9.16.

Since $\mathcal{O}$ results from $\mathcal{O}_0$ by gluing in a Seifert orbifold, $\mathcal{O}$ itself splits along a finite collection of toric 2-suborbifolds into Seifert fibered 3-suborbifolds. Since $\mathcal{O}$ is atoroidal, it must be Seifert fibered. □

Remark. Consider a closed small orbifold $\mathcal{O}$ such that the simplicial volume of $|\mathcal{O}|$ does not vanish. Then the proof shows that there is no collapse and therefore $\mathcal{O}$ is hyperbolic.

Bibliography

[1] C. C. Adams. The noncompact hyperbolic 3-manifold of minimal volume. *Proc. Amer. Math. Soc.* **100** (1987), 601–606.

[2] J. W. Alexander. On the subdivision of 3-space by a polyhedron. *Proc. Nat. Acad. Sci USA* **10** (1924), 6–8.

[3] L. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math.* **72** (1960), 385–404.

[4] E. M. Andreev. On convex polyhedra in Lobačeskiĭ space. *Mat. USSR Sbornik* **10** (1970), 413–440.

[5] E. M. Andreev. On convex polyhedra of finite volume in Lobačeskiĭ space. *Mat. USSR Sbornik* **12** (1971), 255–259.

[6] M. A. Armstrong. The fundamental group of the orbit space of a discontinuoous group. *Math. Proc. Camb. Phil. Soc.* **64** (1968), 299–301.

[7] A. F. Beardon. *The geometry of discrete groups.* Graduate Texts in Mathematics **91**, Springer-Verlag, New York-Berlin, 1983.

[8] L. Bers. Uniformization, moduli and Kleinian groups. *Bull. London Math. Soc.* **4** (1972), 257–300.

[9] M. Bestvina. R-trees in topology, geometry, and group theory. *Handbook of geometric topology,* 55–91. North-Holland, Amsterdam, 2002.

[10] M. Bestvina and G. Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.* **4** (1991), 469–481.

[11] M. Bestvina and M. Feighn. A combination theorem for negatively curved groups. *J. Differential Geom.* **35** (1992), 85–101.

[12] G. Besson, G. Courtois and S. Gallot. Minimal entropy and Mostow's rigidity theorems. *Ergod. Th. & Dynam. Sys.* **16** (1996), 623–649.

[13] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry.* Universitext, Springer, Berlin, 1992.

[14] M. Boileau. Uniformization en dimension trois. Séminaire Bourbaki, Vol. 1998/99. *Astérisque* **266** (2000), Exp. No. 855, 137–174.

[15] M. Boileau and J.P. Otal. Scindements de Heegaard et groupe des homéotopies des petites variétés de Seifert. *Invent. Math.* **106** (1991), 85–107.

[16] M. Boileau, B. Leeb and J. Porti. Uniformization of small 3-orbifolds. *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), 57-62.

[17] M. Boileau, B. Leeb and J. Porti. Geometrization of 3-dimensional orbifolds. Preprint math.GT/0010185 (2003).

[18] M. Boileau and J. Porti. Geometrization of 3-orbifolds of cyclic type. Appendix A by M. Heusener and Porti. *Astérisque* **272** (2001).

[19] F. Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)* **124** (1986), 71–158.

[20] F. Bonahon. Geometric structures on 3-manifolds. *Handbook of geometric topology,* 93–164. North-Holland, Amsterdam, 2002.

[21] F. Bonahon and L. Siebenmann. The classification of Seifert fibred 3-orbifolds. In *Low-dimensional topology (Chelwood Gate, 1982)*, pages 19–85. Cambridge Univ. Press, Cambridge, 1985.

[22] F. Bonahon and L. C. Siebenmann. The characteristic toric splitting of irreducible compact 3-orbifolds. *Math. Ann.* **278** (1987), 441–479.

[23] M. Bonk and B. Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.* **150** (2002), 127–183.

[24] A. Borel. Compact Clifford-Klein forms of symmetric spaces. *Topology* **2** (1963), 111–122.

[25] A. Borel. Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8** (1981), 1–33.

[26] B. H. Bowditch. Geometrical finiteness for hyperbolic groups. *J. Funct. Anal.* **113** (1993), 245–317.

[27] B. H. Bowditch. Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77** (1995), 229–274.

[28] B. H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.* **11** (1998), 643–667.

[29] B. H. Bowditch. Cut points and canonical splitting of hyperbolic groups. *Acta Math.* **180** (1998), 145–186.

[30] B. H. Bowditch. Convergence groups and configuration spaces. Geometric group theory down under (Canberra, 1996), 23–54, de Gruyter, Berlin, 1999.

[31] S. Boyer and X. Zhang. On Culler-Shalen seminorms and Dehn filling. *Ann. of Math. (2)* **148** (1998), 737–801.

[32] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Springer-Verlag, Berlin, 1999.

[33] M. R. Bridson and G. A. Swarup. On Hausdorff-Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)* **40** (1994), 267–289.

[34] M. G. Brin and T. L. Thickstun. Open, irreducible 3-manifolds which are end 1-movable. *Topology* **26** (1987), 211–233.

[35] E. J. Brody. The topological classification of the lens spaces. *Ann. of Math.* **71** (1960), 163–184.

[36] G. Burde and H. Zieschang. *Knots.* de Gruyter Studies in Math., 5. Walter de Gruyter, 1985.

[37] R. D. Canary, D. B. A. Epstein and P. Green. Notes on notes of Thurston. In *Analytical and Geometric Aspects of Hyperbolic Space* (ed. by D. B. A. Epstein), London Math. Soc. Lecture Notes Ser. 111 (1987), Cambridge Univ. Press, Cambridge, pp. 3–92.

[38] J. W. Cannon and D. Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. *Trans. Amer. Math. Soc.*, **330** (1992), 419–431.

[39] J. W. Cannon and E. L. Swenson. Recognizing constant curvature discrete groups in dimension 3. *Trans. Amer. Math. Soc.* **350** (1998), no. 2, 809–849.

[40] C. Cao and G. R. Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Invent. Math.* **146** (2001), 451–478.

[41] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.* **118** (1994), 441–456.

[42] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. part I: *J. Differential Geom.* **23** (1986), 309–346, part II: *J. Differential Geom.* **32** (1990), 269–298.

[43] D. Cooper, C. D. Hodgson and S. P. Kerckhoff. *Three-dimensional orbifolds and cone-manifolds.* With a postface by Sadayoshi Kojima. MSJ Memoirs, **5**. Mathematical Society of Japan, Tokyo, 2000.

[44] D. Cooper, D. D. Long and A. W. Reid. Essential closed surfaces in bounded 3-manifolds. *J. Amer. Math. Soc.* **10** (1997), no. 3, 553–563.

[45] M. Coornaert, T. Delzant and A. Papadopoulos. *Géométrie et théorie des groupes.* Les groupes hyperboliques de Gromov. Lecture Notes in Mathematics, no. 1441. Springer-Verlag, Berlin, 1990.

[46] C. B. Croke and B. Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology* **39** (2000), 549–556.

[47] M. Culler. Lifting representations to covering groups. *Adv. in Math.* **59** (1986), 64–70.

[48] M. Culler, C. McA. Gordon, J. Luecke and P. B. Shalen. Dehn surgeries on knots. *Ann. of Math.(2)* **125** (1987), 237–300.

[49] M. Culler and P. B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)* **117** (1983), 109–146.

[50] M. W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Ann. of Math. (2)* **117**(1983), 293–324.

[51] M. W. Davis and J. W. Morgan. Finite group actions on homotopy 3-spheres. In *The Smith conjecture (New York, 1979)*, pp181–225. Academic Press, Orlando, FL, 1984.

[52] M. Dehn. Über die Topologie des dreidimensionalen Raumes. *Math. Ann.* **69**(1910), 137–168.

[53] W. Dicks and M. J. Dunwoody. *Groups acting on graphs.* Cambridge University Press, Cambridge, 1989.

[54] W. D. Dunbar. Geometric Orbifolds. *Rev. Mat. Univ. Comp. Madrid* **1** (1988), 67–99.

[55] W. D. Dunbar. Hierarchies for 3-Orbifolds. *Topology and its Appl.* **29** (1988), 267–283.

[56] W. D. Dunbar. Classification of solvorbifolds in dimension three. In *Braids (Santa Cruz, CA, 1986),* Contemp. Math. **78**, 207–216, Amer. Math. Soc., Providence, RI, 1988.

[57] W. D. Dunbar. Nonfibering spherical 3-orbifolds. *Trans. Amer. Math. Soc.* **341** (1994), 121–142.

[58] W. D. Dunbar and R. G. Meyerhoff. Volumes of hyperbolic 3-orbifolds. *Indiana Univ. Math. J.* **43** (1994), 611–637.

[59] M. J. Dunwoody. The accessibility of finitely presented groups. *Invent. Math.* **81** (1985) 449–457.

[60] M. J. Dunwoody. An inaccessible group. Geometric group theory, Vol. 1 (Sussex, 1991), 75–78, *London Math. Soc. Lecture Note Ser.,* 181, Cambridge Univ. Press, Cambridge, 1993.

[61] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender subgroups. *Invent. Math.* **135** (1999) 25–44.

[62] M. J. Dunwoody and E. L. Swenson. The algebraic torus theorem. *Invent. Math.* **140** (2000) 605–637.

[63] A. L. Edmonds and C. Livingston. Group actions on fibered three-manifolds. *Comm. Math. Helv.* **58** (1983) 529–542.

[64] D. Epstein et al. *Word processings in groups.* Jones and Bartlett, 1992.

[65] A. Eskin and B. Farb. Quasi-flats and rigidity in higher rank symmetric spaces. *J. Amer. Math. Soc.* **10** (1997), no. 3, 653–692.

[66] B. Farb. The quasi-isometry classification of lattices in semisimple Lie groups. *Math. Res. Lett.* **4** (1997), no. 5, 705–717.

[67] B. Farb and L. Mosher. Problems on the geometry of finitely generated solvable groups. Crystallographic groups and their generalizations (Kortrijk, 1999), 121–134, Contemp. Math., 262, Amer. Math. Soc., Providence, RI, 2000.

[68] M. E. Feighn. Actions of finite groups on homotopy 3-spheres. *Trans. Amer. Math. Soc.* **284** (1984), 141–151.

[69] M. Feighn and G. Mess. Conjugacy classes of finite subgroups of Kleinian groups. *Amer. J. Math.* **113** (1991), 179–188.

[70] C. D. Feustel. On the torus theorem and its applications. *Trans. Amer. Math. Soc.* **217** (1976), 1–45.

[71] C. D. Feustel. On the torus theorem for closed 3-manifolds. *Trans. Amer. Math. Soc.* **217** (1976), 45–58.

[72] S. V. Matveev and A. T. Fomenko. Isoenergetic surfaces of Hamiltonian systems, the enumeration of three-dimensional manifolds in order of growth of their complexity, and the calculation of volumes of closed hyperbolic manifolds. *Russian Math. Surveys* **43** (1998), 3–24.

[73] E. M. Freden. Negatively curved groups have the convergence property. I. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **20** (1995), no. 2, 333–348.

[74] M. Freedman, J. Hass and P. Scott. Least area incompressible surfaces in 3-manifolds. *Invent. Math.* **71** (1983), 609–642.

[75] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. Preprint 1997.

[76] D. Gabai. Foliations and the topology of 3-manifolds III. *J. Differential Geom.* **26** (1987), no. 3, 479–536.

[77] D. Gabai. Convergence groups are Fuchsian groups. *Ann. of Math.* **136** (1992), 447–510.

[78] D. Gabai. Homotopy hyperbolic 3-manifolds are virtually hyperbolic. *J. Amer. Math. Soc.* **7** (1994), 193–198.

[79] D. Gabai. On the geometric and topological rigidity of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **10** (1997), 37–74.

[80] D. Gabai, R Meyerhoff, and N Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math.* **157** (2003), no. 2, 335–431.

[81] D. Gabai and W. H. Kazez. Group negative curvature for 3-manifolds with genuine laminations. *Geom. Topol.* **2** (1998), 65–77.

[82] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)* **55** (1987), no. 2, 331–358.

[83] S. Gadgil and G. Swarup. A proof of a homeomorphism theorem of Waldhausen I. Preprint arXiv:math.GT/00108116 (2001), 54 pp.

[84] S. M. Gersten. Bounded cocycles and combings of groups. *Internat. J. Algebra Comput.* **2** (1992), 307–326.

[85] É. Ghys. Les groupes hyperboliques. in Séminaire Bourbaki, Vol. 1989/90. *Astérisque* **189-190**, 1990.

[86] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov.* Birkhäuser Boston Inc., Boston, MA, 1990.

[87] C. Godbillon. *Éléments de topologie algébrique.* Hermann, Paris, 1971.

[88] W. Goldman and J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), 43–96.

[89] F. González-Acuna and J. M. Montesinos-Amilibia. On the character variety of group representations in $\mathrm{SL}(2, C)$ and $\mathrm{PSL}(2, C)$. *Math. Z.* **214** (1993), no. 4, 627–652.

[90] C.McA. Gordon. Dehn filling: a survey. In *Knot theory (Warsaw, 1995),* Banach Center Publ. **42** (1998) Polish Acad. Sci., Warsaw.

[91] L. Greenberg. Homomorphisms of triangle groups into PSL(2, C). In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, pp. 167–181, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[92] M. Gromov. Hyperbolic manifolds (according to Thurston and Jørgensen). Bourbaki Seminar, Vol. 1979/80. *Lecture Notes in Math.* **842**, pp. 40–53 , Springer, Berlin-New York, 1981.

[93] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73.

[94] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* **56** (1983), 5–99.

[95] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, New York,1987.

[96] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[97] M. Gromov, J. Lafontaine and P. Pansu. *Structures métriques pour les variétés riemanniennes.* CEDIC, Paris, 1981.

[98] V. Guirardel. Reading small actions of a one-ended hyperbolic group on **R**-trees from its JSJ splitting. *Amer. J. Math.* **122** (2000), no. 4, 667–688.

[99] W. Haken. Theorie der Normalflächen. *Acta Math.* **105** (1961), 245–375.

[100] W. Haken. Some results on surfaces in 3-manifolds. In *Studies in Modern topology,* M.A.A. Studies in Math. **5**, 39–98, Prentice Hall, 1968.

[101] R. S. Hamilton. Four-manifolds with positive curvature operator. *J. Diff. Geom.* **24** (1986), 153–179.

[102] Y. Hashizume. On the uniqueness of the decomposition of a link. *Osaka Math. J.* **10** (1958), 283–300; correction, ibid. **11** (1959), 249.

[103] J. Hass. Minimal surfaces in Seifert fiber spaces. *Topology Appl.* **18** (1984), no. 2-3, 145–151.

[104] J. Hass and P. Scott. The existence of least area surfaces in 3-manifolds. *Trans. Amer. Math. Soc.* **310** (1988), No. 1, 87–114.

[105] A. Hatcher. *Notes on basic* 3*-manifold topology.* Available on his homepage. `http://math.cornell.edu/~hatcher.`

[106] J. Hempel. 3*-Manifolds.* Ann. of Math. Studies, No. 86, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1976.

[107] M. Heusener and J. Porti. The variety of characters in $PSL_2(\mathbf{C})$. Preprint. To appear in *Bol. Soc. Mat. Mex.*

[108] H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, On a remarkable polyhedron geometrizing the figure eight knot cone manifolds. *J. Math. Sci. Univ. Tokyo* **2** (1995), 501–561.

[109] C. Hodgson. *Degeneration and Regeneration of Hyperbolic Structures on Three-Manifolds.* Ph.D. Thesis, Princeton University, 1986.

[110] C. Hodgson and S. Kerckhoff. A rigidity theorem for hyperbolic cone manifolds. *J. Diff. Geom.* **48** (1998), 1–59.

[111] C. Hodgson and S. Kerckhoff. Universal bounds for hyperbolic Dehn surgery. Preprint arXiv:math.GT/0204345 (2002).

[112] N. V. Ivanov. Foundations of the theory of bounded cohomology. *J. Sov. Math.* **37** (1987), 1090–1115.

[113] W. Jaco. *Lectures on three-manifold topology.* CBMS Regional Conference Series in Mathematics **43**, American Mathematical Society, Providence, R.I., 1980.

[114] W. Jaco and U. Oertel. An algorithm to decide if a 3-manifold is a Haken manifold. *Topology* **23** (1984) 195–209.

[115] W. Jaco and P. B. Shalen. Seifert fibred spaces in 3-manifolds. *Mem. Amer. Math. Soc.* **220** (1979).

[116] W. Jaco and J. H. Rubinstein. PL equivariant surgery and invariant decompositions of 3-manifolds. *Adv. in Math.* **73** (1989), no. 2, 149–191.

[117] W. Jaco and J. H. Rubinstein. PL minimal surfaces in 3-manifolds. *J. Diff. Geom.* **27** (1988), 493–524.

[118] K. Johannson. *Homotopy equivalences of* 3*-manifolds with boundary.* Lecture Notes in Mathematics **761**, Springer, 1979.

[119] T. Jørgensen. Compact 3-manifolds of constant negative curvature fibering over the circle. *Ann. of Math. (2)* **106** (1977), 61–72.

[120] M. Kapovich. *Hyperbolic manifolds and discrete groups.* Progress in Mathematics, 183. Birkhäuser Boston, Inc., Boston, MA, 2001.

[121] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds. *Geom. Funct. Anal.* **5** (1995), 582–603.

[122] M. Kapovich and B. Leeb. Actions of discrete groups on nonpositively curved spaces. *Math. Ann.* **306** (1996), 341–352.

[123] M. Kapovich and B. Leeb. Quasi-isometries preserve the geometric decomposition of Haken manifolds. *Invent. Math.* **128** (1997), 393–416.

[124] M. Kapovich and B. Leeb. 3-manifold groups and nonpositive curvature. *Geom. Funct. Anal.* **8** (1998), 841–852.

[125] M. Kapovich and J. Millson. On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.* **88** (1998), 5-95.

[126] A. Kawauchi. *A survey of knot theory.* Birkhäuser Verlag, Basel, Boston, Berlin, 1996.

[127] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Studes Sci. Publ. Math.* **86** (1997), 115–197.

[128] H. Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresbericht der Deutschen Mathematiker Vereinigung* **38** (1929), 248–260.

[129] S. Kojima. Deformations of hyperbolic 3-cone-manifolds. *J. Differential Geom.* **49** (1998), 469-516.

[130] S. Kojima. Finite covers of 3-manifolds containing essential surfaces of Euler characteristic = 0. *Proc. Amer. Math. Soc.* **101** (1987), 743–747.

[131] T. Kuessner. *Relative simplicial volume.* PhD. Thesis, Tübingen 2001.

[132] S. Kwasik and R. Schultz. Icosahedral group actions on $\mathbf{R}^3$. *Invent. Math.* **108** (1992), 385–402.

[133] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math.* **76** (1962), 531–538.

[134] P. A. Linnell. On accessibility of groups. *Ann. of Math.* **76** (1962), 531–538. *J. Pure and Applied Algebra* **30** (1983), 39–46.

[135] F. Löbell. Beispiele geschlossener dreidimensionaler Clifford-Kleinscher Räume negativer Krümmung. *Ber. d. Sächs. Akad. d. Wiss.*, **83** (1931), 167–174.

[136] A. Lubotzky and A. R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.* **58** (1985).

[137] J. Luecke. Finite covers of 3-manifolds containing essential tori. *Trans. Amer. Math. Soc.* **310** (1988), 381–391.

[138] C. Maclachlan and A.W. Reid. *The arithmetic of hyperbolic 3-manifolds.* Graduate Texts in Mathematics **219**, Springer-Verlag, New York, 2003.

[139] W. Magnus, A. Karrass and S. Solitar. *Combinatorial group theory.* John Wiley and Sons, New York, London, Sydney 1966.

[140] S. Maillot. Quasi-isometries of groups, graphs and surfaces. *Comment. Math. Helv.* **76** (2001), 20–60.

[141] S. Maillot. Open 3-manifolds whose fundamental groups have infinite center, and a torus theorem for 3-orbifolds. *Trans. Amer. Math. Soc.* **355** (2003), no. 11, 4595–4638.

[142] A. Marden. The geometry of finitely generated Kleinian groups. *Ann. of Math. (2)* **99** (1974), 383–496.

[143] B. Maskit. On Klein's combination theorem III. In *Advances in the theory of Riemann Surfaces*, Ann. of Math. Studies **66** , 297–316, Princeton Univ. Press 1977.

[144] B. Maskit. Kleinian Groups. *Springer Verlag*, Berlin, 1988.

[145] Y. Matsumoto and J. M. Montesinos-Amilibia. A proof of Thurston's uniformization theorem of geometric orbifolds. *Tokyo J. Math.*, **14** (1991), no. 1, 181–196.

[146] K. Matsuzaki and M. Taniguchi. *Hyperbolic manifolds and Kleinian groups.* Oxford Math. Monographs, Oxford, 1998.

[147] G. R. Meyerhoff. The cusped hyperbolic 3-orbifold of minimum volume. *Bull. Amer. Math. Soc. (N.S.)* **13** (1985), 154–156.

[148] D. McCullough. Compact submanifolds of 3-manifolds with boundary. *Quart. J. Math. Oxford Ser. (2)* **37** (1986), no. 147, 299–307.

[149] D. McCullough and A. Miller. Manifold covers of 3-orbifolds with geometric pieces. *Topology Appl.* **31** (1989), 169–185.

[150] D. R. McMillan Jr. Some contractible open 3-manifolds. *Trans. Amer. Math. Soc.* **102** (1962), 373–382.

[151] C. McMullen. Amenability, Poincaré series and holomorphic averaging. *Invent. Math.* **97** (1989), 95–127.

[152] C. McMullen. Iteration on Teichmüller space. *Invent. Math.* **99** (1990), 425–454.

[153] C. McMullen. Riemann surfaces and the geometrization of 3-manifolds. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), 207–216.

[154] W. H. Meeks III and P. Scott. Finite group actions on 3-manifolds. *Invent. Math.* **86** (1986), 287–346.

[155] W. H. Meeks III and S.-T. Yau. Group actions on $\mathbf{R}^3$. In *The Smith conjecture (New York, 1979)*, pages 167–179. Academic Press, Orlando, FL, 1984.

[156] W. H. Meeks III and S. T. Yau. Topology of three-dimensional manifolds and the embedding problems in minimal surface theory. *Ann. of Math. (2)* **112** (1980), 441–484.

[157] W. H. Meeks III, L. Simon, and S.-T. Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)* **116** (1982), 621–659.

[158] G. Mess. The Seifert conjecture and groups which are coarse quasiisometric to planes. Preprint.

[159] J. Milnor. A unique decomposition theorem for 3-manifolds. *Amer. J. Math.* **84** (1962), 1–7.

[160] J. Milnor. A note on the fundamental group. *J. Diff. Geom.* **2** (1968), 1–7.

[161] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Adv. in Math.* **21** (1976), 293–329.

[162] J. Milnor. The Schläfli differential equality. In *Collected Papers* **1**, 293–329, Publish or Perish, Inc. , Huston 1994.

[163] E. E. Moise. Affine structures in 3-manifolds V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)* **56** (1952), 96–114.

[164] J. M. Montesinos. *Classical tessellations and three-manifolds.* Springer-Verlag, Berlin, 1987.

[165] J. W. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture,* Pure Appl. Math. **112**, 37–125, Academic Press, Orlando, 1984.

[166] J. W. Morgan and H. Bass. *The Smith Conjecture.* Pure and Applied Mathematics 112, Academic Press, Orlando, 1984.

[167] J. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures, I. *Ann. of Math. (2)* **120** (1984), 401–476.

[168] J. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures, II: measured laminations in 3-manifolds. *Ann. of Math. (2)* **127** (1988), 403–465.

[169] J. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures, III: actions of 3-manifold groups on trees and Thurston's compactness theorem. *Ann. of Math. (2)* **120** (1984), 467–519.

[170] G. D. Mostow. Strong rigidity of locally symmetric spaces. *Ann. of Math. Stud.* **78**, Princeton University Press, Princeton, NJ, 1973.

[171] D. Mumford. *Algebraic Geometry I: Complex Projective Varieties.* Springer Verlag, Berlin 1976.

[172] R. Myers. Contractible open 3-manifolds which are not covering spaces. *Topology* **27** (1988), no. 1, 27–35.

[173] W. Neumann and G. Swarup. Canonical decompositions of 3-manifolds. *Geom. Topol.* **1** (1998), 21–40.

[174] P. Orlik. *Seifert manifolds.* Lecture Notes in Mathematics **291**, Springer, 1972.

[175] P. Orlik, E. Vogt and H. Zieschang. Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten. *Topology* **6** (1967), 49–64.

[176] J. P. Otal. *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3. Astérisque* No. 235 (1996).

[177] J. P. Otal. Thurston's hyperbolization theorem of Haken manifolds. In *Surveys in differential geometry, Vol. III (Cambridge, MA, 1996)*, 77–194, Int. Press, Boston, MA, 1998.

[178] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)* **129** (1989), 1–60.

[179] C. D. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots. *Ann. of Math. (2)* **66** (1957), 1–26.

[180] C. D. Papakyriakopoulos. On solid tori. *Proc. London Math. Soc.* **7** (1957), 281–299.

[181] A. Parreau. *Degenerescences de sous-groupes discrets de groupes de Lie semisimples et actions de groupes sur des immeubles affines.* Thèse Paris XI, 2000.

[182] F. Paulin. Actions de groupes sur les arbres. Séminaire Bourbaki, Vol. 1995/96. Astérisque No. 241, (1997), Exp. No. 808, 3, 97–137.

[183] F. Paulin. Un groupe hyperbolique est déterminé par son bord. *J. London Math. Soc. (2)* **54** (1996), no. 1, 50–74.

[184] H. Poincaré. Cinquième complément à l'analysis situs. *Rend. Circ. Mat. Palermo* **18** (1904), 45–110.

[185] J. Porti. Torsion de Reidemeister pour les variétés hyperboliques. *Mem. Amer. Math. Soc.* **128** (1997).

[186] J. Porti. Regenerating hyperbolic and spherical cone structures from Euclidean ones. *Topology* **37** (1998), 365–392.

[187] J. Porti. Regenerating hyperbolic cone structures from Nil. *Geometry and Topology* **6** (2002), 815–852.

[188] G. Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.* **21** (1973), 255–286.

[189] J. P. Préaux. Conjugacy problem in the group of an oriented geometric 3-manifold. Preprint (2003).

[190] J. G. Ratcliffe. *Foundations of hyperbolic manifolds.* Graduate Texts in Mathematics **149**, Springer-Verlag, New York, 1994.

[191] E. G. Rieffel. Groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. *J. London Math. Soc. (2)* **64** (2001), no. 1, 44–60.

[192] D. Rolfsen. *Knots and Links.* Publish or Perish, Inc, 1976.

[193] J. H. Rubinstein. Polyhedral minimal surfaces, Heegaard splittings and decision problems for 3-dimensional manifolds. In *Geometric topology (Athens, GA, 1993)*, pages 1–20. Amer. Math. Soc., Providence, RI, 1997.

[194] H. Schubert. Die eindeutige Zerlegbarkeit eines Knotens in Primknoten. *Akad. Wiss. Heidelberg, Mat.-nat. KI.* **3** (1949), 57–104.

[195] P. Scott. On sufficiently large 3-manifolds. *Quart. J. Math. Oxford Ser. (2)* **23** (1972), 159–172; correction, ibid. (2) **24** (1973), 527–529.

[196] P. Scott. Compact submanifolds of 3-manifolds. *J. London Math. Soc. (2)* **7** (1973), 246–250.

[197] P. Scott. Finitely generated 3-manifold groups are finitely presented. *J. London Math. Soc. (2)* **6** (1973), 437–440.

[198] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.* **15** (1983), no. 5, 401–487.

[199] P. Scott. A new proof of the annulus and torus theorems. *Amer. J. Math.* **102** (1980), 241–277.

[200] P. Scott. There are no fake Seifert fibre spaces with infinite π_1. *Ann. of Math. (2)* **117** (1983), no. 1, 35–70.

[201] P. Scott. Strong annulus and torus theorem and the enclosing property of characteristic submanifolds of 3-manifolds. *Quart. J. Math. Oxford (2)* **35** (1984), 485–506.

[202] P. Scott. Homotopy implies isotopy for some Seifert fibre spaces. *Topology* **24** (1985), no. 1, 341–351.

[203] H. Seifert. Topologie der dreidimensionaler gefaserter Raüme. *Acta Math.* **60** (1933), 147-288.

[204] J.-P. Serre. *Arbres, amalgames,* SL_2. *Astérisque* No. 46, Société Mathématique de France, Paris, 1977.

[205] I. Satake. On a generalization of the notion of manifolds. *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 359–363.

[206] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. In *Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960)* pp. 147–164. Tata Institute of Fundamental Research, Bombay.

[207] P. B. Shalen. A torus theorem for regular branched covers of S^3. *Michigan Math. J.* **28** (1981), 347–358.

[208] P. B. Shalen. Representations of 3-manifold groups. *Handbook of geometric topology*, 955–1044. North-Holland, Amsterdam, 2002.

[209] C. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

[210] R. Skora. The degree of a map between surfaces. *Math. Ann.* **276** (1987), 415–423.

[211] T. A. Springer. Aktionen reduktiver Gruppen auf Varietäten. In *Algebraische Transformationsgruppen und Invariantentheorie*, pp. 3–39, DMV Sem. 13, Birkhäuser, Basel, 1989.

[212] J. Stallings. On fibering certain 3-manifolds. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pp. 95–100. Prentice-Hall, Englewood Cliffs, N.J., 1962.

[213] J. Stallings. *Group theory and three-dimensional manifolds.* Yale Math. Monographs **4**, Yale University Press, 1971.

[214] E. Suárez. *Poliedros de Dirichlet de 3-variedades conicas y sus deformaciones.* Ph.D. Thesis, Madrid 1998.

[215] G. A. Swarup. Proof of a weak hyperbolization theorem. *Q. J. Math.* **51** (2000), no. 4, 529–533.

[216] Y. Takeuchi. Partial solutions of the bad orbifold conjecture. *Topology Appl.* **72** (1996), 113–120.

[217] Y. Takeuchi. Waldhausen's classification theorem for finitely uniformizable 3-orbifolds. *Trans. Amer. Math. Soc.* **328** (1991), 151–200.

[218] Y. Takeuchi and M. Yokoyama. Waldhausen's classification theorem for 3-orbifolds. *Kyushu J. Math.* **54** (2000), 371–401.

[219] Y. Takeuchi and M. Yokoyama. The geometric realizations of the decompositions of 3-orbifold fundamental groups. *Topology Appl.* **95** (1999), 129–153.

[220] Y. Takeuchi and M. Yokoyama. PL-least area 2-orbifolds and its applications to 3-orbifolds. *Kyushu J. Math.* **55** (2001), no. 1, 19–61.

[221] C.B. Thomas. *Elliptic structures on 3-manifolds.* London Mathematical Society Lecture Notes Series **104**, Cambridge University press, 1986.

[222] W. Threlfall and H. Seifert. Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes I and II *Math. Ann.* **104** (1930) and **107** (1932), 1-70 and 543-586.

[223] W. Threlfall. Quelques résultats récents de la topologie des variétés. *Enseign. Math.* **35** (1936), 242-255.

[224] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1.* Princeton Mathematical Press, Princeton, NJ, 1997.

[225] W. P. Thurston. *The geometry and topology of* 3*-manifolds.* Princeton Math. Dept., 1979.

[226] W. P. Thurston. *The geometry and topology of* 3*-manifolds.* Version from 1990.

[227] W. P. Thurston. *Hyperbolic structures on* 3*-manifolds: overall logic.* Notes of Summer Workshop at Bowdoin (1980).

[228] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 357–381.

[229] W.P. Thurston, *Three-manifolds with symmetry.* Preprint 1982.

[230] W. P. Thurston. Hyperbolic structures on 3-manifolds, I: Deformations of acylindrical manifolds. *Ann. of Math. (2)* **124** (1986), 203–246.

[231] W. P. Thurston. *Hyperbolic structures on 3-manifolds, II: Surface groups and manifolds which fiber over* S^1. Preprint 1986.

[232] W. P. Thurston. *Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary.* Preprint 1986.

[233] F. C. Tinsley and D. G. Wright. Some contractible open manifolds and coverings of manifolds in dimension three. *Topology Appl.* **77** (1997), no. 3, 291–301.

[234] P. Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.* **391** (1988), 1–54.

[235] P. Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.* **23** (1994), no. 2, 157–187.

[236] E. B. Vinberg. Hyperbolic groups of reflections. *Russian Math. Surveys* **40** (1985), 31–75.

[237] F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)* **87** (1968), 56–88.

[238] F. Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. *Invent. Math.* **3–4** (1967), 308–333, 87–117.

[239] F. Waldhausen. Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten. *Topology* **6** (1967), 505–517.

[240] F. Waldhausen. Über Involutionen der 3 Sphäre. *Topology* **8** (1969), 81–91.

[241] F. Waldhausen. On the determination of some 3-manifolds by their fundamental group alone. In *Proc. International Symposium in Topology*, Hercy-Novi, Yugoslavia, 1968, pp- 331–332. Beograd 1969.

[242] F. Waldhausen. Some problems on 3-manifolds. *Algebraic and geometric topology, Proc. Sympos. Pure Math. XXXI*, Amer. Math. Soc., Providence, R.I. **XXXII** (1978), 313–322.

[243] A. D. Wallace. Modifications and cobounding manifolds. *Can. J. Math.* **12** (1960), 503–528.

[244] C. Weber and H. Seifert. Die beiden Dodekaederräume, *Math. Z.* **37** (1933), 237–253.

[245] H. Weiss. Local rigidity of 3-dimensional cone-manifolds. Dissertation University Tübingen, 2002.

[246] J. H. C. Whitehead. A certain open manifold whose group is unity. *Quart. J. Math.* **6** (1935), 268–279.

[247] J. A. Wolf. *Spaces of constant curvature.* McGraw-Hill Book Co., New York, 1967.

[248] D. G. Wright. Contractible open manifolds which are not covering spaces. *Topology* **31** (1992), no.2, 281–291.

[249] Q. Zhou. 3-*dimensional geometric cone structures.* PhD. Thesis, University of California, L.A. 1989.

[250] Q. Zhou. The moduli space of hyperbolic cone structures. *J. Differential Geom.* **51** (1999), 517–550.

[251] H. Zieschang. Finite groups of mapping classes of surfaces. *Lecture Notes in Mathematics* **875**, Springer, 1981.

[252] H. Zieschang, E. Vogt, and H.-D. Coldewey. Surfaces and planar discontinuous groups. Translated from the German by John Stillwell. *Lecture Notes in Mathematics* **835**, Springer, Berlin, 1980.

[253] B. Zimmermann. Isotopies of Seifert fibered, hyperbolic and Euclidean 3-orbifolds. *Quart. J. Math. Oxford Ser. (2)* **40** (1989), 361–369.

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