1: Lie groups

Matix groups, Lie algebras

1.– Prove that $\mathrm{O}(n)$ is Lie group and that its tangent space at $\mathrm{I} \in \mathrm{O}(n)$ is isomorphic to the space $\mathfrak{so}(n)$ of skew-symmetric matrices

$$\mathfrak{so}(n) = \{A \in \mathrm{M}_{n\times n}(\mathbb{R}) \mid A^t = -A\}$$

(Hint: Compute the differential map at an arbitrary point $\mathrm{O}(n)$ of the map $F : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Sym}(n)$ given by $F(A) = A\,A^t$ and where $\mathrm{Sym}(n)$ denotes the linear space of symmetric matrices.)

2.– Prove that $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong S^3$

3.– For $n \geq 3$, the group $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$. In this exercise we describe $\mathrm{Spin}(3)$ and $\mathrm{Spin}(4)$.

Let $\mathcal{H}$ denote the quaternions. Write a quaternion as $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathcal{H}$, with $a, b, c, d \in \mathbb{R}$. Its trace is $\mathrm{tr}(q) = a$ and its norm is $\|q\| = a^2 + b^2 + c^2 + d^2$. The vector space of quaternions of trace zero is denoted by $\mathcal{H}_0$:

$$\mathcal{H}_0 = \{q \in \mathcal{H} \mid \mathrm{tr}(q) = 0\}$$

and the group of quaternions of norm 1 is denoted by $\mathcal{H}^1$:

$$\mathcal{H}^1 = \{q \in \mathcal{H} \mid \|q\| = 1\}.$$

a) Consider the action of $\mathcal{H}^1$ on $\mathcal{H}_0$ defined as follows: $\forall q \in \mathcal{H}^1$,

$$\begin{array}{rcl} \mathcal{H}_0 & \to & \mathcal{H}_0 \\ x & \mapsto & q\,x\,q^{-1} \end{array}.$$

Prove that $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$

b) Consider the action of $\mathcal{H}^1 \times \mathcal{H}^1$ on $\mathcal{H}$ as follows: $\forall q_1, q_2 \in \mathcal{H}^1$,

$$\begin{array}{rcl} \mathcal{H} & \to & \mathcal{H} \\ x & \mapsto & q_1\,x\,q_2^{-1} \end{array}.$$

Prove that $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$.

c) Restrict this action to $\mathbb{C} \cap \mathcal{H}^1 \cong S^1$ and consider its orbit. Describe the fibration by circles of S^3.

d) Who is the universal cover of $\mathrm{SO}(2)$?

4.– Prove that $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ are non compact.

5.– Prove that $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathcal{H})$ are connected.

6.– Consider the diagonal matrix with p entries equal to one and q entries equal to -1:

$$\mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix},$$

where I_p and I_q denote the identity matrices of size $p \times p$ and $q \times q$. Consider the corresponding isometry groups

$$\begin{aligned} \mathrm{O}(p,q) &= \{A \in \mathrm{GL}(n,\mathbb{R}) | A^t I_{p,q} A = I_{p,q}\} \\ \mathrm{U}(p,q) &= \{A \in \mathrm{GL}(n,\mathbb{C}) | \bar{A}^t I_{p,q} A = I_{p,q}\} \end{aligned}$$

a) How many components do $\mathrm{U}(p,q)$ and $\mathrm{O}(p,q)$ have?

b) Prove that $\mathrm{U}(p,q)$ and $\mathrm{O}(p,q)$ are noncompact.

7.– Prove that there is a retraction $r : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{O}(n)$, i.e. a map r that is the identity on $\mathrm{O}(n)$ and homotpically equivalent to the identity relative to $\mathrm{O}(n)$.

8.– Prove that there is a retraction $r : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{U}(n)$

9.– Let $\mathrm{Sym}(n)$ denote the space of $n \times n$ symmetric matrices with real coefficients, i.e. $\mathrm{Sym}(n) = \{A \in M_{n\times n}(\mathbb{R} \mid A^t = A\}$. and let $\mathrm{Sym}^+(n)$ denote the positive definite ones. Prove that the exponential map $\exp : \mathrm{Sym}(n) \to \mathrm{Sym}^+(n)$ is a homeomorphism.

10.– Prove that the polar decomposition gives a homeomorphism

$$\mathrm{GL}^+(n,\mathbb{R}) \cong \mathrm{Sym}^+(n) \times \mathrm{SO}(n)$$

11.– Prove that there is a homeomorphism

$$\mathrm{GL}(n,\mathbb{C}) \cong \mathbb{R}^{n^2} \times \mathrm{U}(n)$$

12.– Prove that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$

13.– The center of a Lie group is the subgroup of elements that commute with every element of the group.

a) Describe the center Z_n of $\mathrm{GL}(n,\mathbb{C})$.

b) Let $\mathrm{PGL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/Z_n$. Prove that the Lie algebra of $\mathrm{PGL}(n,\mathbb{C})$ is $\mathfrak{sl}(n,\mathbb{C})$.

c) Prove that the action of $\mathrm{PGL}(n,\mathbb{C})$ on the projective space $\mathbb{P}^{n-1}$ is effective (every element different from the identity acts nontrivially).

d) Prove that $\mathrm{SO}(2,1)_0 \cong \mathrm{PSU}(1,1) \cong \mathrm{PGL}(2,\mathbb{R})$. Where the subindex $_0$ denotes the identity component. (Hint: the three groups act naturally by isometries on the hyperbolic plane)

e) Prove that $\mathrm{SO}(3,1)_0 \cong \mathrm{PGL}(2,\mathbb{C})$. (Hint: both groups act naturally by isometries on hyperbolic space)

14.– Prove that $\exp : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is not surjective. Is $\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ surjective?

15.– Prove that if $\phi : G \to H$ is an injective morphism of Lie groups and if $\dim G = \dim H$, then ϕ is an isomorphism.

16.– Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

17.– Prove that $\mathfrak{a}(1)$ is the only 1-dimensional Lie algebra, up to isomorphism. (Here and in the following exercises $\mathfrak{a}(n)$ denotes the n-dimensional abelian Lie algebra).

18.– Prove that a 2-dimensional Lie algebra is isomorphic to either $\mathfrak{a}(2)$ or $\mathfrak{aff}(2)$ (the Lie algebra of affine transformations of the real line).

19.– Verify that the Lie algebra $\mathfrak{su}(2)$ is generated by the following matrices (which are traceless and hermitian)

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and that the Lie bracket is specified by

$$[u_3, u_1] = 2u_2, \qquad [u_1, u_2] = 2u_3, \qquad [u_2, u_3] = 2u_1.$$

The matrices u_i are related to Pauli spin matrices σ_k by $u_1 = i\,\sigma_1$, $u_2 = -i\,\sigma_2$ and $u_3 = i\,\sigma_3$. Notice that Pauli matrices σ_k are hermitian and unitary.

20.– Verify that the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is generated by the following matrices (which are traceless)

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the Lie bracket determined by

$$[e, f] = h, \qquad [f, h] = -2f, \qquad [h, e] = 2e.$$

Show that $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic as complex Lie algebras.

21.– Determine all Lie subgroups of the 2-torus $S^1 \times S^1$.

22.– Prove that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$ with Lie algebra $\mathrm{so}(n)$. Prove that $\mathrm{SU}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ with Lie algebra $\mathfrak{su}(n)$. Prove that $\mathrm{Sp}(2n,\mathbb{K})$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{K})$, for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, with Lie algebra $\mathfrak{sp}(2n,\mathbb{K})$.

23.– Determine the adjoint representations Ad (resp. ad) of the groups $\mathbb{R}^n$ and $\mathrm{Aff}(\mathbb{R})$ (resp. of the Lie algebras $\mathfrak{r}^n$ and $\mathfrak{aff}(\mathbb{R})$).

24.– Determine the adjoint representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

25.– Let $\mathcal{H}^3$ be the Heisenberg group defined in Example 1.11 (4) of the notes. Determine its Lie algebra $\mathfrak{h}^3$ and describe the corresponding exponential map. Prove that, if X, Y are elements of $\mathfrak{h}^3$ then

$$e^X e^Y = e^{X+Y+2[X,Y]}.$$

26.– Consider the bilinear form

$$\begin{aligned} B : \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) &\to \mathbb{R} \\ (a, b) &\mapsto \mathrm{tr}(a \cdot b) \end{aligned}$$

a) Consider the adjoint representation $\mathrm{Ad} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{R}))$. Prove that B is Ad-invariant, namely:

$$B(\mathrm{Ad}_C(a), \mathrm{Ad}_C(b)) = B(a,b), \qquad \forall a,b \in \mathfrak{sl}(2,\mathbb{R}), \forall C \in \mathrm{SL}(2,\mathbb{R}).$$

b) Deduce that there is a natural isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(2,1)$.

27.– The Killing form of a Lie algebra $\mathfrak{g}$ is defined as

$$B(x,y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

a) Prove that it is bilinear, symmetric, and both ad and Ad-invariant.

b) Is it related to the form of the previous exercise?

c) Compute it for $\mathbb{R}^n$ and for $\mathfrak{o}(3)$.

28.– A flag of $\mathbb{R}^n$ is an increasing sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n$$

of linear subspaces of $\mathbb{R}^n$ with $\dim V_i = i$.

a) Show that the set of flags of $\mathbb{R}^n$ has a natural structure of smooth manifold by identifying it to an appropriate homogeneous space and compute its dimension. (This is called the flag manifold and denoted by $\mathrm{Flag}(n)$.)

b) Describe $\mathrm{Flag}(2)$ and $\mathrm{Flag}(3)$.

c) Prove that the set of partial flags (i.e. lines and hyperplanes):

$$\begin{aligned} V_n = & \ \{l \subset \cdots \subset H^{n-1} \subset \mathbb{R}^n \mid \dim l = 1,\ \dim H^{n-1} = n-1\} \\ = & \ \{(l,H) \in \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \mid l \subset H\} \end{aligned}$$

is also a homogeneous manifold.

d) Prove that here is a natural map $\mathrm{Flag}(n) \to V_n$.

29.– Prove that any finite dimensional representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ factors through a representation of $\mathrm{SL}(2,\mathbb{R})$. Namely for every morphism $\rho : \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{GL}(n,\mathbb{R})$ there exists a morphism $\rho' : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{SL}(2,\mathbb{R})} & \xrightarrow{\ \rho\ } & \mathrm{GL}(n,\mathbb{R}) \\ \downarrow \pi & \nearrow \rho' & \\ \mathrm{SL}(2,\mathbb{R}) & & \end{array}$$

(Hint: use the induced representation of the Lie algebra and then complexify, i.e. tensorize by $\otimes\mathbb{C}$. Then use that $\mathrm{SL}(2,\mathbb{C})$ is simply connected).

Deduce that $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is not linear.

30.– A riemannian manifold X is called a *symmetric space* if for each $x \in X$ there exists an isometry $\sigma_x : X \to X$ satisfying:

- $\sigma_x(x) = x$, and
- $d\sigma_x = -\mathrm{Id}|_{T_x X}$.

a) Using that an isometry of X is determined by its action on a point and on the tangent space at this point, prove that $\sigma_x^2 = \mathrm{Id}_X$.

b) Prove that for each $t \in \mathbb{R}$ and every geodesic $\gamma : \mathbb{R} \to X$ satisfying $\gamma(0) = x$, $\sigma_x(\gamma(t)) = \gamma(-t)$.

c) Prove that the group of isometries $\mathrm{Isom}(X)$ acts transitively on X (hint: use the involution centered at a midpont). Prove also that the action of the group of orientation preserving isometries $\mathrm{Isom}^+(X)$ is transitive.

d) Using that $\mathrm{Isom}^+(X)$ is a Lie group, prove that $X = G/K$ for some connected Lie group G and a compact subgroup K.

e) An automorphism $\sigma : G \to G$ is called involutive if $\sigma^2 = \mathrm{Id}_G$. Let $G^\sigma = \{g \in G \mid \sigma(g) = g\}$. Prove that $X = G/K$ as above and there is an involutive morphism $\sigma : G \to G$ such that $G^\sigma \supset K \subset G_0^\sigma$, where G_0^σ denotes the identity component of G^σ.

f) Prove the converse: if G is a Lie group, σ an involutive automorphism of G, and K a compact subgroup of G, with $G^\sigma \supset K \subset G_0^\sigma$, then any G-invariant Riemannian metric on G/K is symmetric.

g) Prove that S^n is a symmetric space.

31.– (Exercise 4.7.) Recall that the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are connected and that $\mathrm{SU}(2) \cong S^3$ can be described as

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & z \end{pmatrix} \mid |w|^2 + |z|^2 = 1 \right\}$$

Let us consider the basis of the Lie algebra $\mathfrak{su}(2)$ given by the matrices

$$i\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

a) Show that the map $\mathrm{SU}(2) \to \mathrm{GL}(3, \mathbb{R})$, given by the correspondence that associates the matrix of Ad_g in the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ to each $g \in \mathrm{SU}(2)$, induces a morphism of Lie groups $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

b) Show that $\ker \varphi = \{1, -1\} = \{1, e^{\pi i \sigma_3}\}$ and deduce in particular that $\mathrm{SO}(3) \cong \mathbb{R}P^3$.

c) Show that representations of $\mathrm{SO}(3)$ are the same as representations of $\mathrm{SU}(2)$ satisfying $e^{\pi i \sigma_3} = \mathrm{Id}$.

32.– (Exercise 4.13.) Let $G = \mathbb{R}$, so $\mathfrak{g} = \mathfrak{r} \cong \mathbb{R}$. A representation V of the Lie algebra $\mathfrak{r}$ is a linear map $\mathbb{R} \to \mathrm{End}(V)$, which is of the form $t \mapsto t\,A$ for a suitable $A \in \mathrm{End}(V)$. The corresponding representation of the group $\mathbb{R}$ is given by $t \mapsto \exp(t\,A)$. Show that such a representation is completely reducible if and only if A is diagonalizable.

33.– (Exercise 4.18.) Using that $\mathbb{C}^n$ is irreducible as representation of $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n,\mathbb{C})$, deduce that the centers of these Lie groups and the corresponding Lie algebras are the following:

$$\begin{array}{ll} Z(\mathrm{SL}(n,\mathbb{C})) = Z(\mathrm{SU}(n)) = \{\lambda \mathrm{Id} \mid \lambda^n = 1\}, & \mathfrak{z}(\mathfrak{sl}(n,\mathbb{C})) = \mathfrak{z}(\mathfrak{su}(n)) = 0 \\ Z(\mathrm{U}(n)) = \{\lambda \mathrm{Id} \mid |\lambda| = 1\}, & \mathfrak{z}(\mathfrak{u}(n)) = \{\lambda \mathrm{Id} \mid \lambda \in i\mathbb{R}\}, \\ Z(\mathrm{SO}(n,\mathbb{C})) = Z(\mathrm{SO}(n,\mathbb{R})) = \{\pm \mathrm{Id}\} & \mathfrak{z}(\mathfrak{so}(n,\mathbb{C})) = \mathfrak{z}(\mathfrak{so}(n,\mathbb{R})) = 0 \end{array}$$

34.– (Exercise 4.20.) It has already been noticed that the irreducible representations of $\mathbb{R}$ are V_λ, $\lambda \in \mathbb{C}$, where each V_λ is a one-dimensional complex vector space with the action of $\mathbb{R}$ given by $\rho(x) = e^{\lambda x} \cdot \mathrm{Id}$. Deduce that the irreducible representations of $S^1 = \mathbb{R}/\mathbb{Z}$ are V_k, $k \in \mathbb{Z}$, where V_k is a one-dimensional complex vector space with the action of S^1 given by $\rho(x) = e^{2\pi i k x} \cdot \mathrm{Id}$ (or $\rho(z) = z^k \cdot \mathrm{Id}$ if we write $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.)

35.– (Exercise 4.23). Prove that each representation of a finite group is unitary, hence completely reducible.

36.– (Exercise 4.30). Determine the characters of the irreducible representations of $S^1 = \mathrm{U}(1)$.

37.– Is the adjoint representation of $\mathrm{SL}(2,\mathbb{C})$ irreducible? Write it in temrs of V_n.

38.– Determine the irreducible representations of $\mathrm{SO}(3,\mathbb{C})$, using that it is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$. Can you construct them explicitely?

39.– Let V_n be the n dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$.

a) Prove that

$$V_n \otimes V_n = \bigoplus_{i=0}^{n} V_{2i}.$$

b) Prove that, if $m \leq n$:

$$V_m \otimes V_n = V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{n-m}.$$