Generalized Curvatures and the Geometry of Data

Geometric and analytic aspects

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Joint work with

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This is captured by a suitable concept of curvature.
And curvature quantifies convexity.
And global convexity is characteristic of nonpositive curvature (**NPC**).

Why is NPC important?

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1. Many important spaces
   - Hyperbolic (and Euclidean) spaces
   - symmetric spaces of noncompact type
   - trees, and more generally, Euclidean (Bruhat-Tits) buildings
   - many moduli spaces, like those of Riemann surfaces or Abelian varieties
   - most groups are hyperbolic (Gromov), together with their Cayley graphs

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2. Representations of $\pi_1(M)$ lead to mappings of M into NPC spaces
3. Relations with topological concepts like hyperconvexity and Topological Data Analysis

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  Convex $\circ$ harmonic map $=$ subharmonic function $\longrightarrow$ Harnack inequalities apply

Preliminaries from metric geometry

(X, d) metric space. $c : [0,1] \longrightarrow X$ continuous (*path*) with $x = c(0), y = c(1)$ has length $l(c) := \sup \sum_{i=0}^{i=n} d(c(t_i), c(t_{i-1}))$. Supremum over all partitions of $[0,1]$.

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Length space if for all x,y,
$d(x,y)=\inf\{l(c):$ c is a path between x and $y\}$.
A length space (X,d) is *geodesic* if any $x,y\in X$ connected by a *shortest geodesic* $c:[0,1]\longrightarrow X$, i.e. $d(x,y)=l(c)$.

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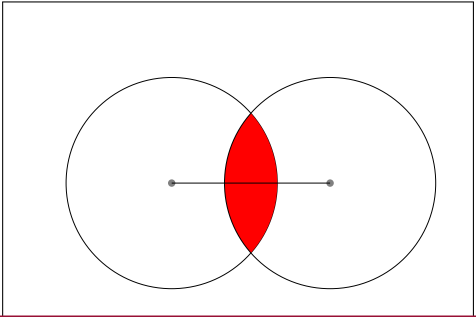
$B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset$
($B(x, r)$ = closed ball with center x, radius r.)

Slogan: **Two balls that can intersect do intersect.**



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Any radii r_i will be > 0 in the sequel.

For $r_1 + r_2 \geq d(x_1, x_2)$,

$$\begin{aligned} \rho((x_1, x_2), (r_1, r_2)) &:= \inf_{x \in X} \max_{i=1,2} \frac{d(x_i, x)}{r_i} \\ \rho(x_1, x_2) &:= \sup_{r_1, r_2} \rho((x_1, x_2), (r_1, r_2)) \end{aligned}$$

Find good points x between x_1 and x_2

Make radii of balls small

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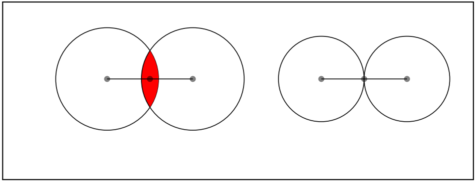

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A key idea now is to extend this to three points.

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Definition

A geodesic length space (X, d) is a *tripod space* if for any three points $x_1, x_2, x_3 \in X$, there exists a *median*, that is, a point $m \in X$ with

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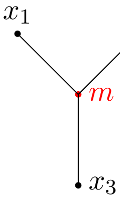

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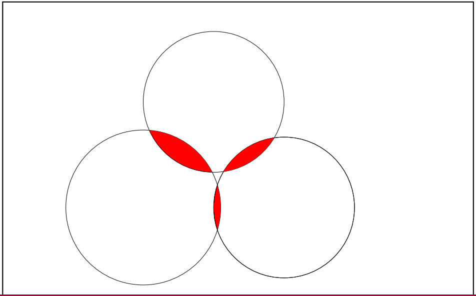
Most metric spaces are *not* tripod spaces. For instance, Riemannian manifolds of dimension > 1 do *not* satisfy tripod property.



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Our strategy: Define *curvature* as deviation from tripod property.

Existence of tripods if for any $x_1, x_2, x_3 \in X$ which do not lie on a geodesic, and $r_i + r_j \geq d(x_i, x_j)$, $1 \leq i < j \leq 3$,

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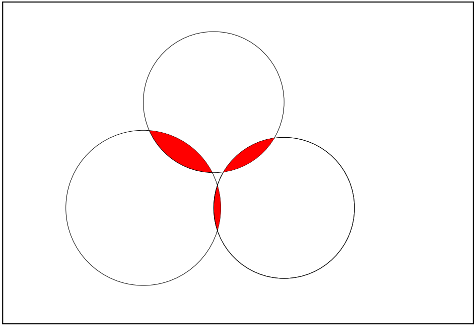

Not satisfied in this example, but for instance in a tripod

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$r_i + r_j = d(x_i, x_j)$ uniquely solved by Gromov products

$$\begin{aligned} r_1 &= \frac{1}{2}(d(x_1, x_2) + d(x_1, x_3) - d(x_2, x_3)), \\ r_2 &= \frac{1}{2}(d(x_1, x_2) + d(x_2, x_3) - d(x_1, x_3)), \\ r_3 &= \frac{1}{2}(d(x_1, x_3) + d(x_2, x_3) - d(x_1, x_2)). \end{aligned} \tag{2}$$

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\rho(x_1, x_2, x_3) := \inf_{x \in X} \max_{i=1,2,3} \frac{d(x_i, x)}{r_i}, \tag{4}
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Existence and uniqueness of weighted circumcenter for triangles in $CAT(0)$ spaces (Alexandrov's generalization of Riemannian manifolds of sect. curv ≤ 0)

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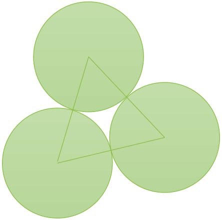
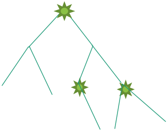
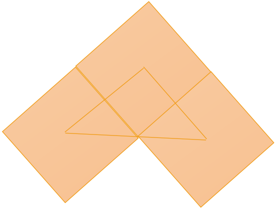
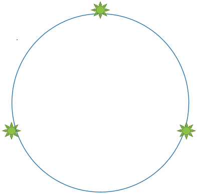
Hyperconvex spaces are tripod spaces.

Hyperconvex spaces

- 
- 
- 
- 

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Relation with Topological Data Analysis (TDA): For metric family $(x_i)_{i\in I}, d$ and $r > 0$, define the Čech complex containing a q-simplex whenever

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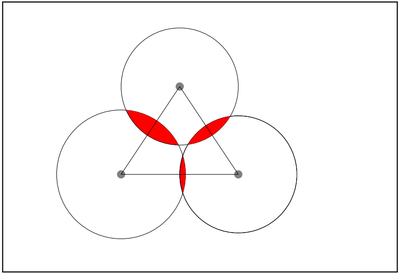

In this example, the triangle is not filled, because no triple intersection.

Topological Data Analysis (TDA): For metric family $(x_i)_{i \in I}, d$ and $r > 0$, define the Čech complex containing a q-simplex whenever

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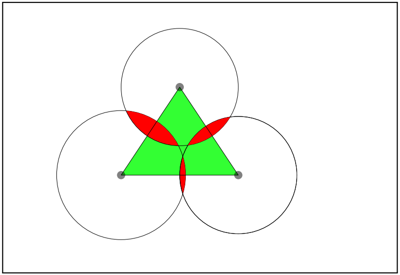
In contrast, in the Vietoris-Rips complex, simplices are filled whenever the balls around their vertices intersect pairwise.



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Variants

Definition

δ-hyperbolic ($\delta \geq 0$) if for any family $\{B(x_i, r_i)\}_{i \in I}$ with $r_i + r_j \geq d(x_i, x_j)$,

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For large radii, δ insignificant, good for asymptotic consideration, while (6) is invariant under scaling the metric d.

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Hilbert spaces are $\sqrt{2}$-hyperconvex. Reflexive and dual Banach spaces are 2-hyperconvex. Therefore, for measure space (X, μ), $L^p(X, \mu), 1 < p < \infty$, are 2-hyperconvex, and if X is finite, $L^1(X, \mu)$ is also 2-hyperconvex. $L^\infty(X, \mu)$ is 1-hyperconvex.

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In geometry, this is quantified by the concept of **curvature.**

From our abstract perspective, **curvature** relates intersection patterns of balls to convexity properties of distance functions.

Non-positive curvature

Definition

The geodesic space (X, d) is a $CAT(0)$ space if for all geodesics $c_1, c_2 : [0, 1] \longrightarrow X$ with $c_1(0) = c_2(0)$

$$d(c_1(t), c_2(s)) \leq \|\bar{c}_1(t) - \bar{c}_2(s)\|, \forall\ t, s \in [0, 1] \tag{7}$$

where $\bar{c}_1, \bar{c}_2 : [0, 1] \longrightarrow \mathbb{R}^2$ are the sides of the comparison triangle in $\mathbb{R}^2$ for $\triangle(c_1(0), c_1(1), c_2(1))$.

Triangles in $CAT(0)$ spaces are not thicker than Euclidean triangles with the same side lengths.

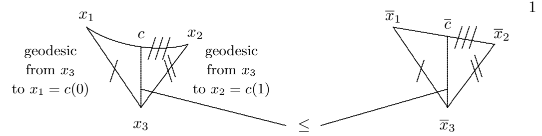

Figure 1: Comparison between a triangle in a space of nonpositive curvature in the sense of Alexandrov and the triangle with the same lengths of corresponding sides (indicated by slashes) in the Euclidean plane.

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A geodesic space (X, d) is a Busemann convex space if for every two geodesics $c_1, c_2 : [0, 1] \longrightarrow X$ with $c_1(0) = c_2(0)$, the distance function $t \mapsto d(c_1(t), c_2(t))$ is convex.

Geodesics in Busemann space diverge at least as fast as in Euclidean space.
Every $CAT(0)$ space is Busemann convex but not conversely.

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(X, d) has non-positive curvature if for each triple (x_1, x_2, x_3) in X with the comparison triangle $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$, one has

$$\rho(x_1, x_2, x_3) \leq \rho(\bar{x}_1, \bar{x}_2, \bar{x}_3),$$

where $\rho(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is similarly defined by

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For any triple of closed balls $\{B(x_i, r_i);\ i = 1, 2, 3\}$ with pairwise intersection, $\bigcap_{i=1,2,3} B(x_i, \rho r_i)$ is non-empty whenever $B(\bar{x}_i, \rho r_i)$, $i = 1, 2, 3$, have a common point. Thus, balls do not need to be enlarged more than in Euclidean case to get triple intersection.

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- Complete $CAT(0)$ spaces have non-positive curvature. Converse not true; in fact, our spaces need not be geodesic, nor have unique geodesics.
- Approximate version applies to discrete spaces.

Theorem

A complete Riemannian manifold (N, g) has non-positive curvature iff it has non-positive sectional curvature.

Proof

Idea of NPC $\Rightarrow$ Sec ≤ 0:
For triple $(a, b, c) \in N$, two Euclidean comparison triangles:
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By NPC, second smaller than first.
But then, geodesics diverge faster in N than in $\mathbb{R}^2$, hence Sec ≤ 0. □

Conclusions

Čech construction assigns to cover $\mathcal{U} = (U_i)_{i \in I}$ of X a simplicial complex $\Sigma(\mathcal{U})$ with vertex set I and a simplex σ_J whenever $\bigcap_{j \in J} U_j \neq \emptyset$ for $J \subset I$. When all intersections are contractible, the homology of $\Sigma(\mathcal{U})$ equals that of X (under some rather general topological conditions on X). When (X, d) is metric space, use covers by distance balls. Now, when (X, d) is a *hyperconvex* metric space, and if we use a cover $\mathcal{U}$ by distance balls, then whenever

$$\bigcap_{j \in J \setminus \{j_0\}} U_j \neq \emptyset \text{ for every } j_0 \in J, \tag{10}$$

then also

$$\bigcap_{j \in J} U_j \neq \emptyset, \tag{11}$$

i.e., whenever $\Sigma(\mathcal{U})$ contains all the boundary facets of some simplex, it also contains that simplex itself. No holes, and no corresponding homology groups.

When only λ-hyperconvexity for $\lambda > 1$ (or δ-hyperbolicity for $\delta > 0$), then nontrivial homology groups may arise. From that perspective, hyperconvex spaces are the simplest model spaces, and homology can be seen as a topological measure for the deviation from such a model.

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Translating geometry into topology

Definition

Let (X, d) be a metric space. We construct a simplicial complex $\check{X}$ whose vertex set is $\{(x, r) : x \in X, r \geq 0\}$. For every finite index set I with more than two points, let $(x_i, r_i)_{i \in I}$ be a set of vertices with all the $r_i > 0$ and x_i are different from each other, $\check{X}$ carries a simplex spanned by these vertices whenever

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The topology of the slices $r \equiv \mathrm{const}$ is evaluated in TDA.

Questions to explore

- Regularity theory for maps into such spaces
- New schemes for Geometric Data Analysis
- Asymptotic geometry of networks (with Areejit Samal)
- Geometry of hyperconvex and tripod spaces and their variants and generalizations

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