A curvature-free $\log(2k-1)$ Theorem

(joint work with Louis Merlin)

Recall the following celebrated result by Anderson - Canary - Culler - Shalen in 1996:

<u>$\log(2k-1)$ Theorem (ACCS - 96')</u>

If Γ is a Kleinian group freely generated by $\gamma_1, \dots, \gamma_k$, then $\forall \tilde{x} \in \mathbb{H}^3$

$$\sum_{i=1}^{k} \frac{1}{1+e^{d(\tilde{x}, \gamma_i \tilde{x})}} \leq \frac{1}{2}.$$

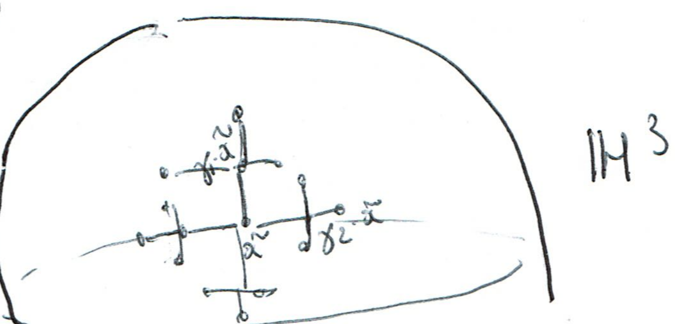
In particular, $\exists\, i_0 \in \{1, \dots, k\}$ s.t.

$$d(\tilde{x}, \gamma_{i_0} \tilde{x}) \geq \log(2k-1).$$

- Kleinian group is a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$
- Γ freely generated by $\gamma_1, \dots, \gamma_k$:

  $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle \simeq \mathbb{F}_k$ free group of rank k

$\rightsquigarrow$ If all displacements are small ($< \log(2k-1)$) then there should exist a relation between the generators of the group.



Together with Louis Merlin, we found the following

Curvature-free $\log(2k-1)$ Theorem:

Let $\tilde{M}$ be a simply connected and complete Riemannian manifold.
If Γ is a discrete subgroup of isometries of $\tilde{M}$ freely generated by $\gamma_1, \dots, \gamma_k$, then $\forall \tilde{x} \in \tilde{M}$ we have

$$\sum_{i=1}^{k} \frac{1}{1 + e^{\delta(\Gamma) \cdot d(\tilde{x}, \gamma_i \cdot \tilde{x})}} \leq \frac{1}{2}.$$

In particular $\exists i_0 \in \{1, \dots, k\}$ s.t.

$$\delta(\Gamma) \cdot d(\tilde{x}, \gamma_{i_0} \cdot \tilde{x}) \geq \log(2k-1).$$

- $\delta(\Gamma)$ = critical exponent of Γ

$$= \lim_{R \to +\infty} \frac{1}{R} \log \# \{ \gamma \in \Gamma \,/\, d(\tilde{x}, \gamma \cdot \tilde{x}) \leq R \}$$

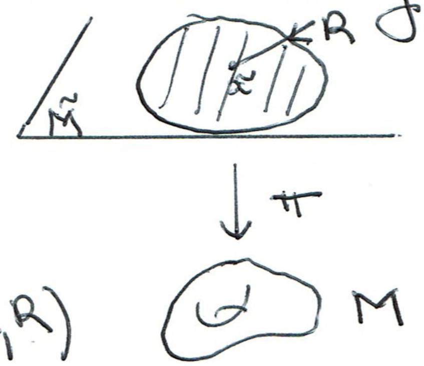
satisfies $\forall s \begin{array}{|l} > \\ < \end{array} \delta(\Gamma)$, $\sum_{\gamma \in \Gamma} e^{-s \cdot d(\tilde{x}, \gamma \cdot \tilde{x})} \begin{array}{|l} CV \\ DV \end{array}$

- If we rescale the Riem. metric by $\lambda > 0$, then $d(\tilde{x}, \gamma_i \cdot \tilde{x}) \rightsquigarrow \lambda \cdot d(\tilde{x}, \gamma_i \cdot \tilde{x})$
 and $\delta(\Gamma) \rightsquigarrow \frac{1}{\lambda} \delta(\Gamma)$.

- This result was previously known for pinched negative curvature $(-1 \leq K \leq -a^2 < 0)$
(Hou - 2001)

- For Kleinian groups, $\delta(\Gamma) \leq 2$ so we do not recover ACCS $\log(2k-1)$ Theorem.
- If Γ Kleinian with $\delta(\Gamma) < 1$, we improve it.

With Louis, we were interested into the following quantity: the volume entropy.



Let M be a closed Riem. mfd

$$\boxed{\mathrm{Ent}(M) = \lim_{R\to+\infty} \frac{1}{R} \log \mathrm{Vol}\, B(\tilde{x}, R)}$$

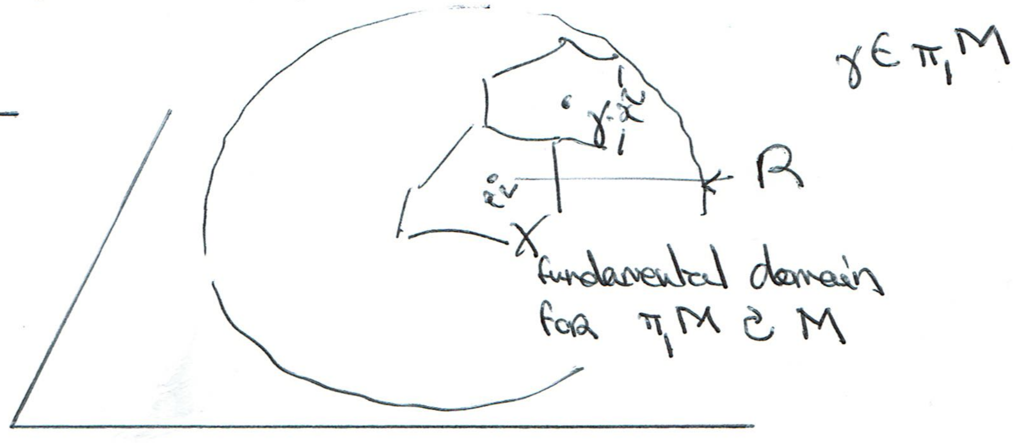
where $B(\tilde{x}, R) = \{\tilde{y} \in \tilde{M} \,/\, d(\tilde{x}, \tilde{y}) \leq R\} \subset \tilde{M}$

Riem. universal cover

By Manning '79, this limit always exists and does not depend on $\tilde{x}$.

So $\mathrm{Vol}\, B(\tilde{x}, R) = e^{\mathrm{Ent}(M) \cdot R + o(R)}$: the volume entropy describes the exponential growth rate of the volume on the universal cover.

Because



we see that

$$\mathrm{Ent}(M) = \lim_{R\to+\infty} \frac{1}{R} \log \#\{\gamma \in \pi_1 M \mid d(\tilde{x}, \gamma\cdot\tilde{x}) \leq R\}$$

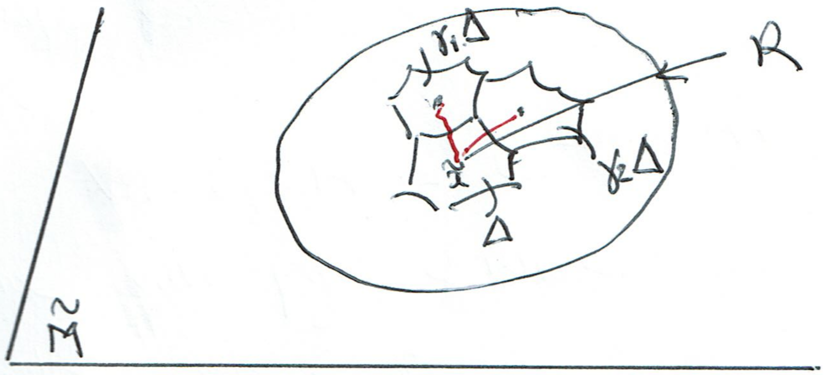
$$\rightsquigarrow \boxed{\mathrm{Ent}(M) = \delta(\pi_1 M)}.$$

(In general, $\Gamma \underset{\text{discrete}}{\subset} \mathrm{Isom}(\tilde{X}) \Rightarrow \delta(\Gamma) \leq \mathrm{Ent}(\tilde{X})$).

Theorem (Besson - Courtois - Gallot '95 & Katok '82):

> If (N^n, hyp) is a hyperbolic mfd, $\forall g$ Riem. metric on N we have
> $$\mathrm{Ent}(N,g)^n \cdot \mathrm{Vol}(N,g) \geq \mathrm{Ent}(N,hyp)^n \cdot \mathrm{Vol}(N,hyp).$$
> with $=$ iff $g = \lambda \cdot hyp$.

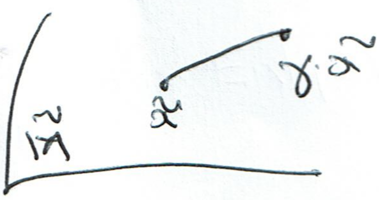
$\rightsquigarrow$ The smaller the volume, the bigger the volume entropy.

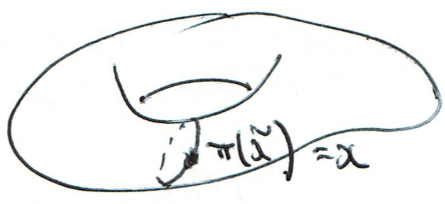


We wanted to understand the interplay between the displacements associated to a finite number of elements and the volume entropy.

For any $\gamma \in \pi_1 M$, we have

$$d(\tilde{x}, \gamma \cdot \tilde{x}) = \inf \{ \ell_g(c) \mid [c] = \gamma \text{ and } c \text{ is a loop based at } \pi(\tilde{x}) = x \}$$





Th (with Louis Merlin)

Let M be a Riem. closed mfd & $k \geq 2$.
If $c_1, \dots, c_k$ is a family of homotopically independent loops based at $x \in M$, then $\langle c_1, \dots, c_k \rangle \simeq \mathbb{F}_k$

$$\sum_{i=1}^{k} \frac{1}{1+e^{\mathrm{Ent}(M) \cdot \lg(c_k)}} \leq \frac{1}{2}.$$

In particular, $\exists\, i_0 \in \{1, \dots, k\}$ s.t.

$$\mathrm{Ent}(M) \cdot \lg(c_{i_0}) \geq \log(2k-1).$$

- Optimal : $\exists \{g_i\}_i$ sobre $\#_{i=1}^{k} (S^1 \times S^2)$: s.t. the sum above goes to $\frac{1}{2}$ when $i \to +\infty$.

⇝ McShane identity '98 :

If S is a hyperbolic punctured tori, then

$$\sum_{\gamma \text{ simple geod.}} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2}.$$

⇝ Collar lemma (Buser version):

If S is a closed hyp. surface, and γ_1, γ_2 two simple closed geod. s.t. $\gamma_1 \cap \gamma_2 = 1$, then

$$\operatorname{sh} \frac{\ell(\gamma_1)}{2} \cdot \operatorname{sh} \frac{\ell(\gamma_2)}{2} > 1$$

hyperbolic sine

In particular,

$$\ell(\gamma_2) \geq 2 \log\left(\frac{4}{\ell(\gamma_1)}\right) + o(1) \text{ when } \ell(\gamma_1) \to 0.$$

Corollary (with L. Merlin).

If c_1, c_2 are two homot. indep. loops based at the same point of a closed Riem. mfd M, then

$$\ell(c_2) \geq \frac{1}{\mathrm{Ent}(M)} \log\left(\frac{4}{\mathrm{Ent}(M)\,\ell(c_1)}\right) + o(1)$$

when $\ell(c_1) \to 0$.

Similar statement by BCG & Sambusetti (2017).