Refined invariants of flopping curves and finite-dimensional Jacobi algebras

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The Atiyah flop

- Let $Y \subset \mathbb{A}^4$ be the variety cut out by $xy - zw = 0$. It has an isolated singularity at the origin. Let $X \to Y$ be the blowup at 0, the exceptional fibre is $\mathbb{P}^1 \times \mathbb{P}^1$. We can contract either copy of $\mathbb{P}^1$, to obtain two resolutions of singularities $X^{\pm} \to Y$ each with exceptional fibre $\mathbb{P}^1$.
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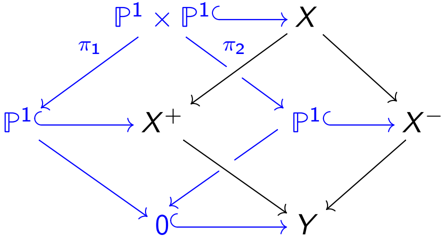

- In terms of the minimal model program (contracting curves C in a threefold Z with $C \cdot K_Z < 0$) both $X^{\pm}$ are just as good, since their canonical bundles are trivial. If the above diagram only models X locally, then X^+ and X^- give non-isomorphic minimal models.

General flops

- A flopping curve E in a smooth threefold X^+ is a rational curve with $E \cdot K_X = 0$ that is the unique exceptional fibre of some contraction $p\colon X^+ \to Y$. This morphism always fits into a diagram like on the previous page.
- (Kawamata): Any two minimal models of a smooth projective threefold are connected by a sequence of flops.
- For the rest of the talk we assume that Y is affine with isolated singularity 0 and study *local* invariants of $C \coloneqq p^{-1}(0) \cong \mathbb{P}^1$.

Enumerative invariants of flops

A trichotomy

$\mathcal{N}_C$	equation	length	n_1	$n_2 \ldots$	$n_{>length}$
$\mathcal{O}_C(-1) \oplus \mathcal{O}(-1)$	xy-zw	1	1	0	0
$\mathcal{O}_C(-2) \oplus \mathcal{O}_C$	$xy - (z^2 - w^{2d})$	1	d	0	0
$\mathcal{O}_C(-3) \oplus \mathcal{O}_C(1)$	*zoo!*	2...6	> 0	> 0	0

Gopakumar–Vafa invariants (after Bryan–Katz–Leung)

The invariants n_i can be defined as follows: there is a deformation

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow & \searrow & \\ \mathcal{Y} & \longrightarrow & B(0;\epsilon) \end{array}$$

over a small disc, with the fibre over 0 equal to $p\colon X \to Y$, and generically the fibre $X_t \to Y_t$ contains only $(-1,-1)$-curves. n_i is the number of such curves with homology class flowing to $i[C]$ over 0.

Jacobi algebras: definition

- Let Q be a quiver, i.e a directed graph, and let $\mathbb{C}Q$ be the free path algebra. E.g. if Q is the quiver with one vertex and three loops, then $\mathbb{C}Q \cong \mathbb{C}\langle x, y, z\rangle$.
- Let $W \in \mathbb{C}Q_{\mathrm{cyc}}$ be a linear combination of cyclic words of length at least three in Q. If a is an arrow of Q, and $W = b_1 b_2 \ldots b_s$ is a single cyclic word, we define

$$\partial W / \partial a = \sum_{b_i = a} b_{i+1} \ldots b_s b_1 \ldots b_{i-1}$$

  and extend linearly to arbitrary W.
- Let I_W be the two-sided ideal generated by all of the partial derivatives of W for a arrows of Q. Then we define

$$\mathrm{Jac}(Q, W) = \widehat{\mathbb{C}Q} / I_W.$$

Finite-dimensional Jacobi algebras

We will restrict to quivers with one vertex, so that Q is determined by the number of loops:

0. $\mathbb{C}Q = \mathbb{C}$, $W = 0$, $\mathrm{Jac}(Q, W) = \mathbb{C}$
1. $\mathbb{C}Q = \mathbb{C}[x]$, either $W = 0$ and $\mathrm{Jac}(Q, W) = \mathbb{C}[\![x]\!]$, or $W \neq 0$ and $\mathrm{Jac}(Q, W) \cong \mathbb{C}[x]/x^d$ where $d = \mathrm{ord}_0(W) - 1$.
2. $\mathbb{C}Q = \mathbb{C}\langle x, y\rangle$. There exist nonzero W (e.g. $x^3 + y^3$) for which $\dim_{\mathbb{C}}(\mathrm{Jac}(Q, W)) = \infty$, but for e.g. $W = x^2y + y^4$ Jacobi algebra is 9-dimensional. No easy classification of isomorphism classes of f.d. Jacobi algebras
3. $\mathbb{C}Q = \mathbb{C}\langle x, y, z\rangle$. Classic example $W = xyz - xzy$ gives $\mathrm{Jac}(Q, W) \cong \mathbb{C}[\![x, y, z]\!]$. Finding finite-dimensional examples seems very hard!

Contraction algebras

Flops $p\colon X \to Y$ provide a source of finite-dimensional Jacobi algebras.

Definition (Donovan–Wemyss)

Recall $C = p^{-1}(0)$. Consider the *noncommutative* deformation functor for $\mathcal{O}_C$, which sends an Artinian local algebra A to the set of iso classes of $A \otimes \mathcal{O}_X$-modules, flat over A, along with an iso $A/\mathfrak{m}_A \otimes \mathcal{O}_X \cong \mathcal{O}_C$. This functor is represented by a finite-dimensional noncommutative algebra A_{con}.

- Example: for the curve in a minimal resolution of $Z(xy - (z^2 - w^{2d}))$, $A_{\mathrm{con}} \cong \mathbb{C}[x]/x^d$.
- (Iyama–Wemyss): A_{con} is flop-invariant.
- One can use A_∞-deformation theory plus work of Hua+Keller to show that there is a Jacobi algebra presentation $A_{\mathrm{con}} \cong \mathrm{Jac}(Q, W)$ for Q a quiver with one vertex and $\mathrm{ext}^1(\mathcal{O}_C, \mathcal{O}_C)$-loops.

The conjectures

Conjecture (Donovan–Wemyss)

The algebra A_{con} is a complete flop-invariant of flopping curves

This has been partially proved by Zheng Hua and Bernhard Keller.

Conjecture (Brown–Wemyss)

(With restrictions on the quiver Q) all finite-dimensional Jacobi algebras arise this way.

Put together, these conjectures amount to the claim that the study of equivalence classes of flopping curves is essentially the same thing as the study of finite-dimensional Jacobi algebras!

Checking the Brown–Wemyss conjecture

$A_{\mathrm{con}} \cong \mathrm{Jac}(Q, W)$ where Q has $e = \mathrm{ext}^1(\mathcal{O}_C, \mathcal{O}_C)$-loops. The number e is determined by the degrees of the normal bundle $\mathcal{N}_C$. For $e =$

0. $\mathcal{N}_C \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. The Atiyah flop is the unique flop. $\mathrm{Jac}(Q, W) \cong \mathbb{C}$ is unique Jacobi algebra for the zero-loop quiver.
1. $\mathcal{N}_C \cong \mathcal{O}_C(-2) \oplus \mathcal{O}_C$. Reid's Pagoda provides a flop for each $d \in \mathbb{Z}_{\geq 2}$. $W \in \mathbb{C}Q = \mathbb{C}[x]$, and there is a f.d. Jacobi algebra for each value of $d = \mathrm{ord}_0(W) - 1$.
2. $\mathcal{N}_C \cong \mathcal{O}_C(-3) \oplus \mathcal{O}_C(1)$. Classifying flopping curves becomes difficult, as is classifying f.d. Jacobi algebras for two-loop quiver.
3. $\mathcal{N}_C \cong \mathcal{O}_C(-4) \oplus \mathcal{O}_C(2)$ (Pinkham): Not allowed! But recall that I didn't present any f.d. Jacobi algebras for the 3-loop quiver, indeed:

Theorem (Iyudu and Smoktunowicz)

If Q is a quiver with one vertex and ≥ 3 loops, there is no $W \in \mathbb{C}Q$ for which $\mathrm{Jac}(Q, W)$ is finite-dimensional.

BPS invariants for Jacobi algebras

Ingredients

1. For A a finitely generated $\mathbb{C}$-algebra, define $\mathrm{Hilb}_n(A) = \{(M, f) \mid M \text{ n-dimensional } A\text{-module}, f : A \twoheadrightarrow M\}$ e.g. $\mathrm{Hilb}_1(A) \cong \mathrm{Spec}(A/([A, A]))$.
2. For X a complex scheme, Behrend defined constructible function $\nu_X : X \to \mathbb{Z}$ dependent on singularities of X. E.g. $\nu_X(p) = d$ if p is unique point of $\mathrm{Spec}(\mathbb{C}[x]/x^d)$.

If A is a (non-formal...) Jacobi algebra, the BPS invariants $\omega_{A,i}$ are defined via

$$\mathcal{Z}_A(T) := \sum_{i\geq 0} \chi(\mathrm{Hilb}_i(A), (-1)^i \nu) T^i = \prod_{i\geq 1} (1 - T^i)^{i\cdot\omega_{A,i}}$$

- $\mathcal{Z}_{\mathbb{C}[x,y,z]}(T) = \prod_{i\geq 1}(1 - T^i)^{-i}$ ($\omega_{A,i} = -1$ for all i)
- $\mathcal{Z}_{\mathbb{C}[x]/x^d}(T) = (1 - T)^d$ ($\omega_{A,1} = d$ and $\omega_{A,i} = 0$ for $i \geq 2$)

Another test for the Brown–Wemyss conjecture

- By work of Katz, Toda, Hua-Toda, for A_{con} the contraction algebra for a flopping curve C, there is an equality $\omega_{A_{\mathrm{con}},i} = n_i$, where n_i are the Gopakumar–Vafa invariants for C.
- The BPS invariants for a Jacobi algebra can be either negative or positive.
- However, the Brown–Wemyss conjecture implies the following

Conjecture

Let $A = \mathrm{Jac}(Q, W)$ be a finite-dimensional Jacobi algebra. Then all of the BPS invariants $\omega_{A,i}$ are positive integers.

The positivity theorem

Theorem

Let $A = \mathrm{Jac}(Q, W)$ be a finite-dimensional Jacobi algebra. Then all of the BPS invariants $\omega_{A,i}$ are positive integers.

- Proof starts with the observation that

$$\mathcal{Z}_{\mathrm{Jac}(Q,W)}(T) = \sum_{i \geq 0} \chi\left(\mathrm{H}(\mathrm{Hilb}_i(\mathbb{C}Q), \phi_{\mathrm{Tr}(W)})[-i]\right) T^i$$

- Via work with Sven Meinhardt, isomorphism of graded vector spaces

$$\bigoplus_{i \geq 0} \mathrm{H}\left(\mathrm{Hilb}_i(\mathbb{C}Q), \phi_{\mathrm{Tr}(W)}[-i]\right) \cong \mathrm{Sym}\left(\bigoplus_{i \geq 1} \mathrm{BPS}_{Q,W,i} \otimes \mathrm{H}(\mathbb{P}^{i-1}, \mathbb{Q})[1]\right)$$

where $\mathrm{BPS}_{Q,W,i}$ are cohomologically graded vector spaces satisfying $\chi(\mathrm{BPS}_{Q,W,i}) = \omega_i$. If the LHS is finite-dimensional, each $\mathrm{BPS}_{Q,W,i}$ must live in even cohomological degree, so $\chi(\mathrm{BPS}_{Q,W,i}) \geq 0$.

(Okke van Garderen): There are non-iso flops with isomorphic $\mathrm{BPS}_{A_{\mathrm{con}},i}$ $\forall i$.