Kähler moduli spaces on non-Kähler Calabi-Yau manifolds

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Based on joint work with Rubio and Tipler arXiv:2004.11399

Moduli spaces of Calabi-Yau metrics play an important role in geometry and string theory

$$hol(g) \subset SU(n).$$

Our understanding of these moduli spaces is via a *separation of variables:* any such metric determines a complex n-dimensional manifold X endowed with a holomorphic volume form Ω

$$K_X := \Lambda^n T^* X \cong_\Omega \mathcal{O}_X.$$

Theorem (Yau 1977)

Let (X, Ω) be a compact Calabi-Yau manifold of dimension n. Then, for any Kähler class $\tau \in H^{1,1}(X, \mathbb{R})$ there exists a unique Kähler metric g such that $[\omega] = \tau$ and

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}.$$

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Thus, the moduli problem for Calabi-Yau metrics in dimension 6 reduces to study moduli spaces of Kähler Calabi-Yau three-folds ($h^{0,2}(X) = 0$)

$$\begin{array}{ccc} H^2(X,\mathbb{C}) \cong H^{1,1}(X) & \longrightarrow & \mathcal{M}_{g,CY} \\ & & \downarrow \\ & & \mathcal{M}_{X,\Omega} \end{array}$$

Both fibres and base of $\mathcal{M}_{g,CY}$ have an interesting special Kähler metric which plays an important role in mirror symmetry, with Kähler potential

$$K = -\log \int_X \|\Omega\| \omega^3.$$

Motivated by Type II string theory, fundamental work by Bridgeland shows that the *Kähler moduli* $\mathcal{K} \subset H^{1,1}(X)$ admits a *stringy deformation*:

$$Stab(X) \to \mathcal{K}.$$

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In this talk, we are concerned with *stringy deformations* of the '*Kähler moduli*' of a compact Calabi-Yau manifold, motivated by the heterotic string.

Key ingredients in our story are:

- Our (X, Ω) may be non-Kähler!
- X is decorated with a holomorphic vector bundle E, satisfying

$$c_1(E) = 0, \qquad c_2(E) = c_2(X).$$

The Calabi problem

(through the eyes of the heterotic string)

In the 1950s, E. Calabi asked the question of whether one can prescribe the volume form μ of a Kähler metric on a compact complex manifold X.

For metrics on a fixed Kähler class $[\omega_0] \in H^{1,1}(X, \mathbb{R})$, the *Calabi Problem* reduces to solve the Complex Monge-Ampère equation for a smooth function φ on X:

$$(\omega_0 + 2i\partial\bar{\partial}\varphi)^n = n!\mu.$$

Theorem (Yau 1977)

Let X be a compact Kähler manifold with smooth volume μ. Then there exists a unique Kähler metric with $\omega^n = n!\mu$ in any Kähler class.

Provided that X admits a holomorphic volume form Ω, taking μ as below reduces the holonomy of the metric to $SU(n)$ (Calabi-Yau metric)

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In 2011, Joel Fine gave a moment map interpretation of the Calabi problem using a gauge theory framework.

'Despite the fact that Yau has long since resolved the Calabi conjecture, this moment-map picture does raise interesting questions ...'

- `Fine, J. Symp. Geom.` **12**, `2011.`

In this talk, we shall give a different moment map picture, *through the eyes of the heterotic string*.

- `GF, Rubio, Tipler, arXiv:2004.11399`

X compact complex manifold of dimension n, possibly non-Kähler. Define

$$\Omega^{1,1}_{>0} = \{\omega \mid \omega(, J) > 0\} \subset \Omega^{1,1}_{\mathbb{R}}$$

and consider the tangent bundle

$$T\Omega^{1,1}_{>0} = \{(\omega, b)\} \cong \Omega^{1,1}_{>0} \times \Omega^{1,1}_{\mathbb{R}},$$

endowed with the complex structure

$$\mathbf{J}(\dot{\omega}, \dot{b}) = (-\dot{b}, \dot{\omega}).$$

Consider the partial action of the additive group of complex two-forms

$$\begin{aligned} \Omega^2_{\mathbb{C}} \times T\Omega^{1,1}_{>0} &\to T\Omega^{1,1}_{\mathbb{R}} \\ (B, (\omega, b)) &\mapsto (\omega + Re\ B^{1,1}, b + Im\ B^{1,1}). \end{aligned} \tag{1}$$

We study a Hamiltonian action of the subgroup of $i\Omega^2 \subset \Omega^2_{\mathbb{C}}$ for a natural family of Kähler structures on $T\Omega^{1,1}_{>0}$.

To define our family of Kähler structures, we fix a smooth volume form μ on X. For any $\omega \in \Omega^{1,1}_{>0}$, we define the *dilaton function* f_ω by

$$\omega^n = n! e^{2f_\omega} \mu.$$

Definition: the *dilaton functional* on $T\Omega^{1,1}_{>0}$ is

$$M(\omega, b) := \int_X e^{-f_\omega} \frac{\omega^n}{n!}.$$

There is a pseudo-Kähler structure $\Omega := -d\mathbf{J}d \log M$. The associated metric is (the subscript 0 means primitive):

$$\begin{aligned} g = {} & \frac{1}{2M} \int_X (|\dot\omega_0|^2 + |\dot b_0|^2) e^{-f_\omega} \frac{\omega^n}{n!} \\ & + \frac{1}{2M} \left(\frac{1}{2} - \frac{n-1}{n} \right) \int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2) e^{-f_\omega} \frac{\omega^n}{n!} \\ & + \left(\frac{1}{2M} \right)^2 \left(\left(\int_X \Lambda_\omega \dot\omega e^{-f_\omega} \frac{\omega^n}{n!} \right)^2 + \left(\int_X \Lambda_\omega \dot b e^{-f_\omega} \frac{\omega^n}{n!} \right)^2 \right). \end{aligned}$$

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Proposition (GF, Rubio, Tipler)

The $i\Omega^2$-action on $(T\Omega^{1,1}_{>0}, \Omega)$ is Hamiltonian, with equivariant moment map μ.

$$\langle \mu(\omega, b), iB \rangle = \frac{1}{2M} \int_X B \wedge e^{-f_\omega} \frac{\omega^{n-1}}{(n-1)!}.$$

Upon restriction to the subgroup $i\Omega^2_{ex} \subset i\Omega^2$ of imaginary exact 2-forms, zeros of the moment map are given by conformally balanced metrics $d(e^{-f_\omega}\omega^{n-1}) = 0$.

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An interesting upshot of our picture is that the balanced property for a compact complex manifold arises as a balancing condition, analogue to the *zero center of mass*.

Assume X pluriclosed. Fix a real closed three-form $H_{\mathbb{R}} \in \Omega^3$ ($\sim$ NS flux in string theory), $dH_{\mathbb{R}} = 0$, and consider the complex subspace

$$\mathcal{W} = \{(\omega, b) \mid d^c\omega - db = H_{\mathbb{R}}\} \subset T\Omega^{1,1}_{>0}.$$

Observe: $i\Omega^2_{ex}$ are *symmetries* for $H_{\mathbb{R}}$.

Proposition (GF, Rubio, Tipler)

The $i\Omega^2_{ex}$-action on $(\mathcal{W}, \Omega$ is Hamiltonian. Zeros of μ are given by solutions of

$$d(e^{-f_\omega}\omega^{n-1}) = 0, \qquad dd^c(\omega + ib) = 0. \tag{2}$$

Observe: Equation (2) implies ω Kähler, $d\omega = 0$, and $f_\omega = const$.

Thus, the symplectic reduction $\mathcal{M} = \mu^{-1}(0)/i\Omega^2_{ex}$ can be identified with the moduli space of (complexified) solutions of the Calabi problem on X (for varying $c \in \mathbb{R}$)

$$\omega^n = n!c\mu, \qquad d\omega = 0.$$

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Theorem (GF, Rubio, Tipler)

The moduli space $\mathcal{M} := \mu^{-1}(0)/i\Omega^2_{ex}$ of (complexified) solutions of the Calabi problem on X inherits a Kähler structure with Kähler potential $-\log \int_X \frac{\omega^n}{n!}$.

Remark: by Yau's Theorem $\mathcal{M} \subset H^{1,1}(X)$.

Remark: When (X, Ω) is a Calabi-Yau three-fold and we take

$$\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}$$

we recover the Weil-Petersson metric on the complexified Kähler moduli of X.

- Candelas, De la Ossa, Moduli space of Calabi-Yau manifolds, Nuclear Phys. **B 355** (1991)

Kähler moduli for Hull-Strominger

The mathematical study of the **Hull-Strominger system** was initiated by Fu, Li, Tseng, and Yau more than 15 years ago.

$$F \wedge \omega^2 = 0 \qquad F^{2,0} = F^{0,2} = 0$$

$$d(\|\Omega\|\omega^2) = 0 \qquad dd^c\omega - \alpha \operatorname{tr} R \wedge R + \alpha \operatorname{tr} F \wedge F = 0$$

- Li, Yau, JDG 70 (2005).
- Fu, Yau, JDG 78 (2008).
- Fu, Tseng, Yau, CMP 289 (2009).

Basic ingredients:

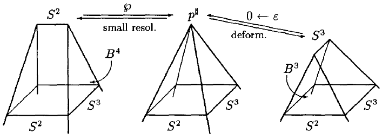
- A hermitian form ω on a Calabi-Yau threefold (X, Ω) (possibly non-Kähler).
- A unitary connection A on a bundle over X, with curvature $F = F_A$.
- A connection ∇ on $T\underline{X}$, with curvature $R = R_\nabla$

Due to its origins in heterotic string theory, ∇ is often required to be Hermite-Yang-Mills:

$$R \wedge \omega^2 = 0, \qquad R^{2,0} = R^{0,2} = 0.$$

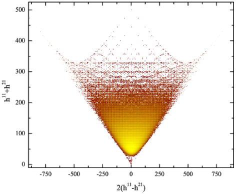
- Strominger, Nucl. Phys. B 274 (1986).
- Hull, Turin 1985 Proceedings (1986).

These equations provide a promising approach to the geometrization of *transitions* and *flops* in the passage from Kähler to non-Kähler Calabi-Yau three-folds ($\sim$ *Reid's Fantasy*) ...



- M. Reid, Math. Ann. **278** (1987) 329--334

... and relate to a conjectural generalization of *mirror symmetry* and *GW* theory, where the Calabi-Yau is endowed with a bundle E such that $c_2(E) = c_2(X)$.



- Melnikov, Plesser, *A* $(0,2)$*-mirror map*, JHEP 1102 (2011) • Donagi, Guffin, Katz, Sharpe, Asian J. Math. 18 (2014) • Garcia-Fernandez, Crelle's J. (2000)

Many non-Kähler solutions of the Hull-Strominger system has been constructed over the last 15 years

• DG: Yau, Li, Fu, Tseng, Fernandez, Ivanov, Ugarte, Villacampa, Grantcharov, Fino, Vezzoni, Andreas, GF, Rubio, Tipler, Fei, Phong, Picard, Zhang, Shahbazi, ... • Hep-th: De la Ossa, Svanes, Anderson, Gray, Sharpe, Ashmore, Minasian, Strickland-Constable, Waldram, Tennyson, Candelas, McOrist, Larfors, ...

(1) $d\Omega = 0,\ F_A^{0,2} = 0,\ R_\nabla^{0,2} = 0$ (2) $F_A \wedge \omega^2 = 0,\ R_\nabla \wedge \omega^2 = 0$
(3) $d(\|\Omega\|\omega^2) = 0,$ (4) $dd^c\omega = \alpha\, \mathrm{tr}\, R_\nabla^2 - \alpha\, \mathrm{tr}\, F_A^2,$

There are two important cohomological quantities attached to a solution:

- The *balanced class* in Bott-Chern cohomology, which enables to define an algebraic stability, equivalent to (2):

$$[\|\Omega\|\omega^2] \in H^{2,2}_{BC}(X) = \frac{\mathrm{Ker}\, d : \Omega^{2,2} \to \Omega^{3,2} \oplus \Omega^{2,3}}{\mathrm{Im}\, dd^c : \Omega^{1,1} \to \Omega^{2,2}}$$

- The *string class* in $H^3(P, \mathbb{R})$, where $P = P_G \times_M P_M$, with P_M the frame bundle of M

$$\tau = [p^* d^c\omega + \alpha CS(\nabla) - \alpha CS(A)] \in H^3(P, \mathbb{R}).$$

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Theorem (GF, Rubio, Tipler)

Let (X, Ω) be a Calabi-Yau n-manifold (possibly non-Kähler) endowed with a hermitian bundle (E, h) such that

$$h^{0,1}_A(X) = 0, \qquad h^{0,2}_{\bar\partial}(X) = 0, \qquad c_1(E) = 0, \qquad c_2(E) = c_2(X).$$

The moduli space of solutions of Hull-Strominger with fixed string class satisfying **Condition A** inherits a (possibly degenerate) pseudo-Kähler structure

$$\begin{aligned}
g_\alpha = & -\frac{\alpha}{M}\int_X \langle\dot\theta \wedge J\dot\theta\rangle \wedge e^{-f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\
& + \frac{1}{2M}\int_X (|\dot\omega_0|^2 + |\dot b_0|^2)e^{-f_\omega}\frac{\omega^n}{n!} \\
& + \frac{1}{2M}\left(\frac{1}{2} - \frac{n-1}{n}\right)\int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2)e^{-f_\omega}\frac{\omega^n}{n!} \\
& + \left(\frac{1}{2M}\right)^2\left(\left(\int_X \Lambda_\omega \dot\omega e^{-f_\omega}\frac{\omega^n}{n!}\right)^2 + \left(\int_X \Lambda_\omega \dot b e^{-f_\omega}\frac{\omega^n}{n!}\right)^2\right).
\end{aligned}$$

with Kähler potential $K = -\log\int_X \|\Omega\|_\omega \omega^n$.

By construction, the moduli space $\mathcal{M}$ has a natural map to the classical moduli space of holomorphic principal G-bundles $\mathcal{M}_{bundles}$, with $G = SL(r, \mathbb{C}) \times SL(n, \mathbb{C})$,

$$\mathcal{M} \to \mathcal{M}_{bundles}. \tag{3}$$

Conjecturelly, the fibres can be identified with (a quotient of) a subspace of the Aeppli cohomology group

$$H_A^{1,1}(X) = \frac{ker\ \partial\bar{\partial}}{Im\ \partial \oplus \bar{\partial}}$$

Theorem (GF, Rubio, Tipler)

Assume $h_A^{0,1}(X) = 0$, $h_{\bar{\partial}}^{0,2}(X) = 0$, and **Condition A**. Then, the metric g_α along the fibres of (3) is given by

$$g_\alpha = \frac{1}{2M}\left(\frac{1}{2M}(Re\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Re\ \dot{\mathfrak{a}} \cdot Re\ \dot{\mathfrak{b}} + \frac{1}{2M}(Im\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Im\ \dot{\mathfrak{a}} \cdot Im\ \dot{\mathfrak{b}}\right)$$

Here, $\dot{\mathfrak{b}} \in H_{BC}^{n-1,n-1}(X)$, $\dot{\mathfrak{a}} \in H_A^{1,1}(X)$ are 'complexified variations'–obtained via *gauge fixing*–of the balanced class $\mathfrak{b}$ and the Aeppli class $\mathfrak{a}$ of a solution and $\cdot$ is

$$H_A^{1,1}(X) \otimes H_{BC}^{n-1,n-1}(X) \to \mathbb{C}.$$

Remark: Aeppli classes for solutions of the Bianchi identity are defined via Bott-Chern secondary classes BCh_2

$$\mathfrak{a}_1 - \mathfrak{a}_0 = [\omega_1 - \omega_0 - BCh_2] \in H_A^{1,1}(X, \mathbb{R}).$$

Our formula for the moduli metric along the fibres of $\mathcal{M} \to \mathcal{M}_{bundles}$ shows that g_α is 'semi-topological': fibre-wise it can be expressed in terms of classical cohomological quantities.

$$g_\alpha = \frac{1}{2M}\left(\frac{1}{2M}(Re\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Re\ \dot{\mathfrak{a}} \cdot Re\ \dot{\mathfrak{b}} + \frac{1}{2M}(Im\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Im\ \dot{\mathfrak{a}} \cdot Im\ \dot{\mathfrak{b}}\right)$$

When (X, Ω) is a Kähler CY3 $H_A^{1,1}(X) \cong H^{1,1}(X)$ and, as $\alpha \to 0$, we recover *Strominger's formula* for the metric on the *complexified Kähler moduli* of X.

- Strominger, Phys. Rev. Lett. 55 (1985)

Observe: the holomorphic prepotential seems to break as soon as we split the Kähler class into Aeppli $\mathfrak{a}$ and Bott-Chern $\mathfrak{b}$.

$$H^{1,1}(X) \to \mathbb{C} \colon [\alpha] \mapsto \int_X \alpha^3 + \textit{quantum corrections} (\sim GW).$$

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$$\mathfrak{a}_1 - \mathfrak{a}_0 = [\omega_1 - \omega_0 - BCh_2] \in H^{1,1}_A(X, \mathbb{R}).$$

Our formula for the moduli metric along the fibres of $\mathcal{M} \to \mathcal{M}_{bundles}$ shows that g_α is 'semi-topological': fibre-wise it can be expressed in terms of classical cohomological quantities.

$$g_\alpha = \frac{1}{2M}\left(\frac{1}{2M}(Re\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Re\ \dot{\mathfrak{a}} \cdot Re\ \dot{\mathfrak{b}} + \frac{1}{2M}(Im\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Im\ \dot{\mathfrak{a}} \cdot Im\ \dot{\mathfrak{b}}\right)$$

When (X, Ω) is a Kähler CY3 $H^{1,1}_A(X) \cong H^{1,1}(X)$ and, as $\alpha \to 0$, we recover *Strominger's formula* for the metric on the *complexified Kähler moduli* of X.

- Strominger, Phys. Rev. Lett. 55 (1985)

Observe: the holomorphic prepotential seems to break as soon as we split the Kähler class into Aeppli $\mathfrak{a}$ and Bott-Chern $\mathfrak{b}$.

$$H^{1,1}(X) \to \mathbb{C} \colon [\alpha] \mapsto \int_X \alpha^3 + \textit{quantum corrections } (\sim GW).$$

Proposition (GF, Rubio, Tipler)

Let (X, Ω) be a Kähler Calabi-Yau n-manifold endowed with a stable holomorphic vector bundle E such that

$$c_1(E) = 0, \qquad c_2(E) = c_2(X).$$

Then, there exists $\epsilon_0 > 0$ and a smooth family of solutions of Hull-Strominger parametrized by $\alpha \in [0, \epsilon_0[$ such that **Condition A** holds for small $\alpha > 0$.

Example: Let X be a complete intersection Calabi-Yau threefold. By a result of Huybrechts, TX has unobstructed deformations parametrized by $H^1(EndTX)$. Since TX is stable for any Kähler class, any pair of small deformation E of TX is also stable. For the quintic hypersurface

$$h^1(EndTX) = 224$$

and we obtain a family of deformations of the special Kähler metric on $H^{1,1}(X)$ parametrized by a non-empty open subset of

$$H^1(EndTX) \times [0, \epsilon_0[.$$

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Moltes gràcies!