STABILITY OF SYZYGY ON ABELIAN VARIETIES

jw F. Caucci.

(X, L) a polarized smooth variety. Suppose that L is globally generated.

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \xrightarrow{ev.} L \to 0$$

M_L syzygy bundle
kernel bundle.

Stability of M_L?

Stability $\rightsquigarrow$ slope stability: $\forall F \hookrightarrow M_L$ if $\mu_L(F) = \dfrac{c_1(F) \cdot L^{\dim X - 1}}{\mathrm{rk}(F)}$

we have. $\mu_L(F) < \mu_L(M_L)$

<u>Known results</u>:

(1) On $\mathbb{P}^n_k$ $M_{\mathcal{O}_{\mathbb{P}^n}(d)}$ slope semistable if $d > 0$ (in arbitrary char).

(2) X smooth proj curve M_L is semistable $\deg L \geq 2g$ (Ein - Lazarsfeld)

(3) X smooth surface M_{L^d} is semistable for $d \gg 0$.

(Ein - Lazarsfeld - Mustopa)

generalizing result Camere

They also study 3fold with Picard rk 1. M_L^d is semistable for $d \gg 0$.

<u>Conjecture</u> [ELM] X smooth proj of arbitrary dim. M_L^d is semist. for $d \gg 0$.

Let X be an abelian variety.

Let L be globally generated line bundle. (we allow char k to be positive)

<u>Thm</u> [Caucci - L] If $\forall Z \subseteq X$ nonzero abelian subv we have

$$(L^{\dim Z} \cdot Z) \geq \frac{g!}{(g - \dim Z)!}$$

then M_L is slope stable wrt L.

where $g = \dim X$

Note if X is simple (no abelian subv) then M_L is slope stable as soon as L is g.g.

<u>Thm</u> [CL] M_{L^2} is slope semistable $\forall$ abelian variety.

This solves ELM conjecture for abelian varieties.

Why is M_L interesting?

Why stability of M_L?

(N_p) Property: Given L g.g. $\Phi_L : X \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}^r$ $r = h^0(L) - 1$.

How is the ideal $I_{\Phi_L(X)/\mathbb{P}^r}$

Let X be a curve. | If $\deg(L) \geq 2g+1$ $\rho_m : \mathrm{Sym}^m H^0(L) \twoheadrightarrow H^0(L^{\otimes m})$ are surj $\forall m \gg 0$

$\Phi_L(X)$ is a proj. normal variety.

It allows you to count the number of hypers. of degree m passing through $\Phi_L(X)$ which $\dim \ker \rho_m$ via RR.

If $\deg L \geq 2g+2$ (X curve), the $I_{\Phi(L)/\mathbb{P}^r}$ is generated by quadrics.

Now, let. $S = \mathrm{Sym}^m H^0(L)$

$$R(L) = \bigoplus_m H^0(L^{\otimes m})$$ which f.g over S.

The generalization is that $\cdots \to E_{r-1} \to E_{r-2} \to \quad \cdots \to E_1 \to E_0 \to R(L) \to 0$

minimal free resolution

And for $\deg L \geq 2g+p+1$ (X curve)

The matrices of this minimal resolution only have linear entries.

Equivalently

Defn For $p \geq 0$ we say that L satisfies property (N_p) iff

$$E_0(L) \simeq S \quad \text{and} \quad E_i(L) \simeq \bigoplus S(-i-1) \quad \forall\, 1 \leq i \leq p.$$

Proposition If L is g.g. and $H^1(L) = 0$

L satisfies the (N_1) property iff $\quad H^1(X, \Lambda^{p+1} M_L \otimes L^k) = 0 \quad \forall k \geq 1.$

Idea of the proof of this prop. use the Koszul resolution.

Suppose X is a curve:

$$H^1(\Lambda^{p+1} M_L \otimes L^k) \overset{?}{=} 0$$

$$\wr\!\wr$$

$$\operatorname{Ext}^1(\mathcal{O}_X, \Lambda^{p+1} M_L \otimes L^k) \underset{SD}{=} \operatorname{Hom}(\Lambda^{p+1} M_L, L^{-k} \otimes K_C)^\vee$$

If M_L is stable $\underset{\text{Char } k = 0}{\Longrightarrow}$ $\Lambda^{p+1} M_L$ is stable (polystable)

If $\mu(\Lambda^{p+1} M_L) = \dfrac{-(p+1)\deg L}{h^0(L)-1} > \mu(L^{-k} \otimes K_C)$

$\Big\}$ $\Longrightarrow$ stability would provide $H^1(\Lambda^{p+1} M_L \otimes L^k) = 0$ $\forall k \geq 1$.

Some condition to guarantee property N_p (not the best one directly ...)

Idea of the proof of Thm (M_L stable).

Hodge-type inequality + Green's strategy to prove cohomological stability (Ein-Lazarsfeld).

Idea of the proof of Thm (M_{L^2} is semistable $\forall X$ abelian variety)

Semistable reduction: we use that simple abelian var are dense in the moduli space

↳ to construct a semistable bundle via properness

↳ check that this semistable bundle is isom. to M_{L^2}.

Use L^2 to ensure it is glob.-gen.

For example if X is an abelian 3fold.

$$H^1(X, \Lambda^{p+1} M_L \otimes L^k) \neq 0$$

$$\shortparallel$$

$$\mathrm{Ext}^1(\mathcal{O}_x, \Lambda^{p+1} M_L \otimes L^k) \underset{\substack{\text{3fold} \\ K_X = 0}}{=} \mathrm{Ext}^2(\Lambda^{p+1} M_L \otimes L^k, \mathcal{O}_x)^\vee$$

$$= \mathrm{Hom}(\Lambda^{p+1} M_L \otimes L^k, \mathcal{O}_x[2])^\vee$$

$$\mathcal{O}_x[1] \to E_p \to \Lambda^{p+1} M_L \otimes L^k \to \mathcal{O}_x[2]$$

Use Bridgeland stab cond
$\mathcal{O}_x[1]$ stable.

dist. triang.

semistable ?

Given Bridgeland stab cond on abelian 3fold

or tilt stab on abelian 3 fold.

Suppose $\Lambda^{p+1} M_2 \otimes L^k \in \mathrm{Coh}^\beta(X)$

we know it is tilt-semistable

for $\alpha >> 0$

<u>Question</u>: How long it remains stable

$\alpha \searrow 0$?