Clifford algebras, spinors and applications
Problem Sheet 2

31 January 2017

Problem 1.

We dealt with Clifford algebras of direct sums of quadratic vector spaces. Now we want to look at subspaces:

a) Show that a vector subspace W of a quadratic vector space (V, Q) naturally inherits the structure of a quadratic subspace.

b) Is $Cl(W, Q_W)$ naturally a subalgebra of $Cl(V, Q)$?

Problem 2.

At some point we introduced the quadratic vector space $V + V^*$. We just need to give a vector space V and we have a canonical quadratic form in $V + V^*$: for $X + \alpha \in V + V^*$,

$$Q(X + \alpha) = i_X \alpha.$$

This vector space has a very special basis: take any basis $\{e_i\}$ of V, and take its dual basis $\{e^i\}$ in V^*, i.e., the one such that $e^i(e_j) = \delta^i_j$. Their union $\{e_i\} \cup \{e^i\}$ is then a basis of $V + V^*$, and hence a generating set of the Clifford algebra $Cl(V + V^*)$.

a) Compute, in the Clifford algebra, $(e_i)^2$, $(e^i)^2$, $e_i e_j + e_j e_i$, $e^i e^j + e^j e^i$, and $e_i e^j + e^j e_i$.

Consider $V^* \subset V + V^*$ and the subalgebra $Cl(V^*) \subset Cl(V + V^*)$.

b) Do we have a model for $Cl(V^*)$ that we knew before Clifford algebras?

c) Is $Cl(V^*)$ a left ideal? In other words, is it preserved under the action of elements $X + \alpha \in V + V^*$ by (left) Clifford multiplication?

Define $S = Cl(V^*) e_\Omega$, where e_Ω is a volume element for $Cl(V)$, i.e., $e_\Omega = e_1 \ldots e_n$.

d) Is S a left ideal?

e) Describe the action of $X + \alpha \in V + V^*$ on S using part b) and well-known operators.