Clifford algebras, spinors and applications
Problem Sheet 3

14 February 2017

Problem 1.

We had two descriptions of the group Spin:

$$Spin(V, Q) = \{g \in (Cl^{ev}(V, Q))^{\times} \ : \ gVg^{-1} = V,\, \Delta(g) = \pm 1\},$$

$$Spin(V, Q) = \{v_1 \dots v_{2r} \ ; \ r \geq 0,\, v_j \in V,\, Q(v_j) = \pm 1\}.$$

With the first definition, we deduced that

$$Spin(2, 0) = \{a + be_1e_2 \ : \ a^2 + b^2 = 1\}.$$

Its elements are not expressed as a product of unit vectors, but as a linear combination of elements in the Clifford algebra, which does not trivially fit the second definition.

a) Is $a + be_1e_2$ a product of an even number of unit vectors? How?

b) Conversely, can we write any product of an even number of unit vectors as $a + be_1e_2$ with $a^2 + b^2 = 1$?

Problem 2.

Describe $Spin(4, 0)$ and $Spin^+(1, 3)$ using the Clifford algebra. Recall that the subscript $+$ refers to $\Delta(g) = +1$. How different are $Spin(r, s)$ and $Spin(s, r)$?

Problem 3

We proved with all detail what the irreducible representations of $Cl(r, s)$ are, but we did not prove that every representation of $Cl(r, s)$ is necessarily the sum of irreducibles, i.e., is completely reducible. You are very welcome to search around in any references to answer these two questions.

a) Find a representation (of an algebra) that is not completely reducible.

b) Prove that any representation of $Cl(r, s)$ is completely reducible.