Dirac structures and generalized geometry
Problem Sheet 1

Friday 8th January 2016,
hand in by Thursday 14th

Let M be a smooth manifold with tangent bundle TM.

Problem 1

A Lie algebroid A over M is a vector bundle $A \to M$ together with:

- a Lie bracket $[\ ,\ ]$ on $\Gamma(A)$.

- a bundle map $\rho : A \to TM$, known as anchor, such that, for $f \in \mathcal{C}^\infty(M)$ and $u, v \in \Gamma(A)$, the Leibniz rule is satisfied:

$$[u, fv] = f[u, v] + \rho(u)(f)w,$$

and ρ is a Lie algebra homomorphism on sections, i.e.,

$$[\rho(u), \rho(v)] = \rho([u, v]).$$

Let us look for some examples:

a) The tangent bundle itself is an example of a Lie algebroid, what are the bracket and the anchor?

b) What is a Lie algebroid over a point $M = \{*\}$?

c) When is a distribution of A a Lie algebroid?

We can see that the previous definition has too much information:

d) Prove that $[\rho(v), \rho(w)] = \rho([v, w])$ is a consequence of the other properties. (*Hint: Consider the Jacobi identity for u, v, fw and apply the Leibniz rule.*)

Problem 2

The bracket of the Lie algebroid is responsible for some structure not only on the sections of A, but also on the the sections of the exterior algebra $\wedge^\bullet A^*$. To start with, we have an exterior derivative $d_A : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ defined by

$$(d_A\varphi)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho(X_i) \left(\varphi(X_0, X_1, \ldots, \hat{X}_i, \ldots X_k) \right) + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_k),$$

where $\varphi \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma(A)$, and $\hat{X}_i$ denotes that X_i is missing. Moreover, the Lie derivative by $X \in \Gamma(A)$, $L_X : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$, is defined by

$$(L_X\varphi)(X_1, \ldots, X_k) = \rho(X)(\varphi(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \varphi(X_1, \ldots, [X, X_i], \ldots, X_k).$$

Prove that they satisfy the Cartan formula

$$L_X\varphi = d_A(i_X\varphi) + i_X(d_A\varphi).$$

Problem 3

In this course we will look very often at sections of $\mathbb{T}M = TM + T^*M$. Let us fix some notation for them:

$$u = X + \alpha, \qquad v = Y + \beta, \qquad w = Z + \gamma \qquad \in \Gamma(\mathbb{T}M).$$

Let us define a bracket in $\Gamma(\mathbb{T}M)$ by

$$[u, v] = [X + \alpha, Y + \beta] := [X, Y] + \mathcal{L}_X\beta - i_Y d\alpha.$$

a) Check that $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$.

b) Compute $[u, fv]$, for $f \in \mathcal{C}^\infty(M)$.

c) Check that $[\rho(u), \rho(v)] = \rho([u, v])$.

d) Looking at the definition in Problem 1, when does a subbundle $L \subset \mathbb{T}M$ become a Lie algebroid by restricting the bracket and the anchor map to L?