Dirac structures and generalized geometry
Problem Sheet 4

Thursday 28th January 2016,
hand in by Thursday 4th

Problem 1

Show that choosing a splitting $TM + T^*M = C_+ + C_-$ into positive and negative-definite subspaces orthogonal to each other is equivalent to giving an orthogonal automorphism $G \in \mathrm{O}(TM + T^*M)$ squaring to the identity such that $\langle \_, G\_\rangle$ is positive-definite. This map G is called a generalized metric.

Problem 2

Recall that a Kähler structure on a manifold M can be defined as a pair (g, J) consisting of a positive-definite metric g and a complex structure J such that $g(J\_, \_)$ defines a symplectic structure.

a) How would you define a generalized Kähler structure?

Whatever definition you have given, it should be equivalent to having a pair of commuting generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ such that $-\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

b) Prove that your definition is equivalent to this one.

c) Show that a usual Kähler structure can be seen as a generalized Kähler structure.

d) Consider the B-transforms of the complex and sympectic structures coming from a Kähler structure. Do they always fit into a generalized Kähler structure?

Problem 3

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a $2n$-dimensional manifold. Each one of the generalized complex structures $\mathcal{J}_i$ determines a decomposition $(T+T^*)_{\mathbb{C}} = L_i + \overline{L_i}$. As we have two of them, we can consider the $\pm i$-eigenbundles of $L_1^{\pm}$ with respect to $\mathcal{J}_2$:

$$(T+T^*)_{\mathbb{C}} = L_1^+ + L_1^- + \overline{L_1^+} + \overline{L_1^-}.$$

a)What are L_2 and $C_\pm \otimes \mathbb{C}$ in this decomposition?

b) Are $L_1^{\pm}$ necessarily closed under the Dorfman bracket?

c) Prove that a generalized Kähler structure is equivalent to giving two complex rank n subbundles L_1^+, L_1^- that are isotropic, orthogonal to each other, positive and negative-definite (respectively) w.r.t. $\langle\ ,\bar{\ }\rangle$, and involutive, such that $L_1^+ \perp \overline{L_1^-}$ and $L_1^+ + L_1^-$ is involutive.

Problem 4

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a $2n$-dimensional manifold. Since $C_\pm \cap T^*M = 0$, we have that $C_\pm$ can be seen as the graph of

$$g : TM \to T^*M,$$

where g is symmetric and positive-definite, and b is skew-symmetric.

a) Is b necessarily closed?

b) Show that $C_\pm$ are stable under $\mathcal{J}_i$, and hence define complex structures on $C_\pm$ (without mentioning integrability).

The maps $\mathrm{Id} + g \pm b$ give isomorphisms $TM \cong C_\pm$ in such a way that we can translate the complex structures $\mathcal{J}_i$ on $C_\pm$ to an almost complex structure on M. We could have up to four different ones.

c) Are they all necessarily different?

d) When do a metric g, a 2-form b and as many almost complex structures on M as you found in b) determine an almost generalized Kähler structure (same definition as in Problem 2 but with generalized almost complex structures).