Generalized Geometry, an introduction
Assignment 1

Universitat Autònoma de Barcelona
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Problem 1. Consider the vector space $\mathbb{R}^4$ with the symplectic form

$$\omega((x_1, y_1, x_2, y_2), (x'_1, y'_1, x'_2, y'_2)) = \sum_{i=1}^{2} (x_i y'_i - y_i x'_i).$$

- Find a subspace U such that $U = U^\omega$.
- Find a plane U, i.e., a subspace isomorphic to $\mathbb{R}^2$, such that $U + U^\omega = \mathbb{R}^4$.

Problem 2. A subspace U of a symplectic vector space (V, ω) such that $U^\omega \subseteq U$ is called a coisotropic subspace. Prove that the quotient U/U^ω naturally inherits a symplectic structure. This is called the coisotropic reduction.

Problem 3. Given a symplectic form, we have an isomorphism $V \to V^*$. We invert this isomorphism to get a map $V^* \to V$, which we can see as a map

$$\pi : V^* \times V^* \to k.$$

Prove that the map π is bilinear, non-degenerate and skew-symmetric.

Problem 4. Consider a real vector space with a linear complex structure (V, J) and its complexification $V_{\mathbb{C}}$. Prove that $iJ = Ji$. When does the map

$$ai + bJ,$$

for $a, b \in \mathbb{R}$, define a linear complex structure on $V_{\mathbb{C}}$, seen as a real vector space?

Problem 5. Consider a real vector space with a linear complex structure (V, J). Prove that the map

$$J^* : V^* \to V^*,$$

given by

$$J^* \alpha(v) = \alpha(Jv),$$

for $\alpha \in V^*$ and $v \in V$, defines a linear complex structure on V^*. Given a basis (v_i) with dual basis (v^i), prove that

$$J^* v^i = -(Jv_i)^*.$$

Problem 6. The invertible linear transformations of $\mathbb{R}^n$ are called the general linear group and denoted by $\mathrm{GL}(n, \mathbb{R})$. If they moreover preserve the euclidean metric, we have the orthogonal group $\mathrm{O}(n, \mathbb{R})$. For a vector space V, we write $\mathrm{GL}(V)$ and $\mathrm{O}(V, g)$ when V comes with a linear riemannian metric g.

- Show that a basis $\{v_i\}$ determines a linear riemannian metric by $g(v_i, v_j) = \delta_{ij}$, and hence any vector space admits a linear riemannian metric.

- * Can two different bases determine the same riemannian metric? If so, describe the space of bases determining the same riemannian metric.

- * Conversely, given a linear riemannian metric, how can you describe the space of all metrics?

Problem 7. * Describe the space of linear complex structures on a given even-dimensional real vector space.