Generalized Geometry, an introduction
Assignment 2

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Problem 1. Let $e^1, e^2 \in V^*$ be linearly independent. From $e^1 \wedge e^2 = e^1 \otimes e^2 - e^2 \otimes e^1$, we see that $e^1 \wedge e^2 = -e^2 \wedge e^1$. Is it also true that for $\alpha \in \wedge^p V^*$, $\beta \in \wedge^q V^*$,

$$\alpha \wedge \beta = -\beta \wedge \alpha?$$

Problem 2. Let V be 4-dimensional with basis (e_1, e_2, e_3, e_4) and dual basis (e^1, e^2, e^3, e^4). Let

$$\omega = e^1 \wedge e^2, \qquad \omega' = e^1 \wedge e^2 + e^2 \wedge e^3, \qquad \omega'' = e^1 \wedge e^2 + e^3 \wedge e^4$$

be elements of $\wedge^2 V^*$, that is linear presymplectic structures. Regard them as maps $V \to V^*$ and tell if any of them is a linear symplectic structure.

Problem 3. Let V be an n-dimensional vector space.

- Compute the dimension of the vector spaces $\otimes^p V$ and $\wedge^p V$.
- Use the notation $\omega^m := \underbrace{\omega \wedge \ldots \wedge \omega}_{m \text{ times}}$. Let $n = 2m$. Prove that the 2-form $\omega \in \wedge^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0$.

Problem 4. The contraction by X is the linear map $i_X : \otimes^k V^* \to \otimes^{k-1} V^*$ linearly extending the correspondence

$$\alpha_1 \otimes \ldots \otimes \alpha_k \mapsto \alpha_1(X) \alpha_2 \otimes \ldots \otimes \alpha_k.$$

- Prove that the contraction maps $\wedge^k V^*$ onto $\wedge^{k-1} V^*$ and find a formula for $i_X(\alpha_1 \wedge \ldots \wedge \alpha_k)$.

- Prove that $i_X i_X \alpha = 0$ for $\alpha \in \wedge^k V^*$. What about $i_X i_X \varphi$ for $\varphi \in \otimes^k V^*$?
- Find the relation between $i_X(\alpha \wedge \beta)$, $i_X\alpha \wedge \beta$ and $\alpha \wedge i_X\beta$.

Problem 5. For $X \in V$, we defined the contraction map $i_X : \wedge^k V^* \mapsto \wedge^{k-1} V^*$. For $\alpha \in V^*$ define now, for $\varphi \in \wedge^k V^*$,

$$\alpha\wedge : \wedge^k V^* \mapsto \wedge^{k+1} V^*$$
$$\varphi \mapsto \alpha \wedge \varphi.$$

Note that for $\lambda \in k$ we have $i_X \lambda = 0$ and $\alpha \wedge \lambda = \lambda\alpha$.

In generalized linear algebra, for $X + \alpha \in V + V^*$ and $\varphi \in \wedge^\bullet V^*$, define the action

$$(X + \alpha) \cdot \varphi := i_X \varphi + \alpha \wedge \varphi.$$

- Prove that $(X + \alpha) \cdot ((X + \alpha) \cdot \varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.

Define the annihilator of $\varphi \in \wedge^\bullet V^*$ by

$$\mathrm{Ann}(\varphi) = \{X + \alpha \mid (X + \alpha) \cdot \varphi = 0\}.$$

We want to use annihilators to describe maximally isotropic subspaces in a similar way as we described a complex structure as

$$span(\bar{z}_1, \ldots, \bar{z}_m) = \mathrm{Ann}(z^1 \wedge \ldots \wedge z^m).$$

- Prove that $\mathrm{Ann}(\varphi)$ is always an isotropic subspace, for $\varphi \in \wedge^\bullet V^*$.
- Show that $\mathrm{Ann}(1) = V$ and $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\lambda\varphi)$ for $\lambda \neq 0$.
- Let vol_V be a non-zero element of $\wedge^{\dim V} V^*$, show that $\mathrm{Ann}(vol_V) = V^*$.
- What is the relation between $\{\alpha \in \wedge^k V^* \mid i_X\alpha = 0 \text{ for } X \in E\}$ and $\wedge^k \mathrm{Ann}(E)$?
- Let $E \subseteq V$ be a subspace. Find φ such that $\mathrm{Ann}(\varphi) = E + \mathrm{Ann}(E)$.
- Let $\omega \in \wedge^2 V^*$, find φ such that $\mathrm{Ann}(\varphi) = gr(\omega)$.

If you feel adventurous, you can try also:

- * Let $\pi \in \wedge^2 V$, find φ such that $\mathrm{Ann}(\varphi) = gr(\pi)$.
- ** Prove that $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\psi)$ if and only if $\varphi = \lambda\psi$ for $\lambda \in k^*$.
- *** When does φ define a maximally isotropic subspace?