Generalized Geometry, an introduction
Assignment 4

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Problem 1. We have defined the Dorfman bracket by

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - i_Y d\alpha.$$

We want to find some of its symmetries. Consider a diffeomorphism $f : M \to M$, that is, a smooth map with smooth inverse. We use the notation $f_* : T \to T$ for the differential and $f_* = (f^*)^{-1}$. Consider the orthogonal bundle map

$$f_* := \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix} : T + T^* \to T + T^*.$$

- Recall the properties of push-forwards and pull-backs and prove that

$$[f_* u, f_* v] = f_*[u,v].$$

For $B \in \Omega^2(M)$, define $e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : T + T^* \to T + T^*$.

- When do we have $[e^B u, e^B v] = e^B[u,v]$?

Problem 2. Let $M = \mathbb{R}^3$ with coordinates (x,y,z) and consider the coordinate vector fields $\{\partial_x, \partial_y, \partial_z\}$, which generate T at every point. Consider the 1-forms $\{dx, dy, dz\}$, which are dual to the coordinate vector fields and generate T^* at every point. Define the subbundle

$$L := span(\partial_y + z dx, \partial_x - z dy, dz) \subset T + T^*.$$

- Prove that L is a maximally isotropic subbundle.
- Is L involutive with respect to the Dorfman bracket?
- Describe $L \cap T$.

Problem 3. Let $\mathcal{J}$ be an almost generalized complex structure (a bundle map $\mathcal{J} \in \mathrm{O}(T+T^*)$ with $\mathcal{J}^2 = -1$) and L the $+i$-eigenspace of $\mathcal{J}$ in $(T+T^*)_{\mathbb{C}}$.

• Prove that the following expression defines a tensor (the Nijenhuis tensor), that is, it is $\mathcal{C}^\infty(M)$-linear, where $u, v \in \Gamma(T + T^*)$,

$$N_{\mathcal{J}}(u, v) = [\mathcal{J}u, \mathcal{J}v] - \mathcal{J}[\mathcal{J}u, v] - \mathcal{J}[u, \mathcal{J}v] - [u, v].$$

• Prove that $\mathcal{J}$ is a generalized complex structure if and only if $N_{\mathcal{J}}$ vanishes.
• Compare this to the case of usual complex structures.

Problem 4.

Let L be the $+i$-eigenspace of the generalized complex structure $\mathcal{J}$ and $B \in \Omega^2_{cl}$. • What is the $\mathcal{J}$-operator corresponding to $e^B L$?

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two anticommuting generalized almost complex structures,

$$\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1.$$

• Show that, for $t \in [0, \frac{\pi}{2}]$, $\mathcal{J}_t$ is a generalized almost complex structure:

$$\mathcal{J}_t = \sin t \mathcal{J}_1 + \cos t \mathcal{J}_2.$$

A Kähler manifold is a complex manifold (M, J) together with a riemannian metric such that $\omega := g(J\cdot, \cdot)$ is a closed 2-form (see Section 1.7 of the lecture notes to see the linear version of this). A hyperKähler manifold is a manifold M together with three anticommuting usual complex structures $\{I, J, K\}$ (that is, they satisfy the relations of quaternions, $IJ = -JI$, etc.) and a riemannian metric such that the following are closed two forms.

$$\omega_I := g(I\cdot, \cdot), \qquad \omega_J := g(J\cdot, \cdot), \qquad \omega_K := g(K\cdot, \cdot).$$

• Show that, when we regard $\omega_I, \omega_J, \omega_K : T \to T^*$, we have

$$\omega_I I = -I^*\omega_I, \qquad \omega_J I = I^*\omega_J.$$

Consider the generalized complex structures

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_{\omega_J} := \begin{pmatrix} 0 & -\omega_J^{-1} \\ \omega_J & 0 \end{pmatrix}.$$

• Prove that $\mathcal{J}_I$ and $\mathcal{J}_{\omega_J}$ anticommute.

Consider $\mathcal{J}_t$ as above for $\mathcal{J}_1 = \mathcal{J}_I$ and $\mathcal{J}_2 = \mathcal{J}_{\omega_J}$.

• Prove that $\mathcal{J}_t$ is a generalized complex structure. (Hint: use a B-field related to ω_K to transform $\mathcal{J}_t$ into a generalized complex structure whose integrability is easy.)