Generalized geometry, an introduction

$$\mathcal{J} = L = \varphi$$

Roberto Rubio
Weizmann Institute of Science

August 9, 2018

Disclaimer and acknowledgments

These are introductory lecture notes about Dirac and generalized geometry. They reflect the contents of the course "Generalized Geometry, an introduction" taught at the Weizmann Institute of Science in the second semester of 2017/2018, and are strongly based on the course "Introduction to Generalized Geometry", taught at IMPA in 2015.

These are the first iteration of a future set of improved and more complete lecture notes, so the text has not been carefully proofread. If you find any typos, mistakes, or if there is something not sufficiently clear, I will appreciate if you can let me know by email: roberto . rubio @ weizmann . ac . il

These notes greatly benefited from the interaction with the students and interested listeners of the courses they are based one. These were:

- Guilherme de Almeida, Miquel Cueca, Hudson Lima, Nicolás Martínez-Alba, Inocencio Ortiz, Luis Pánfilo Yapu, Ricardo Ramos, Davide Stefani and Juan José Villareal at IMPA in 2015.
- Ran Calman, Hagai Helman, Tsuf Lichtman, Michal Pnueli, Aviv Shalit, Omer Shamir, Michal Shavit, Raz Slutsky and Misha Yutushui at the Weizmann Institute in 2018.

I take this opportunity to thank all of them, and also IMPA and the Weizmann Institute of Science, for making both courses possible.

Contents

Chapter 1

Linear algebra

Linear algebra deals with vector spaces, their linear transformations, their representation as matrices, their eigenvalues and eigenvectors, and extra structures, such as a euclidean metric.

A vector space over a field k is an abelian group $(V, +)$ together with a compatible map $k \times V \to V$. Compatible means that whatever properties held in school for $(\mathbb{R}^2, +, \cdot)$, like distributivity, will hold now in abstract.

We will work with **finite-dimensional vector spaces over the fields** $k = \mathbb{R}, \mathbb{C}$. This chapter is not going to teach you linear algebra, but we will look and explore some aspects of it that will be relevant later. You can review the basics in your favorite linear algebra book. If you do not have one, the beginnings of Chapters 1, 9 and 11 of [Rom08] may help you. For multilinear algebra, Section 1.5, you may want to look at [Gre78].

1.1 The vector spaces we know

If we had to start the theory of vector spaces ourselves, we would look at $(\mathbb{R}^2, +, \cdot)$ and make an abstract definition by forgetting about $\mathbb{R}^2$ and instead consider a set V. In this process we have to make an effort to forget a couple of things we used every now and then in out teenage years: a basis (that is, coordinates), and a euclidean metric, the so-called scalar product given, for $(x, y), (x', y') \in \mathbb{R}^2$, by

$$\langle (x, y), (x', y') \rangle = xx' + yy',$$

whose associated quadratic form is

$$||(x, y)||^2 = x^2 + y^2.$$

Both bases and the Euclidean metric were really helpful so we will not dismiss them. We will take bases when needed, which will look like $\{e_1, e_2\}$ and not like $\{(1,0),(0,1)\}$. On the other hand, the generalization of the euclidean metric on $\mathbb{R}^n$ to an arbitrary vector space is a symmetric non-degenerate bilinear map

$$g : V \times V \to k.$$

Symmetric means $g(u,v) = g(v,u)$ for any $u,v \in V$, and non-degenerate that the map $g(u,\cdot) : V \to k$ is a non-zero map for any $u \in V$. Thanks to g, we can talk about orthogonal vectors: those such that $g(u,v) = 0$, which we sometimes write as $u \perp v$.

To be completely honest, the analogue of the euclidean metric requires k to be $\mathbb{R}$, and then we can require it to be positive definite, that is, $g(v,v) > 0$ for $v \neq 0$. We will call such g a **linear riemannian metric**[1] on the vector space V. Thanks to the bilinearity, when we choose a basis $\{v_i\}$ of V, a metric can be described with the symmetric matrix whose (i,j)-entry is $g(v_i, v_j)$.

A linear riemannian metric allows us to define orthonormal bases, those bases $\{e_i\}$ such that

$$g(e_i, e_j) = \delta_{ij}, \tag{1.1}$$

that is, the metric corresponds to the identity matrix with respect to this basis. Recall that the Gram-Schmidt orthogonalization process takes any basis and gives an orthonormal basis. Conversely, given any basis $\{e_i\}$ we can define a linear riemannian metric g by (1.1), for which $\{e_i\}$ becomes an orthonormal basis. Thus, any vector space admits a linear riemannian metric.

On the other hand, a linear riemannian metric uniquely determines a complement to any vector subspace $U \subseteq V$, the orthogonal complement. Indeed, the set

$$U^\perp = \{v \in V \mid g(v,U) = 0\}$$

is a vector subspace and satisfies

$$U \oplus U^\perp = V. \tag{1.2}$$

Finally, given a subspace $U \subseteq V$, the restriction of g to U gives a riemannian metric on U.

Fine print 1.1. On complex vector spaces, we cannot talk about positive definiteness, as

$$g(iv, iv) = i^2 g(v,v) = -g(v,v).$$

[1] This is not a very standard terminology, but it is very descriptive. The usual term is real inner product.

But if we consider hermitian forms h, where we replace bilinearity with sesquilinearity,

$$h(\lambda u, v) = \lambda h(u, v), \qquad h(u, \lambda v) = \bar{\lambda} h(u, v), \qquad h(u, v) = \overline{h(v, u)},$$

we can talk again about being positive definite.

1.2 Linear symplectic structures

We have reviewed some things that happened to a linear riemannian metric, that is, a symmetric non-degenerate bilinear map $V \times V \to k$ which is moreover positive definite.

We now wonder what happens if we start with a skew-symmetric non-degenerate bilinear map

$$\omega : V \times V \to k.$$

We call this ω a **linear symplectic structure**.

Example 1.1. The vector space $\mathbb{R}^2$ has the symplectic form

$$\omega((x, y), (x', y')) = xy' - yx'.$$

In general, the vector space $\mathbb{R}^{2n}$ has the symplectic form

$$\omega((x_1, y_1, \ldots, x_n, y_n), (x'_1, y'_1, \ldots, x'_n, y'_n)) = \sum_{i=1}^{n} (x_i y'_i - y_i x'_i).$$

Fine print 1.2. If we want to be very precise, we would talk about skew-symmetric ($\omega(u, v) = -\omega(v, u)$) or alternating ($\omega(u, u) = 0$) forms. In all generality, any alternating form is skew-symmetric, but the converse is only true when $\operatorname{char} k \neq 2$.

Any vector space admits a linear riemannian metric, but does any vector space admit a linear symplectic structure? Is there any analogue of an orthonormal basis, which we obviously do not have, as $\omega(v, v) = 0$? Is there any analogue of the orthogonal complement?

We will answer all these questions at the same time. We start with the **symplectic complement** of a subspace $U \subseteq V$,

$$U^\omega = \{v \in V \mid \omega(v, U) = 0\}.$$

Note that $\langle u \rangle^\omega$ cannot be all of V (as ω is non-degenerate) and contains $\langle u \rangle$ (as $\omega(u, u) = 0$), so

$$\langle u \rangle + \langle u \rangle^\omega \neq V.$$

However, we can prove the following.

Lemma 1.2. *For any subspace $U \subset V$ of a symplectic subspace (V, ω),*

- *we have* $\dim U + \dim U^\omega = \dim V$.
- *we have* $U \oplus U^\omega = V$ *if and only if* $\omega_{|U}$ *is a linear symplectic form. In this case* U *is called a symplectic subspace.*
- *we have* $(U^\omega)^\omega = U$. *In particular,* U *is symplectic if and only if* U^ω *is symplectic.*

Proof. We regard ω as a map $V \to V^*$. The image of U^ω by this map corresponds to

$$\operatorname{Ann} U = \{\alpha \in V^* \mid \alpha(U) = 0\}.$$

This is actually the kernel of the restriction map $V^* \to U^*$, which is moreover surjective. Thus, the composition

$$V \to V^* \to U^*$$

is surjective with kernel U^ω, so

$$\dim V = \dim U^\omega + \dim U^* = \dim U^\omega + \dim U.$$

For the second part, as we know $\dim U + \dim U^\omega = \dim V$, the condition $U \oplus U^\omega = V$ is equivalent to $U \cap U^\omega = \{0\}$. This latter condition is equivalent to $\omega(v, U) \neq \{0\}$ for any $v \in U$, which is the same as $\omega_{|U}$ being non-degenerate, and hence a linear symplectic form (as the restriction is clearly bilinear and skew-symmetric).

Finally, as $\omega(u, U^\omega) = 0$ for all $u \in U$, we have $U \subseteq (U^\omega)^\omega$. We must have equality as

$$\dim V = \dim U + \dim U^\omega = \dim U^\omega + \dim(U^\omega)^\omega.$$

$\square$

With the previous lemma, we can find a normal form for a symplectic structure. Take a non-zero vector $u \in V$. As $\omega(u, \cdot) : V \to k$ is not the zero map, there exists $v \in V$ such that $\omega(u_1, v_1) = 1$. The subspace

$$U_1 := \langle u_1, v_1 \rangle$$

is symplectic (as the restriction of ω is non-degenerate), so we have $V = U_1 \oplus U_1^\omega$ with U_1^ω symplectic. By repeating this argument with U_1^ω we obtain u_2, v_2 with $\omega(u_2, v_2) = 1$, and, for $U_2 = \langle u_2, v_2 \rangle$, we consider U_2^ω inside U_1^ω.

This process can be repeated until we get to U_m, where $\dim V = 2m$. By considering the basis $(u_1, v_1, \ldots, u_m, v_m)$, the matrix of ω is given by

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}.$$

By reordering the basis to $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ we get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.3}$$

where 0 and 1 denote the zero and identity $m \times m$ matrices. This is actually the matrix of the symplectic form ω of Example 1.1 with respect to the canonical basis, so any symplectic structure, upon a choice of a basis, looks like the one in the example.

Fine print 1.3. An argument by induction would be more elegant. Once we find U_1, we apply the induction hypothesis on U_1^ω and reorder the basis.

We have found a very strong constraint: a vector space V admitting a linear symplectic structure must be even dimensional. Conversely, any even-dimensional vector space admits a linear symplectic structure, as we just have to take a basis and define ω by the matrix in (1.3).

Problem: A subspace U such that $U^\omega \subset U$ is called a coisotropic subspace. Prove that the quotient U/U^ω naturally inherits a symplectic structure. This is called the coisotropic reduction.

1.3 Linear presymplectic and Poisson structures

We have seen that the restriction of a symplectic form to a subspace is not necessarily symplectic, but we still have a skew-symmetric bilinear map

$$\omega : V \times V \to k,$$

which is possibly degenerate. This is called a **linear presymplectic structure** on V.

We can find a normal form for a linear presymplectic structure ω as follows. Regard ω as a map $V \to V^*$, consider its kernel $\ker \omega$ and find a complement W, that is, a subspace $W \subset V$ such that $V = \ker \omega \oplus W$. The restriction of ω to W gives a symplectic form, so we can find a basis of W such that $\omega_{|W}$ is given by a matrix as in (1.3). Add the basis of W to any basis of $\ker\omega$. As $\omega(\ker\omega, W) = \{0\}$, the form ω is represented by

$$\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}.$$

On the other hand, given a symplectic form, we have an isomorphism $V \to V^*$. If we invert this isomorphism to get a map $V^* \to V$, which we can see as a map

$$\pi : V^* \times V^* \to k.$$

Problem: Prove that the map π is bilinear, non-degenerate and skew-symmetric.

We thus get a non-degenerate (or invertible, if you wish) map $\pi : V^* \times V^* \to k$. If we drop the non-degeneracy condition we get the linear version of a Poisson structure.

Fine print 1.4. We do not highlight this term as a linear Poisson structure commonly refer to the structure defined in $\mathfrak{g}^*$, the dual of a Lie algebra. We will probably go back to it.

The bottom line of this section is that a symplectic form can degenerate in two ways, as a map $\omega : V \times V \to k$, thus giving linear presymplectic structures (which appear naturally on subspaces) or as a map $\pi : V^* \times V^* \to k$ which will give rise to Poisson structures at the global level (as we will see later on).

1.4 Linear complex structures

Linear riemannian metrics and symplectic structures did not look so different, but complex structures on a vector space have a different flavour.

A **linear complex structure** on a real vector space V is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{Id}$.

Example 1.3. In $\mathbb{R}^2$, the endomorphism given by $J((a,b)) = (-b,a)$ is a linear complex structure.

Some of the questions we answered for symplectic structures may be asked again now: any real vector space admits a complex structure?, is there any sort of normal basis for a complex structure?, what happens with subspaces?

The latter question has an easy answer: if we look at a subspace U, we will get a complex structure, as long as $J(U) \subseteq U$, which would of course imply $J(U) = U$. The resulting complex structure is denoted by $J_{|U}$.

As for the normal basis, we can start with any non-zero vector $v \in V$ and add $Jv \in V$. They are not linearly dependent as $Jv = \lambda v$ would imply $J(Jv) = \lambda^2 v$, that is $-v = \lambda^2 v$, which is not possible. We already see that we cannot have a linear complex structure on a 1-dimensional vector space. If V were $\langle v, Jv\rangle$, we are done. Otherwise, choose any vector $w \in V \setminus \langle v, Jv\rangle$. Again Jw is linearly independent from v, Jv and w. If we had $Jw = aw + bv + cJv$, we would also have

$$-w = J(Jw) = aJw + bJv - cv = a^2 w + (ab - c)v + (ac + b)Jv,$$

which would mean that v, Jv, w are linearly dependent. Thus $\{v, Jv, w, Jw\}$ are linearly independent. This process can be repeated inductively, and we get that any linear complex structure admits a basis

$$(v_1, Jv_1, \ldots, v_m, Jv_m).$$

If we order the basis like $(Jv_1, \ldots, Jv_m, v_1, \ldots, v_m)$, the endomorphism J is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.4}$$

This has plenty of information. First of all, a vector space must be even dimensional in order to admit a linear complex structure. Conversely, any even dimensional vector space admits a linear complex structure J by just choosing a basis and defining J by (1.4).

Complexifying a vector space with a complex structure can be a bit confusing at first glance. The underlying set is $V \times V$, but in order to make it more user friendly, we use a formal element i and write it as

$$V_{\mathbb{C}} = V \times V = \{u + iv \mid u, v \in V\}.$$

The sum is clear, and the scalar product, for $a + ib \in \mathbb{C}$, is given by

$$(a + ib)(u + iv) = (au - bv) + i(av + bu).$$

Note that the i in ib is an imaginary number, whereas the i in iv or $i(av+bu)$ is a formal element, which gains meaning from its properties. For instance, we can define a conjugation operation on $V_{\mathbb{C}}$ given by

$$\overline{u + iv} = u - iv.$$

Finlly, a real basis for V becomes a complex basis for $V_{\mathbb{C}}$, so

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}.$$

The endomorphism $J \in \mathrm{End}(V)$ extends to an endomorphism of $V_{\mathbb{C}}$ by the correspondence

$$u + iv \mapsto Ju + iJv,$$

which, for the sake of simplicity, we will also denote by J.

The complex version of J has an advantage over the real one. They both satisfy $J^2 + \mathrm{Id} = 0$, and hence their minimal polynomial is $x^2 + 1$. As the roots are imaginary, $\pm i$, we can diagonalize it only for the complex version. So we will get a $+i$-eigenspace and a $-i$-eigenspace.

The fact that J comes from a real endomorphism gives us a lot of information. Given a $+i$-eigenvector $u + iv$, such that $J(u + iv) = -v + iu$, we get

$$J(\overline{u + iv}) = J(u - iv) = -v - iu = (-i)(u - iv) = (-i)\overline{u + iv},$$

so $u - iv$ is a $+i$-eigenvector. Moreover, they are linearly independent, as V cannot contain any eigenvector (the eigenvalues are complex). The eigenvectors thus come in pairs. Actually, we can be more precise about them. If we denote by $V^{1,0}$ the $+i$-eigenspace in $V_{\mathbb{C}}$ and by $V^{0,1}$ the $-i$-eigenspace in $V_{\mathbb{C}}$, we have

$$\begin{aligned} V^{1,0} &= \{v - iJv \mid v \in V\}, \\ V^{0,1} &= \{v + iJv \mid v \in V\}. \end{aligned}$$

Note that $\dim_{\mathbb{R}} V^{1,0} = \dim_{\mathbb{R}} V$. One can define isomorphisms

$$(V, J) \cong (V^{1,0}, J_{|V^{1,0}}), \qquad (V, -J) \cong (V^{0,1}, J_{|V^{0,1}}),$$

of real vector spaces with linear complex structures. We mean that the map $\varphi : V \to V^{1,0}$ given by

$$\varphi(v) = v - iJv$$

satisfies $\varphi \circ J = J_{|V^{1,0}} \circ \varphi$, and analogously for $V^{0,1}$ and the map $v \to v + iJv$.

Fine print 1.5. Note that $V_{\mathbb{C}}$ has two complex structures, the complexification of J and the product by i. These two structures coincide on $V^{1,0}$, whereas they are opposite to each other on $V^{0,1}$.

We have seen that a linear complex structure J determines a complex subspace $V^{1,0} \subset V_{\mathbb{C}}$ with $\dim V^{1,0} = \dim V$ satisfying $V^{1,0} \cap \overline{V^{1,0}} = 0$. Actually, this information will completely describe a linear complex structure.

Lemma 1.4. *A subspace* $L \subset V_{\mathbb{C}}$ *such that* $\dim_{\mathbb{R}} L = \dim_{\mathbb{R}} V$ *and* $L \cap \bar{L} = \{0\}$ *determines a decomposition* $V_{\mathbb{C}} = L \oplus \bar{L}$ *and the subspace* $V \subset V_{\mathbb{C}}$ *corresponds to the set*

$$\{l + \bar{l} \mid l \in L\}.$$

Proof. The decomposition follows from $L + \bar{L} \subseteq V_{\mathbb{C}}$, the fact that L and $\bar{L}$ are disjoint and the condition on the dimension of L. Similarly we have $\{l + \bar{l} \mid l \in L\} \subset V$ and the dimensions are the same. □

Proposition 1.5. *Given a subspace* $L \subset V_{\mathbb{C}}$ *such that* $\dim_{\mathbb{R}} L = \dim V$ *and* $L \cap \bar{L} = \{0\}$*, there exists a unique linear complex structure* J *such that* $L = V^{1,0}$.

Proof. By using the previous lemma, we define an endomorphism J of $V = \{l + \bar{l} \mid l \in L\}$ by

$$l + \bar{l} \mapsto il - i\bar{l}. \tag{1.5}$$

This map indeed maps V to V as $il \in L$ and $\overline{il} = -i\bar{l}$.

In order to see that it is unique, the condition $L = V^{1,0}$ determines uniquely the complexification of any such J. It must be the map $l' + \bar{l} \mapsto il' - i\bar{l}$. The only option for J is the restriction of this map to V, which is of course the definition in (1.5). □

1.5 Tensor and exterior algebras

We have talked a lot about bilinear maps, to which we required some extra condition as non-degeneracy, symmetry or skew-symmetry. If we forget, for the moment, about these extra conditions, what can we say about all the bilinear maps on a vector space? This is a very sensible question, as we know that the bilinear maps V^* do have the structure of a vector space. This structure is unique as we require the compatibility conditions

$$(\varphi + \psi)(v) = \varphi(v) + \psi(v), \qquad (\lambda\varphi)(v) = \lambda\varphi(v),$$

for φ, $\psi \in V^*$, u, $v \in V$.

It makes sense to endow the set of bilinear maps with the structure of a vector space by defining

$$(B + B')(u, v) = B(u, v) + B'(u, v), \qquad (\lambda B)(u, v) = \lambda B(u, v),$$

for B, B' bilinear maps $V \times V \to k$ and u, $v \in V$. So the bilinear maps on a vector space are actually a vector space, and we would like to have a name for it. Actually, since they are maps, we will give a name to and describe its dual.

Take a basis $\{e_i\}$ of V, a bilinear map B is completely determined by the images $B(e_i, e_j)$, that is, it is a linear map on the pairs $\{(e_i, e_j)\}$. This is actually a basis of the space we are interested (the dual space to the space of bilinear maps). Note that this is not $V \times V$, as there n^2 elements in $\{(e_i, e_j)\}$, whereas the dimension of $V \times V$ is $2n$, as $\{(e_i, 0), (0, e_i)\}$ gives a basis.

Let us name this space:

$$V \otimes V,$$

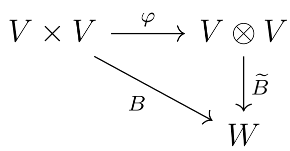
and the elements of the basis will be denoted by $\{e_i \otimes e_j\}$ instead of $\{(e_i, e_j)\}$. It may look a bit artificial at first glance, but we can make it look better. To start with, we define a map $\psi : V \times V \to V \otimes V$ by extending bilinearly the correspondence

$$\psi : (e_i, e_j) \mapsto e_i \otimes e_j.$$

Thus, if we have $u = \sum_i u_i e_i$, $v = \sum_j v_j e_j$,

$$u \otimes v = \sum_{ij} u_i v_j (e_i \otimes e_j).$$

The vector space $V \otimes V$ together with the map ψ satisfy a so-called universal property, which sums up what we just did: any bilinear map $B : V \times V \to k$ factorizes through the map φ giving rise to $\tilde{B} : V \otimes V \to k$. Actually this is valid for bilinear maps $B : V \times V \to W$ to another vector space W.



This is called the universal property of the tensor product, and the vector space of bilinear maps is $(V \otimes V)^*$.

It is never repeated enough that an element of $V \otimes V$ is not necessarily of the form $u \otimes v$ for some $u, v \in V$, but a linear combination of these elements.

We could do the same for two different vector spaces V, W and maps $B : V \times W \to k$ in order to get get $V \otimes W$, or for more vector spaces. It can be checked that

$$V \otimes (W \otimes U) \cong (V \otimes W) \otimes U,$$

so it makes sense to talk about $V^{\otimes 3} = V \otimes V \otimes V$, and, in general, $V^{\otimes n}$. Their duals, $(V^{\otimes n})^*$ are spaces of n-linear maps.

We can put together all the powers $\otimes^n V$ into an infinite-dimensional vector space

$$\otimes^\bullet V = \oplus_{i=0}^{\infty} V^{\otimes i},$$

where $V^{\otimes 0} = k$ and $V^{\otimes 1} = V$. When we take an infinite direct sum of vector spaces, its elements are finite linear combinations of the infinite-dimensional basis.

In this vector space we can define a linear operation, also denoted by $\otimes$, by extending bilinearly the definition

$$(e_{i_1} \otimes \ldots \otimes e_{i_r}) \otimes (e_{j_1} \otimes \ldots \otimes e_{j_s}) \mapsto (e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e_{j_1} \otimes \ldots \otimes e_{j_s}).$$

This product is compatible with the vector space structure of $\otimes^\bullet V$, is moreover associative, non-commutative when $\dim V \geq 2$, and endows $\otimes^\bullet V$ with the structure of an algebra. Its name is the **tensor algebra** of V.

Let us go back to bilinear maps. We always asked for some extra hypothesis, like symmetry or skew-symmetry. Can we do the same now? We could, by saying that the bases are, respectively, $\{(e_i, e_j)\}_{i \leq j}$ or $\{(e_i, e_j)\}_{i<j}$, as the images of these elements completely determine a symmetric and an skew-symmetric map. But then there would not be any relation to the previous space.

The usual way to deal with symmetric and skew-symmetric maps is as follows. A bilinear map is a map $B : V \otimes V \to k$. If it is symmetric, it is because it vanishes on the subspace spanned by

$$\{e_i \otimes e_j - e_j \otimes e_i\},$$

whereas if it is skew-symmetric it vanishes on the subspace spanned by

$$\{e_i \otimes e_i\}.$$

By defining the quotient vector spaces

$$\operatorname{Sym}^2 V = \frac{V \otimes V}{span(\{e_i \otimes e_j - e_j \otimes e_i\})}, \qquad \wedge^2 V = \frac{V \otimes V}{span(\{e_i \otimes e_i\})},$$

we get that a symmetric bilinear map is an element of the vector space $(\operatorname{Sym}^2 V)^*$ and a skew-symmetric one, an element of the vector space $(\wedge^2 V)^*$.

Analougously as we did with the tensor algebra, one considers

$$\operatorname{Sym}^\bullet V = \frac{\otimes^\bullet V}{gen(\{e_i \otimes e_j - e_j \otimes e_i\})}, \qquad \wedge^\bullet V = \frac{\otimes^\bullet V}{gen(\{e_i \otimes e_i\})},$$

where $gen(\{a_k\})$ denotes the ideal of the algebra $\otimes^\bullet V$ generated by the elements $\{a_k\}$, and talk about the **symmetric and the exterior algebras**, respectively.

From now on, we will consider the vector space V^* as it is the one we will use thoughout. The exterior algebra $\wedge^\bullet V^*$ is formally defined as the quotient of $\otimes^\bullet V^*$ by an ideal I. For $\alpha_1, \ldots, \alpha_k \in V^*$ we denote by $\alpha_1 \wedge \ldots \wedge \alpha_k$ the element $[\alpha_1 \otimes \ldots \otimes \alpha_k] \in \otimes^\bullet V^*/I$. This is not very practical, so we define maps $Alt_k : \wedge^k T^* \to \otimes^k T^*$ by

$$Alt_k(\alpha_1 \otimes \ldots \otimes \alpha_k) := \sum_{\sigma \in \Sigma_k} sgn(\sigma) \alpha_{\sigma 1} \otimes \ldots \otimes \alpha_{\sigma k} \in \otimes^\bullet V^*. \tag{1.6}$$

For instance, for α, $\beta \in V^*$ we have $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$. The map

$$Alt = \oplus_{n=0}^\infty Alt_n : \otimes^\bullet V^* \to \otimes^\bullet V^*$$

has exactly as its kernel the ideal $gen(\{e_i \otimes e_i\})$, so its image inside $\otimes^\bullet V^*$ can be identified with $\wedge^\bullet V^*$. Note that this is a choice and we could have well taken any multiple of the map Alt (actually you may have seen (1.6) with different constants). This identification is given by

$$\alpha_1 \wedge \ldots \wedge \alpha_k \mapsto \sum_{\sigma \in \Sigma_k} sgn(\sigma) \alpha_{\sigma 1} \otimes \ldots \otimes \alpha_{\sigma k} \in \otimes^\bullet V^*.$$

We will mainly regard $\wedge^\bullet V^*$ as a subalgebra of $\otimes^\bullet V^*$.

The product operation on $\wedge^\bullet V^* \subset \otimes^\bullet V^*$ corresponds to the **wedge product** defined as follows: for decomposable

$$\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \in \wedge^p V^*, \qquad \beta = \beta_1 \wedge \ldots \wedge \beta_q \in \wedge^q V^*,$$

where $\alpha_j, \beta_j \in V^*$, the product is given by

$$(\alpha_1 \wedge \ldots \wedge \alpha_p) \wedge (\beta_1 \wedge \ldots \wedge \beta_q) = \alpha_1 \wedge \ldots \wedge \alpha_p \wedge \beta_1 \wedge \ldots \wedge \beta_q,$$

and then it is extended linearly.

On $\otimes^\bullet V^*$ there is an important operation known as **contraction**. For any element of V, define a map

$$i_X : \otimes^k V^* \to \otimes^{k-1} V^*.$$

by extending linearly the correspondence

$$\alpha_1 \otimes \ldots \otimes \alpha_k \mapsto \alpha_1(X) \alpha_2 \otimes \ldots \otimes \alpha_k.$$

The contraction preserves the subalgebra $\wedge^\bullet V^*$ and acts as

$$i_X(\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{j=1}^k (-1)^j \alpha_j(X) \alpha_1 \wedge \ldots \wedge \widehat{\alpha_j} \wedge \ldots \wedge \alpha_k.$$

where $\widehat{\alpha_j}$ denotes that α_j is missing.

The constructions given before were meant to give an intuitive introduction. Before continuing we must address an important issue: we have chosen a basis to define all of them. One could argue that in case of choosing a different basis everything is naturally isomorphic. Let us spell it out for the case of $V \otimes W$, as everything can be reduced to it. Say we defined $V \otimes W$ using bases (e_i) of V, and (f_j) of W. If we use instead bases (v_i) and (w_j), the correspondence $e_i \otimes f_j \mapsto v_i \otimes w_j$ would give an isomorphism of the two versions of $V \otimes W$. This, of course, is not fully satisfactory. The definitive answer to this issue is the **free vector space** of a set. This is, if you will pardon the expression, a hell of a vector space. Call the set S, its free vector space, $\mathcal{F}(S)$, has as a basis the set S, which may well be infinite, but we only allow linear combinations of a finite number of elements. Formally, one should consider the set of maps $f : S \to k$ of finite support, that is, which are non-zero just on a finite number of elements of S. As an example, for a finite set S of n elements, $\mathcal{F}(S) \cong k^n$. If S is infinite, $\mathcal{F}(S)$ will be infinite dimensional. Just as we defined $\mathrm{Sym}^2 V$ and $\wedge^2 V$ as quotients of $\otimes^2 V$, we can define the tensor product $V \otimes W$ as a quotient of $\mathcal{F}(V \times W)$ by a subspace describing the bilinearity we want. This subspace is

$$\begin{aligned} Z = span(\{&(v, w) + (v', w) - (v + v', w), (v, w) + (v, w') - (v, w + w'), \\ &(\lambda v, w) - \lambda(v, w), (v, \lambda w) - \lambda(v, w) \mid v \in V, w \in W, \lambda \in k\}), \end{aligned}$$

so that we get the definition

$$V \otimes W = \frac{\mathcal{F}(V \times W)}{Z}.$$

We will not use directly, but this is the proper way to do things.

A not least important question is: why did we care so much about all this? The answer is that it offers a powerful language for the mathematics we want to do. Before doing so, let us highlight one property of the tensor product: we have isomorphisms

$$(\otimes^2 V)^* \cong \otimes^2 V^*, \qquad (\mathrm{Sym}^2 V)^* \cong \mathrm{Sym}^2 V^*, \qquad (\wedge^2 V)^* \cong \wedge^2 V^*.$$

For convenience we will use the latter expressions.

A linear presymplectic structures is just $\omega \in \wedge^2 V^*$. If it is symplectic, we just have to add the adjective non-degenerate. For linear riemannian structures, we would talk about positive-definite and non-degenerate $g \in \mathrm{Sym}^2 V^*$. And for the linear version of Poisson structures, we just have $\pi \in \wedge^2 V$.

Example 1.6. The vector space $\mathbb{R}^{2n}$ with canonical basis $\{e_i\}$ and dual basis $\{e^i\}$ has a canonical symplectic structure given by

$$\omega = \sum_{i=1}^{n} e^i \wedge e^{i+n}.$$

This is the structure described as a map in Example 1.1.

Is this formalism of any use for linear complex structures? Well, an endomorphism of V is an element of the vector space $V \otimes V^*$. If we add the condition $J^2 = -\operatorname{Id}$, we would get a linear complex structure. We will not use this much, though.

Consider (V, J) and a basis (x_k, y_k) with $y_k = Jx_k$, for $1 \leq k \leq \dim V/2$. We use the notation

$$z_k = x_k - iJx_k = x_k - iy_k, \qquad \bar{z}_k = x_k + iJx_k = x_k + iy_k$$

for the complex basis of $V_{\mathbb{C}}$ in terms of eigenspaces. Denote the dual basis of (x_k, y_k) by (x^i, y^i), we then have the dual basis

$$z^j = x^k + iJy^k, \qquad \bar{z}^k = x^k - iJy^k.$$

The linear complex structure J on V is equivalently given by the subspace $\bar{L} = \mathit{span}(\{\bar{z}_k\})$. A very important fact for us is that L can be described by

$$\varphi = z^1 \wedge \ldots \wedge z^k \in \wedge^{\dim V/2} V^* \tag{1.7}$$

by using the definition of annihilator of a form

$$\operatorname{Ann}(\varphi) = \{v \in V \mid i_v \varphi = 0\}.$$

Now the important, non-trivial question is...

What are the elements $\varphi \in \wedge^{\dim V/2} V^*$ such that
$\operatorname{Ann}(\varphi)$ defines a linear complex structure?

1.6 Linear transformations

The invertible linear transformations of a vector space V are called the general linear group and denoted by $\operatorname{GL}(n, \mathbb{R})$, that is,

$$\{f : V \to C \mid f \text{ is invertible and } f(u + v) = f(u) + f(v), f(\lambda c) = \lambda f(c)\}.$$

When V comes with a linear riemannian metric g, we have

$$\mathrm{O}(V,g) = \{f \in \mathrm{GL}(V) \mid g(f(u), f(v)) = g(u,v)\}.$$

For a symplectic form ω we have

$$\mathrm{Sp}(V,\omega) = \{f \in \mathrm{GL}(V) \mid \omega(f(u), f(v)) = \omega(u,v)\}.$$

And finally, for a complex structure J,

$$\mathrm{GL}(V,J) = \{f \in \mathrm{GL}(V) \mid J \circ f = f \circ J\}.$$

For the case of $\mathbb{R}^n$, we talk about $\mathrm{GL}(n,\mathbb{R})$, or about $\mathrm{O}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$, for the standard linear riemannian, symplectic or complex structure, respectively, of $\mathbb{R}^n$.

By choosing a basis of V (or in the case of $\mathbb{R}^n$ by taking the standard basis), we can see $\mathrm{GL}(V)$ as $n \times n$ matrices with non-zero determinant. For an orthonomal basis,

$$\mathrm{O}(n,\mathbb{R}) \cong \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^T A = \mathrm{Id}\}.$$

For a basis such that the symplectic form is given by the matrix J, we have $\omega(u,v) = u^T J v$, so

$$\mathrm{Sp}(2n,\mathbb{R}) \cong \{A \in \mathrm{GL}(2n,\mathbb{R}) \mid A^T J A = J\}.$$

And for a basis such that the endomorphism J is given by J, we have

$$\mathrm{GL}(n,\mathbb{C}) \cong \{A \in \mathrm{GL}(2n,\mathbb{R}) \mid A^{-1} J A = J\}.$$

The groups $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are defined in terms of the matrix J. If you look closer, you will realize that

$$\mathrm{O}(2n,\mathbb{R}) \cap \mathrm{Sp}(2n,\mathbb{R}) = \mathrm{O}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}). \tag{1.8}$$

This intersection has a name, $\mathrm{U}(n)$, and corresponds to the linear transformations preserving a hermitian metric.

Before this section we only dealt with vector spaces. What kind of objects are $\mathrm{GL}(V)$, $\mathrm{O}(V,g)$, $\mathrm{Sp}(V,\omega)$ and $\mathrm{GL}(V,J)$? We can see them, by choosing a basis, inside $\mathbb{R}^{n^2}$, but they are not vector subspaces, as they do not include the zero, and the sum of two elements is not necessarily inside the group. But they are not missing a structure, as they actually have two, which we describe intuitively. Let us talk about about $\mathrm{GL}(V)$, which we see as the subset of $\mathbb{R}^{n^2}$ such that the determinant (a polynomial on the entrances of the matrices) does not vanish.

- It is an affine algebraic set, that is, it can be regarded as the vanishing set of a polynomial. Wait, we said it is the subset of $\mathbb{R}^{n^2}$ where a polynomial does not vanish. So, what? We can regard any matrix A as the pair $(A, \det A)$ inside $\mathbb{R}^{n^2+1}$. The group $\mathrm{GL}(V)$ corresponds to the pairs (A, x) in $\mathbb{R}^{n^2+1}$ such that $\det A - x = 0$, the vanishing of a polynomial. As $\mathrm{GL}(V)$ is moreover a group and the group operation and the inverse are algebraic maps, we have that $\mathrm{GL}(V)$ is an affine algebraic group.
- On $\mathrm{GL}(V)$ we have a topology coming from $\mathbb{R}^{n^2}$, The elements close to a matrix A in $\mathrm{GL}(V)$ are given by $A + X$ with X a matrix with small entries. If $\det(A) \neq 0$, then $\det(A + X) \neq 0$ for X small enough, so $\mathrm{GL}(V)$ locally looks like a ball in $\mathbb{R}^{n^2}$. This intuitively says that it has the structure of a differentiable manifold. As it is a group and the inverse and group operations are smooth, we get the structure of a **Lie group**. This is the structure we will use.

We saw that we can define a linear riemannian metric on V by choosing a basis $\{e_i\}$ and setting

$$g(e_i, e_j) = \delta_{ij}.$$

This correspondence defines a map from the set of bases of V, let us denote it by $Bas(V)$, to the set of linear riemannian metrics on V. This map is clearly onto, as any linear riemannian metric admits an orthonormal basis. In order to study its kernel, we need a better understanding of the set of (ordered) bases, let us denote it by $Riem(V)$,

Again, the set of bases $Bas(V)$ is not naturally a vector space (how would you sum two bases?) but comes with some extra structure. Given two bases, there is one and only one element in $\mathrm{GL}(V)$ sending one to the other. In other words, the group $\mathrm{GL}(V)$ acts on $Bas(V)$ (that is, there is a map $\rho : \mathrm{GL}(V) \to Maps(Bas(V), Bas(V)))$ transitively (there is one element in $\mathrm{GL}(V)$ sending one basis to the other) and faithfully (there is only one). This structure is called a $GL(V)$-torsor. It is almost a group: if we choose any element of $Bas(V)$ and we declare it to be the identity, we would have a group. But we do not have a preferred choice of identity.

In order to describe the map $Bas(V) \to Riem(V)$, we choose $b \in Bas(V)$, whose image is some $g \in Riem(V)$. Before we said that $Bas(V) = \mathrm{GL}(V) \cdot b$. Note now that b' maps to g if and only if $b' \in \mathrm{O}(V, g) \cdot b$, as, for $b = \{e_i\}$,

$$g(Ae_i, Ae_j) = g(e_i, e_j) = \delta_{ij} \longleftrightarrow A \in \mathrm{O}(V, g).$$

We thus get an isomorphism

$$Riem(V) \simeq \frac{\mathrm{GL}(V)}{\mathrm{O}(V, g)} \simeq \frac{\mathrm{GL}(n, \mathbb{R})}{\mathrm{O}(n, \mathbb{R})},$$

which we usually express as the latter. Analogously, the space of linear symplectic structures and linear complex structures on a vector space V are parametrized by the homogeneous spaces

$$\frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{Sp}(2n,\mathbb{R})}, \qquad \frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})}.$$

1.7 Hermitian structures

Consider a hermitian metric on a complex vector space V, that is, a map $h : V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear on the first component and satisfies $h(v,u) = \overline{h(u,v)}$, which implies that is anti-linear on the second component. The usual example is $h(u,v) = u^T\overline{v}$. The unitary group for the hermitian metric h is defined by

$$\mathrm{U}(n) := \{M \in \mathrm{GL}(n,\mathbb{C}) \mid h(Mu,Mv) = h(u,v), \text{ for } u,v \in \mathbb{C}^n\}.$$

Decompose an arbitary hermitian metric h into real and imaginary parts:

$$h(u,v) = R(u,v) + iI(u,v).$$

From $h(Ju,v) = ih(u,v)$, we get $R(Ju,v) = -I(u,v)$ so

$$h(u,v) = R(u,v) - iR(Ju,v).$$

And from $h(v,u) = \overline{h(u,v)}$, we get that $R(u,v) = R(v,u)$, so R is a usual real metric, let us denote it then by g, and $I(u,v) = -g(Ju,v)$ becomes a linear symplectic form, which we call $-\omega$. We thus have

$$h(u,v) = g(u,v) - i\omega(u,v).$$

The condition of $g(Ju,v)$ being a linear symplectic form is equivalent to

$$g(Ju,Jv) = -g(J(Jv),u) = g(v,u) = g(u,v),$$

that is, the complex structure J is orthogonal with respect to g.

We can thus conclude:

- On a vector space with a linear riemannian metric g and a linear complex structure J such that J is orthogonal, we have that an automorphisms preserves g and J if and only if it preserves J and

  $$h(u,v) = g(u,v) - ig(Ju,v),$$

  that is

  $$\mathrm{O}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{U}(n).$$

- On a vector space with a linear riemannian metric g and a symplectic form ω such that $g^{-1} \circ \omega$ is a linear complex structure (when we look at g, ω as maps $V \to V^*$), we have that an automorphisms preserves g and ω if and only if it preserves J and

$$h(u, v) = g(u, v) - i\omega(u, v),$$

that is

$$\mathrm{O}(2n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n).$$

This is the statement of the identity (1.8) for general g, ω and J. Of course, when we look at the matrix form, everything fits nicely. If the metric is represented by the identity matrix Id (the one that makes $A^{-1} = A^t$), we have that J given as in (1.4) is orthogonal, and the resulting symplectic form ω is given $\mathrm{Id} \cdot J$, that is, also by J.

Chapter 2

Generalized linear algebra

Being completely fair, the first part of this chapter should be called Dirac linear algebra, as it is the base for linear Dirac structures.

Let us summarize some of the things we have done. Symplectic forms are a skew-symmetric analogue of an inner product, which we can regard as "skew-symmetric" isomorphisms $V \to V^*$ or $V \to V^*$. When we drop the hypothesis of being isomorphisms we get linear versions of presymplectic structures, say ω, and Poisson structures, say π.

Linear presymplectic and Poisson structures look different, but if we look at their graphs,

$$gr(\omega) = \{X + \omega(X) \mid X \in V\}, \qquad gr(\omega) = \{\pi(\alpha) + \alpha \mid \alpha \in V^*\},$$

they look quite similar:

$$gr(\omega), gr(\pi) \subset V + V^*, \qquad \dim gr(\omega) = \dim gr(\pi) = \dim V.$$

And this is not all, but to say more we need to talk about $V + V^*$, which we do next.

2.1 The generalized vector space $V + V^*$

For a vector space V, consider the vector space $V + V^*$. We will denote its elements by $X + \alpha, Y + \beta$, with $X, Y \in V$ and $\alpha, \beta \in V^*$. This vector space comes equipped with a canonical pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(i_X \beta + i_Y \alpha).$$

Given a subspace $W \subseteq V + V^*$, define the orthogonal complement for this pairing in the usual way by

$$W^\perp = \{u \in V + V^* \mid \langle u, W \rangle = 0\}.$$

We say that W is **isotropic** if $W \subset W^\perp$. We say that W is **maximally isotropic** when it is isotropic and is not strictly contained in an isotropic subspace. This is a general definition, for any pairing. In this case we will see that maximally isotropic subspaces correspond to those W such that $W = W^\perp$.

To start with, take a basis (e_i) of V and its dual basis (e^i) of V^*. By considering the basis (e_i, e^i) of V^* the canonical pairing is given by

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{2.1}$$

If we consider the basis $(e_i + e^i, e_i - e^i)$, it is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so we see that the signature is (n, n), as we have an orthogonal basis of n vectors of positive length and n vectors of negative length.

This type of arguments allows us to show that the dimension of a maximally isotropic subspace must be at most $\dim V$. Otherwise, we could choose a basis in such a way that the matrix of $\langle \cdot, \cdot \rangle$ is given by

$$\begin{pmatrix} & & & * \\ & 0 & & * \\ & & & * \\ * & * & * & * \end{pmatrix},$$

with a zero block bigger than an $n \times n$-matrix, which is not possible as the matrix must be invertible.

The dimension of any maximally isotropic subspace is always $\dim V$, as we next prove.

Lemma 2.1. *For a vector space V, we have that $\dim V$ is the dimension of any maximally isotropic subspace of $V + V^*$.*

Proof. Let L be a maximally isotropic subspace. We first show that $L^\perp$ is semidefinite, that is, $Q(v) \leq 0$ or $Q(v) \geq 0$ for all $v \in L$. Otherwise, choose a complement C so that $L^\perp = L \oplus C$. If C has two vectors v, w with $Q(v) > 0$ and $Q(w) < 0$, a suitable linear combination $v + \lambda w$ would be null, and $L \oplus \mathit{span}(v + \lambda w)$ would be isotropic containing L. Secondly, as $L^\perp \cap V = \{0\}$, we have that

$$\dim L^\perp = \dim(L^\perp \oplus V) - \dim V \leq 2n - n = n,$$

On the other hand, as the pairing is non-degenerate, we have $\dim L^\perp = 2n - \dim L$, so the previous equation becomes

$$2n - \dim L \leq n,$$

that is, $\dim L \geq n$, so $\dim L = n$. □

Fine print 2.1. In general one can show that the dimension of a maximally isotropic subspace for a non-degenerate symmetric pairing of signature (m, n) is $\min(m, n)$.

Thus, $V + V^*$ is a vector space of dimension $2n$ with a symmetric pairing of signature (n, n) and the choice of two maximally isotropic subspaces, V and V^*.

We now go back to linear presymplectic and Poisson structures.

Proposition 2.2. *Let $\omega \in \otimes^2 V^*$ and $\pi \in \otimes^2 V$, regarded as maps $V \to V^*$ and $V^* \to V$. Denote by gr the graph of these maps.*

- *We have $\omega \in \wedge^2 V^*$ if and only if $gr(\omega)$ is maximally isotropic in $V + V^*$.*
- *We have $\pi \in \wedge^2 V$ if and only if $gr(\pi)$ is maximally isotropic in $V + V^*$.*

Let L be a maximally isotropic subspace of $V + V^*$.

- *We have $L \cap V^* = \{0\}$ if and only if $L = gr(\omega)$ for a unique $\omega \in \wedge^2 V^*$.*
- *We have $L \cap V = \{0\}$ if and only if $L = gr(\pi)$ for a unique $\pi \in \wedge^2 V$.*

Proof. We prove the statements for ω, as they are analogous for π. The first statement follows from

$$\langle X + \omega(X), Y + \omega(Y) \rangle = 0 \leftrightarrow \omega(X, Y) = -\omega(Y, X).$$

For the statement about L, if $L \cap V^* = \{0\}$, for each $X \in V$, there is at most one $\alpha \in V^*$ such that $X + \alpha \in L$. As $\dim L = \dim V$, there must be exactly one, so $L = gr(\omega)$ for $\omega : X \mapsto \alpha$ whenever $X + \alpha \in L$. Finally, from the first part, $\omega \in \wedge^2 V^*$. The converse is easy. □

2.2 The symmetries

When we do linear algebra, we have the and the group of automorphisms $\mathrm{GL}(V)$. When doing generalized linear algebra, we want transformations preserving the canonical pairing,

$$\mathrm{O}(V + V^*, \langle \cdot, \cdot \rangle) := \{g \in \mathrm{GL}(V + V^*) \mid \langle gu, gv \rangle = \langle u, v \rangle \text{ for } u, v \in V + V^*\},$$

where we will omit the pairing $\langle \cdot, \cdot \rangle$ and write just $\mathrm{O}(V + V^*)$. Regard the elements of $\mathrm{GL}(V + V^*)$ as a matrix

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

where the entries are endomorphisms $A : V \to V$, $D : V^* \to V^*$, $B : V \to V^*$ and $C : V^* \to V$. By polarization, $g \in \mathrm{GL}(V + V^*)$ belongs to $\mathrm{O}(V + V^*)$ if and only if

$$i_{AX+C\alpha}(BX + D\alpha) = i_X\alpha$$

for any $X + \alpha \in V + V^*$. By choosing $\alpha = 0$ we must have $A^*B : V \to V^*$ defines a skew 2-form, by choosing $X = 0$ we must have $C^*D : V^* \to V$ defines a skew 2-vector, and when $i_X\alpha \neq 0$, we get $A^*D + C^*B = \mathrm{Id}$.

Fine print 2.2. The same identities can be found by using matrices and the matrix representation $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ for the canonical pairing.

We can describe some special elements:

- When $B, C = 0$, we get, for any $A \in \mathrm{GL}(V)$, the element

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}.$$

- When $A = D = 1$ and $B = 0$, we get the elements, for β skew,

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

In the latter case, when we have $C = 0$ instead of $B = 0$, we get one of the most important ingredients of our theory.

Definition 2.3. The elements of the form

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in \mathrm{O}(V + V^*),$$

for $B \in \wedge^2 V^*$, are called ***B*-fields**.

2.3 Maximally isotropic subspaces

Based on Proposition 2.4, maximally isotropic subspaces contain, as particular cases, the linear versions of presymplectic and Poisson structures.

Not all maximally isotropic subspace are necessarily linear presymplectic or Poisson. For instance, $E+\operatorname{Ann} E$ for $E \subseteq V$ any subspace. This includes, as particular cases, the subspaces V and V^*, but when E is a proper subspace, $E+\operatorname{Ann} E$ is neither linear presymplectic nor Poisson.

In this section we shall describe all maximally isotropic subspaces. Let $L \subset V+V^*$ be a maximally isotropic subspace. Denote by π_V the projection $\pi_V : V+V^* \to V$. Define

$$E := \pi_V(L).$$

We first prove that $\operatorname{Ann}(E) \subseteq L$: for $\beta \in \operatorname{Ann}(E)$, we have

$$2\langle \beta, X+\alpha\rangle = i_X\beta = 0,$$

so $\operatorname{Ann}(E) \subset L^\perp$, that is $span(L, \operatorname{Ann}(E))$ is an isotropic subspace. As $L \subset span(L, \operatorname{Ann}(E))$ and L is maximally isotropic, we must have $\operatorname{Ann}(E) \subseteq L$. Moreover, it is then easy to check that $\operatorname{Ann}(E) = L \cap V^*$.

We are going to define a map that sends $X \in E$ to all the possible $\alpha \in V^*$ such that $X+\alpha \in L$. Since α may not be unique, we see what the possible options are. Let $X+\alpha \in L$. We have that $X+\beta \in L$ if and only if $\beta \in \alpha + \operatorname{Ann}(E)$, as $\alpha - \beta = (X+\alpha) - (X+\beta) \in L \cap V^* = \operatorname{Ann}(E)$.

The previous observation allows to define the map

$$\varepsilon : E \to \frac{V^*}{\operatorname{Ann}(E)} \to E^* \tag{2.2}$$

$$X \mapsto \alpha + \operatorname{Ann}(E) \mapsto \alpha_{|E}, \tag{2.3}$$

whenever $X+\alpha \in L$. By the isotropy of L, this map satisfies $\varepsilon \in \wedge^2 E^*$.

We then have that L equals the subspace

$$L(E,\varepsilon) := \{X+\alpha \mid X \in E, \alpha_{|E} = \varepsilon(X)\}.$$

The converse statement is also true and we sum both up in the following proposition.

Proposition 2.4. *For any $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$, the subspace $L(E,\varepsilon)$ is maximally isotropic and any maximally isotropic $L \subset V+V^*$ is of this form, with $E = \pi_V(L)$ and ε defined as in* (2.2).

As a consequence of this, the dimension of any maximally isotropic subspace is exactly $\dim V$.

Fine print 2.3. This statement about the dimension can be proved directly by showing first that if L is maximally isotropic, then $L^\perp$ is semi-definite (that is, the quadratic form Q satisfies $Q \leq 0$ or $Q \geq 0$) and then intersecting L with a maximal positive-definite or negative-definite subspace.

Global versions or maximally isotropic subspaces, together with an integrability condition, will be defined as Dirac structures. Hence, it makes sense to have the following definition.

Definition 2.5. A **linear Dirac structure** is a maximally isotropic subspaces of $V + V^*$.

Note that the image of a maximally isotropic space by an element $g \in \mathrm{O}(V + V^*)$ is again maximally isotropic. With the notation of Section 2.2, we have, for $B \in \wedge^2 V^*$,

$$\left(\begin{smallmatrix} 1 & 0 \\ B & 1 \end{smallmatrix}\right) L(E, \varepsilon) = L(E, \varepsilon + i^* B),$$

where the injection $i : E \to V$ gives the map $i^* : \wedge^2 V^* \to \wedge^2 E^*$, which is a restriction to the elements of E. On the other hand, for $A \in \mathrm{GL}(V)$,

$$\left(\begin{smallmatrix} A & 0 \\ 0 & (A^{-1})^* \end{smallmatrix}\right) L(E, \varepsilon) = L(AE, (A^{-1})^* \varepsilon),$$

where $A^{-1} : AE \to E$ gives $(A^{-1})^* : \wedge^2 E^* \to \wedge^2 (AE)^*$.

2.4 Annihilators of forms

In this section we show an alternative way of looking at maximally isotropic subspaces as the annihilator of a form, as we did for complex structures in (1.7), that is,

$$span(\bar{z}_1, \ldots, \bar{z}_m) = \mathrm{Ann}(z^1 \wedge \ldots \wedge z^m).$$

For $X \in V$, we defined the contraction map $i_X : \wedge^k V^* \mapsto \wedge^{k-1} V^*$. For $\alpha \in V^*$ define now, for $\varphi \in \wedge^k V^*$,

$$\begin{aligned} \alpha\wedge : \wedge^k V^* &\mapsto \wedge^{k+1} V^* \\ \varphi &\mapsto \alpha \wedge \varphi. \end{aligned}$$

Note that for $\lambda \in k$ we have $i_X \lambda = 0$ and $\alpha \wedge \lambda = \lambda \alpha$.

In generalized linear algebra, for $X + \alpha \in V + V^*$ and $\varphi \in \wedge^\bullet V^*$, define the action

$$(X + \alpha) \cdot \varphi := i_X \varphi + \alpha \wedge \varphi. \tag{2.4}$$

Define the annihilator of $\varphi \in \wedge^\bullet V^*$ by

$$\mathrm{Ann}(\varphi) = \{X + \alpha \mid (X + \alpha) \cdot \varphi = 0\}.$$

Denote $(X + \alpha) \cdot ((X + \alpha) \cdot \varphi)$ by $(X + \alpha)^2 \cdot \varphi$. We have that

$$\begin{aligned} (X + \alpha)^2 \cdot \varphi &= (X + \alpha) \cdot (i_X \varphi + \alpha \wedge \varphi) = i_X(\alpha \wedge \varphi) + \alpha \wedge i_X \varphi \\ &= i_X \alpha \cdot \varphi - \alpha \wedge i_X \varphi + \alpha \wedge i_X \varphi = i_X \alpha \cdot \varphi = \langle X + \alpha, X + \alpha \rangle \varphi. \end{aligned}$$

As a consequence of this, $\mathrm{Ann}(\varphi)$ is always an isotropic subspace, as we have $Q(v) = 0$ for $v \in V + V^*$, which implies, by so-called polarization (in any characteristic other than 2)

$$2\langle u, v\rangle = Q(u+v) - Q(u) - Q(v),$$

that is, the pairing $\langle\cdot,\cdot\rangle$ restricted to $\mathrm{Ann}(\varphi)$ is identically zero.

Some of the maximally isotropic subspaces we know can be easily recovered as annihilators of forms. For instance,

$$\mathrm{Ann}(1) = V, \qquad \mathrm{Ann}(vol_V) = V^*,$$

for any choice of volume form $vol_V \in \wedge^{\dim V} V^* \setminus \{0\}$.

A slightly more involved example is

$$E + \mathrm{Ann}(E) = \mathrm{Ann}(vol_{\mathrm{Ann}\,E}).$$

Indeed, $E + \mathrm{Ann}(E) \subset \mathrm{Ann}(vol_{\mathrm{Ann}\,E})$, and since $E + \mathrm{Ann}(E)$ is maximally isotropic, they must be equal.

Note that φ is never unique, as $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\lambda\varphi)$ for $\lambda \neq 0$. The converse is also true, but we will have to prove it later.

The most important example is perhaps

$$gr(\omega) = \{X + i_X\omega : X \in V\},$$

which is described as $\mathrm{Ann}(\varphi)$ for

$$\varphi = e^{-\omega} := \sum_{j=0}^{\lfloor \dim V/2 \rfloor} \frac{(-\omega)^j}{j!} = 1 - \omega + \frac{\omega^2}{2!} - \frac{\omega^3}{3!} + \dots.$$

The exponential of a form plays a very important role in our theory. For $B \in \wedge^2 V^*$, recall the notation

$$e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in \mathrm{O}(V + V^*).$$

On the other hand, we shall consider the action on $\varphi \in \wedge^\bullet V^*$ given by

$$e^{-B} \wedge \varphi = \varphi - B \wedge \varphi + \frac{1}{2!} B^2 \wedge \varphi - \dots = \sum_{j=0} (-1)^j \frac{B^j}{j!} \wedge \varphi.$$

Lemma 2.6. *Let $\varphi \in \wedge^\bullet V^*$ and consider the isotropic subspace $L = \mathrm{Ann}(\varphi)$. We have*

$$e^B\,\mathrm{Ann}(\varphi) = \mathrm{Ann}(e^{-B} \wedge \varphi).$$

Proof. Let $X+\alpha$ such that $(X+\alpha)\cdot\varphi = 0$, we then have

$$(X+\alpha+i_XB)\cdot\sum_{j=0}\frac{(-1)^j}{j!}B^j\wedge\varphi$$
$$=\sum_{j=1}\frac{(-1)^j}{(j-1)!}i_XB\wedge B^{j-1}\wedge\varphi+\sum_{j=0}\frac{(-1)^j}{j!}i_XB\wedge B^j\wedge\varphi = 0,$$

so $e^BL\subset \mathrm{Ann}(e^{-B}\wedge\varphi)$. For the converse, consider $Y+\beta\in\mathrm{Ann}(e^{-B}\wedge\varphi)$, write it as $Y+(\beta-i_YB)+i_YB$. It follows that $Y+(\beta-i_YB)\in\mathrm{Ann}(\varphi)$. □

This lemma together with the example $L(E,0)$ is the key of the following proposition.

Proposition 2.7. *The annihilator* $\mathrm{Ann}(\varphi)$ *of a form* $\varphi\in\wedge^\bullet V^*$ *is a maximally isotropic if and only if it can be written as*

$$\varphi=\lambda e^B\wedge(\theta_1\wedge\ldots\wedge\theta_r),$$

for $\lambda\in\mathbb{R}^*$, $B\in\wedge^2V^*$. *That is,* φ *is the* B*-transform of a decomposable form.*

Proof. All isotropic subspaces can be written as $L(E,\varepsilon)$ for $E\subseteq V$ and $\varepsilon\in\wedge^2E^*$. The map $i^*:\wedge^2V^*\to\wedge^2E^*$ is surjective, as any element $\wedge^2E^*$ can be extended to $\wedge^2V^*$ by choosing a complement of E in V, or, completing a basis of E to a basis of V. Thus, there exists $B\in\wedge^2V^*$ such that $i^*B=\varepsilon$ and we have

$$L(E,\varepsilon)=e^BL(E,0).$$

Recall that $L(E,0)=\mathrm{Ann}(vol_{\mathrm{Ann}\,E})$, so, by Lemma 2.6,

$$L(E,\varepsilon)=e^{-B}\wedge vol_{\mathrm{Ann}\,E},$$

where $vol_{\mathrm{Ann}\,E}$ is a decomposable form and the result follows. We include the λ in the statement for the case $r=0$ and to remind that the whole line has the same annihilator. □

Fine print 2.4. The extension of $\wedge^2E^*$ to $\wedge^2V^*$ by choosing a complement W corresponds to the identity

$$\wedge^2V^*=\wedge^2(E+W)^*\cong\wedge^2E^*\oplus E^*\otimes W^*\oplus\wedge^2W^*.$$

Note that φ as in Proposition (2.7) has a parity: it is either an even or an odd form.

Before we review all this by looking at its complex version, when doing linear generalized complex structure, it is a good idea to tell about what is going on behind the scenes.

2.5 The Clifford algebra and the Spin group

This section could be somehow completely hidden, but it would be a shame not to take this opportunity to see the Clifford action and spinors in action.

Think about the action $(X+\alpha)\cdot\varphi = i_X\varphi + \alpha\varphi$. Since it is linear, we have a map

$$\otimes^\bullet(V+V^*) \to \mathrm{End}(\wedge^\bullet V^*),$$

where $(v_1 \otimes \ldots \otimes v_r)\cdot\varphi = v_1 \cdot (\ldots(v_r \cdot \varphi))$, for $v_i \in V+V^*$. Moreover $(X+\alpha)\otimes(X+\alpha)$ and $Q(X+\alpha)$ act on the same way, so the ideal generated by the elements $(X+\alpha)\otimes(X+\alpha) - Q(X+\alpha)$ is in the kernel of the action, so we have a map

$$\frac{\otimes^\bullet(V+V^*)}{gen((X+\alpha)\otimes(X+\alpha) - Q(X+\alpha) \mid X+\alpha \in V+V^*)} \to \mathrm{End}(\wedge^\bullet V^*).$$

The left-hand side is the Clifford algebra, which we define this in general for a vector space with a quadratic form (W,Q):

$$\mathrm{Cl}(W,Q) := \frac{\otimes^\bullet W}{gen(w\otimes w - Q(w) \mid w \in W)},$$

which we usually denote by $\mathrm{Cl}(W)$. We use the notation

$$[w_1 \otimes w_2 \otimes \ldots \otimes w_{r-1} \otimes w_r] = w_1 w_2 \ldots w_{r-1} w_r,$$

and equally for the product

$$(w_1 \ldots w_r)(z_1 \ldots z_s) = w_1 \ldots w_r z_1 \ldots z_s.$$

Note that, by polarization and being B the corresponding bilinear form, we have the following identity on the Clifford algebra

$$vw = -wv + 2B(v,w). \tag{2.5}$$

By choosing a basis $\{e_i\}$, this means that any element can be written as a linear combination of decomposable elements, where no element from the basis appears more than once. In other words, the products of elements of the basis with no repeated elements form a basis, together with 1 form a basis of $\mathrm{Cl}(W)$. Actually, we can write a map

$$\wedge^\bullet W \to \mathrm{Cl}(W) \tag{2.6}$$

$$v_1 \wedge \ldots \wedge v_r \mapsto v_1 \ldots v_r. \tag{2.7}$$

This map is an isomorphism, but only of vector spaces in general, not of algebras. As an algebra $\mathrm{Cl}(W)$ is generated, analogously to the tensor, symmetric or exterior algebra, by $1 \in k$ and any basis of W, but the product is not the same.

Actually, to be exact, there is one and only one case where the map (2.6) is an isomorphism of algebras: when $Q = 0$. We do have

$$\wedge^\bullet W = \mathrm{Cl}(W, 0),$$

and thus the Clifford algebra can be regarded as a generalization, actually a quantization, of the exterior algebra.

We describe now the structure of $\mathrm{Cl}(W)$ that we will need later. To start with, the $\mathbb{Z}$-grading of $\otimes^\bullet W$ (given by the degree of the tensors) endows $\mathrm{Cl}(W)$ with a $\mathbb{Z}_2$-graded algebra. Indeed, $\mathrm{Cl}(W) = \mathrm{Cl}_0(W) \oplus \mathrm{Cl}_1(W)$, where

$$\mathrm{Cl}_0(W) = [\otimes^{ev} W], \qquad \mathrm{Cl}_1(W) = [\otimes^{odd} W]$$

and $\mathrm{Cl}_i(W)\mathrm{Cl}_j(W) \subseteq \mathrm{Cl}_{i+j}(W)$ for $i, j \in \mathbb{Z}_2$. We will use the decomposition of an element $\alpha \in \mathrm{Cl}(W)$ as

$$\alpha = \alpha_0 + \alpha_1.$$

Secondly, the algebra automorphism extending the map $-\mathrm{Id}$ on W, which acts on $\otimes^\bullet W$ by Id on $\otimes^{ev} W$ and $-\mathrm{Id}$ on $\otimes^{odd} W$, is defined also on $\mathrm{Cl}(W)$. We denote it by $\widetilde{\cdot}$:

$$\widetilde{\alpha} = \widetilde{\alpha_0 + \alpha_1} = \widetilde{\alpha_0} + \widetilde{\alpha_1} = \alpha_0 - \alpha_1.$$

The same happens with the reversing map $\cdot^T$, the antiautomorphism defined by

$$(v_1 \otimes \ldots \otimes v_r)^T = v_r \otimes \ldots \otimes v_1.$$

As it preserves the ideal, it is also defined on $\mathrm{Cl}(W)$ as

$$(v_1 \ldots v_r)^T = v_r \ldots v_1. \tag{2.8}$$

The following observation is key in the understanding of the Pin and Spin groups. Given two vectors $u, v \in W$, such that $Q(u) \neq 0$, the reflection is defined as

$$R_u(v) = v - 2\frac{B(u, v)}{Q(u)}u.$$

This can be rewritten in terms of the Clifford algebra as, by (2.5),

$$v - \frac{1}{Q(u)}(uv + vu)u = v - \frac{1}{Q(u)}(uvu + vQ(u)) = -uv\frac{u}{Q(u)} = -uvu^{-1} = \widetilde{u}vu^{-1},$$

as any $u \in W$ with $Q(u) \neq 0$ is invertible with inverse $u^{-1}/Q(u)$.

The trick of the tilde automorphism allows us to define this as a homomorphism from the group

$$\Gamma = \{v_1 \dots v_r \mid v_i \in V, Q(v_i) \neq 0\},$$

which is a subgroup of the group of units of $\mathrm{Cl}(W)$ to $\mathrm{GL}(W)$. Moreover, all the reflections are orthogonal maps, so we actually have a group homomorphism, for $g = v_1 \dots v_r$,

$$\begin{aligned} \widetilde{\mathrm{Ad}} : \Gamma &\to \mathrm{O}(W) \\ g &\mapsto (x \mapsto \tilde{g} x g^{-1}). \end{aligned}$$

By Cartan-Dieudonné theorem, every orthogonal transformation (in a space with a non-degenerate pairing) is a composition of reflections, by non-null vectors, so we actually get that $\widetilde{\mathrm{Ad}}$ is a surjective map.

We shall compute the kernel of $\widetilde{\mathrm{Ad}}$. A useful observation to do that is that given any element $\gamma \in \mathrm{Cl}(W)$ and a vector $v \in W$, we can always write

$$\gamma = \alpha + v\beta,$$

where α and β are elements of $\mathrm{Cl}(W)$ that can be expressed without v.

Lemma 2.8. *We have that* $\ker \widetilde{\mathrm{Ad}} = \{\pm 1\}$.

Proof. The kernel of $\widetilde{\mathrm{Ad}}$ consists of those g such that $\tilde{g} v = v g$ for any $v \in W$. By writing $g = g_+ + g_-$, the sum of even and odd parts, we get the conditions

$$g_+ v = v g_+, \qquad g_- v = v g_-.$$

Write $g_+ = \alpha + v\beta$, with $\alpha \in \mathrm{Cl}^{ev}(W)$ and $\beta \in \mathrm{Cl}^{odd}(W)$. We then get

$$\alpha v = v \alpha, \qquad v \beta v = v v \beta,$$

where, by parity, the first identity is always satisfied and the second one is never satisfied. Thus $g_+ = \alpha$, which does not contain v in its expression. As the same argument applies for any v, we have that g_+ must be a scalar in k^*. Analogously for g_- one obtains that g_- does not contain v in its expression, but as $g_- \in \mathrm{Cl}_{odd}(W)$, we have $g_- = 0$. Thus $\ker \widetilde{\mathrm{Ad}} \subset k^*$. Conversely, any scalar is in $\ker \widetilde{\mathrm{Ad}}$, as for a non-null vector v we have $(\frac{\lambda}{Q(v)} v) v = \lambda \neq 0$. □

For the last part of this section, assume that W is a real vector space and consider the subgroups

$$\mathrm{Pin}(W) = \{g = v_1 \dots v_r \mid v_i \in V, Q(v_i) = \pm 1\}$$

and

$$\mathrm{Spin}(W) = \{g = v_1 \dots v_{2s} \mid v_i \in V, Q(v_i) = \pm 1\} = \mathrm{Pin}(W) \cap \mathrm{Cl}_0(W).$$

Proposition 2.9. *The groups* $\mathrm{Pin}(W)$ *and* $\mathrm{Spin}(W)$ *are double covers of* $\mathrm{O}(W)$ *and* $\mathrm{SO}(W)$*, respectively.*

Proof. We consider the restriction of $\widetilde{\mathrm{Ad}}$ to $\mathrm{Pin}(W)$. Its kernel is the intersection of $\mathbb{R}^* \subset \Gamma$ with $\mathrm{Pin}(W)$. Let $\lambda = v_1 \dots v_r \in \Gamma \cap \mathrm{Pin}(W)$. We have

$$\lambda^2 = \lambda^T \lambda = (v_1 \dots v_r)^T (v_1 \dots v_r) = v_r \dots v_1 v_1 \dots v_r = Q(v_1) \dots Q(v_r) = \pm 1.$$

So $\lambda^2 = \pm 1$ and $\lambda \in \mathbb{R}^*$, so the only possibilities are $\lambda = \pm 1$.

For the statement about $\mathrm{Spin}(W)$, recall that reflections are orientation-reversing transformations, so in order to preserve the orientation we need an even number of them.

□

Fine print 2.5. The Clifford group is sometimes defined by $\Gamma = \{g \in \mathrm{Cl}(W)^\times \mid \widetilde{\mathrm{Ad}}_g W = W\}$. One can prove that this is equivalent to the definition we gave above. For more details about this, see [Gar11][Sec. 8.1]. For a very good and straightforward introduction to Clifford algebras and the Spin group, see [FO17].

We finish this section by relating the action of $V + V^*$, and hence $\mathrm{Cl}(V + V^*)$, on $\wedge^\bullet V^*$, to the Clifford product when we regard $\wedge^\bullet V^*$ inside the Clifford algebra. The way to do this is by $\wedge^\bullet V^* = \mathrm{Cl}(V^*)$, since V^* is isotropic. So, we actually have an action

$$\mathrm{Cl}(V \oplus V^*) \otimes \mathrm{Cl}(V^*) \to \mathrm{Cl}(V^*). \tag{2.9}$$

One can easily check that $\mathrm{Cl}(V^*)$ is a subalgebra of $\mathrm{Cl}(V \oplus V^*)$. How is the action (2.9) related to the Clifford product? First, let us derive some identities for the union of dual bases $\{e_i\} \cup \{e^i\}$, which is a basis of $V \oplus V^*$:

$$e_i^2 = 0, \qquad (e^i)^2 = 0 \qquad e_i e^i = 1 - e^i e_i, \qquad e_i e^j = -e^j e_i.$$

Now, let us see if there is any relation to the Clifford product. For any vector e_1 and the differential form $1 \in \wedge^\bullet V^*$, our initial action is $i_{e_1} 1 = 0$, while the Clifford product is $e_1 1 = e_1$. They are not the same, but they are actually related. We have that $\wedge^\bullet V^*$ is isomorphic to $\mathrm{Cl}(V^*) \cdot \det V \subset \mathrm{Cl}(V \oplus V^*)$, where $\det V \subset \mathrm{Cl}(V) \subset \mathrm{Cl}(V \oplus V^*)$ is a one-dimensional vector space generated by $e_1 \dots e_n$. Let us check naively that we do not have the same issue as before. The element $1 \in \wedge^\bullet V^*$ corresponds to

$$e_1 \dots e_n \in \mathrm{Cl}(V \oplus V^*).$$

The action of e_1 by the Clifford product is $e_1 e_1 \dots e_n = 0 = 0(e_1 \dots e_n) \in \mathrm{Cl}(V \oplus V^*)$, so the corresponding form is 0 and everything fits. Another example. The element $e^1 \in \wedge^\bullet V^*$ corresponds to $e^1 e_1 \dots e_n \in \mathrm{Cl}(V^*) \cdot \det$. The action of e_1 by Clifford product is

$$e_1 \cdot (e^1 e_1 \dots e_n) = (1 - e^1 e_1) e_1 \dots e_n = 1 e_1 \dots e_n,$$

which corresponds to $1 \in \wedge^\bullet V^*$. One can formally check that this works, that is,

$$(X + \alpha) \cdot \varphi = (X + \alpha)\varphi \det V,$$

where the products on the right-hand side are Clifford products.

2.6 Linear generalized complex structures

Linear generalized complex structures, as introduced in their global version in [Hit12] and developed in [Gua11], are the first truly generalized concept following the philosophy "whatever you do for V do it for $V + V^*$ and require the canonical pairing to be preserved".

A **linear generalized complex structure** is an endomorphism $\mathcal{J}$ of $V + V^*$ such that $\mathcal{J}^2 = -1$ that moreover is orthogonal for the canonical pairing, that is, $\langle \mathcal{J} u, \mathcal{J} v \rangle = \langle u, v \rangle$, for $u, v \in V + V^*$.

Example 2.10. As first, and fundamental, examples, consider a linear complex structure J and a linear symplectic structure ω. The endomorphisms

$$\mathcal{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \qquad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{2.10}$$

are linear generalized complex structures.

It will useful to note that, for $v \in V + V^*$,

$$\langle Jv, v \rangle = 0. \tag{2.11}$$

This is a consequence of the orthogonality of $\mathcal{J}$, as

$$\langle Jv, v \rangle = \langle J^2 v, Jv \rangle = \langle -v, Jv \rangle = -\langle Jv, v \rangle.$$

One of the first questions we answered about linear symplectic and complex structures was what vector spaces admit such a structure.

Proposition 2.11. *A vector space V admits a linear generalized complex structure if and only if $\dim V$ is even.*

Proof. Take a null vector $v_1 \in V + V^*$. The vector $\mathcal{J}v_1$ is null by the orthogonality of $\mathcal{J}$, and orthogonal to v_1 by (2.11), so $N_1 = span(v_1, \mathcal{J}v_1)$ is an isotropic subspace. If it is not maximal, take a vector $v_2 \in N_1^\perp$. Again, $\mathcal{J}v_2$ is null and orthogonal to v_2. Moreover $\mathcal{J}v_2$ is orthogonal to N_1, by orthogonality of $\mathcal{J}$ and $\mathcal{J}N_1 = N_1$. Thus, $N_2 = span(v_1, \mathcal{J}v_1, v_2, \mathcal{J}v_2)$ is an isotropic subspace. This process can be repeated until we obtain

$$N_m = span(v_1, \mathcal{J}v_1, \ldots, v_m, \mathcal{J}v_m),$$

an even-dimensional maximally isotropic subspace. Since the dimension of a maximally isotropic subspace of $V + V^*$ is $\dim V$, we have that $\dim V$ must be even, $2m$. Conversely, by Example (2.10), any even dimensional space admits a linear generalized complex structure, as it admits both linear symplectic and complex structures. □

Just as for usual complex structures, we can describe linear generalized complex structures by looking at the $+i$-eigenspaces, which we will denote by L. For Example 2.10 we have

$$L_J = V^{0,1} \oplus (V^{1,0})^*, \qquad L_\omega = \{X - i\omega(X) \mid X \in V\}. \tag{2.12}$$

These are subspaces of $(V + V^*)_{\mathbb{C}}$. This complex vector space also comes with a pairing, which is the $\mathbb{C}$-linear extension of the pairing on $V + V^*$. For quadratic complex vector space, we do not have a concept of signature as $\langle iv, iv \rangle = -\langle v, v \rangle$, and actually, all non-degenerate symmetric bilinear pairings are equivalent. The spaces in (2.12), apart from satisfying $L_J \cap \overline{L_J} = L_\omega \cap \overline{L_\omega} = \{0\}$, are both maximally isotropic subspaces of $(V + V^*)_{\mathbb{C}}$. The theory of maximally isotropic subspaces and forms that we developed for linear Dirac structures applies here as well. Maximally isotropic subspaces are describe by $L(E, \varepsilon)$, where $E \subset V_{\mathbb{C}}$ is a subspace and $\varepsilon \in \wedge^2 E^*$. The subspace $L(E, \varepsilon)$ fits into the short exact sequence of vector spaces

$$0 \to \operatorname{Ann} E \to L(E, \varepsilon) \to E \to \{0\},$$

which shows that

$$\dim_{\mathbb{C}} L(E, \varepsilon) = \dim_{\mathbb{C}} E + \dim_{\mathbb{C}}(\operatorname{Ann} E) = \dim_{\mathbb{C}} V_{\mathbb{C}} = \dim V.$$

Lemma 2.12. *The $+i$-eigenspace L of a linear generalized complex structure is a maximally isotropic subspace of $(V + V^*)_{\mathbb{C}}$.*

Proof. By using (2.11) and the fact that J is orthogonal, we have

$$\langle x - iJx, x - iJx \rangle = \langle x, x \rangle + i^2 \langle Jx, Jx \rangle - 2i \langle x, Jx \rangle = \langle x, x \rangle - \langle x, x \rangle = 0,$$

that is, all the vectors are null. By polarization, L is isotropic. As $\dim_{\mathbb{C}} L = \dim V$, the result follows. □

So we could say that the condition $\dim_{\mathbb{C}} L = \dim V$ we had for linear complex structures seen as subspaces becomes being maximally isotropic for linear generalized complex structures. It only remains to check that the operator $\mathcal{J}$ defined by i on L and $-i$ on $\overline{L}$, as we did in Proposition 1.5 for linear complex structures, is orthogonal. We use that $\overline{L}$ is also maximally isotropic and any element $v \in V + V^*$ can be written as $v = l + \bar{l}$, with $l \in L$. The orthogonaly of $\mathcal{J}$ follows then from

$$\langle \mathcal{J}(l+\bar{l}), \mathcal{J}(l+\bar{l})\rangle = \langle il - i\bar{l}, l - i\bar{l}\rangle = 2\langle l, \bar{l}\rangle = \langle l+\bar{l}, l+\bar{l}\rangle.$$

Thus, a linear generalized complex structure can be equivalently given by a maximally isotropic subspace $L \subset (V+V^*)_{\mathbb{C}}$ such that $L \cap \overline{L} = \{0\}$.

Fine print 2.6. For a general subbundle $L \subset (V+V^*)_{\mathbb{C}}$, the quantity $\dim L \cap \overline{L}$ is called the real index.

Apart from the operator $\mathcal{J}$ and the subspace L, we also want to describe linear generalized complex structures by using forms and annihilators. The action (2.4) we had complexifies to

$$(V+V^*)_{\mathbb{C}} \to \mathrm{End}(\wedge^\bullet V_{\mathbb{C}}^*).$$

So we will deal with complex forms, unlike the real forms we used for linear Dirac structures. For instance, we have

$$L_J = \mathrm{Ann}(vol_{(V^{1,0})^*}), \qquad L_\omega = \mathrm{Ann}(e^{i\omega}),$$

where $vol_{(V^{1,0})^*}$ is any non-zero $(m,0)$-form for $n = 2m$ (recall that $\dim V$ must be always even by Proposition 2.11).

From Proposition 2.7, we have that the forms whose annihilator is a maximally isotropic subspace are exactly

$$\varphi = \lambda e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r, \tag{2.13}$$

with $\lambda \in \mathbb{C}^*$, $B, \omega \in \wedge^2 V^*$ and $\theta_i \in V^*$.

These φ above give complex Dirac structures, but linear generalized complex structures satisfy the extra condition $L \cap \overline{L} = \{0\}$. How is this reflected on φ? We see it next.

2.7 The Chevalley pairing

Let T be the reversing operator on $\wedge^\bullet V^*$ given, for $\alpha_j \in V^*$, by

$$(\alpha_1 \wedge \ldots \wedge \alpha_t)^T = \alpha_T \wedge \ldots \wedge \alpha_1.$$

We define a pairing $(\cdot,\cdot)$ on $\wedge^\bullet V^*$ with values on $\det V^* = \wedge^{top} V^*$ by

$$(\varphi,\psi) = (\varphi^T \wedge \psi)_{top},$$

where *top* denotes the top exterior power or component.

Lemma 2.13. *For $v \in V + V^*$ we have $(v\cdot\varphi,\psi) = (\varphi, v\cdot\psi)$. Consequently, for $x \in \mathrm{Cl}(V+V^*)$,*

$$(x\cdot\varphi,\psi) = (\varphi, x^T\cdot\psi),$$

and for $g \in \mathrm{Spin}(V+V^*)$,

$$(g\cdot\varphi, g\cdot\psi) = \pm(\varphi,\psi). \tag{2.14}$$

Proof. We start with the first identity. By linearity, we can do it for forms of pure degree. We have

$$(X\cdot\varphi^s,\psi^t) = (\varphi^s, X\cdot\psi^t)$$

when $s+t = \dim V + 1$, as $(\varphi^s)^T\wedge\psi^t = 0$ and $i_X(\varphi^s)^T = (-1)^s(i_X\varphi^s)^T$; and

$$(\alpha\cdot\varphi^s,\psi^t) = (\varphi^s,\alpha\cdot\psi^t)$$

when $s+t = \dim V - 1$, by commutation relations. The second identity follows by repeated application of the first one, and the third identity follows from $g^T g = \pm 1$ for $\mathrm{Spin}(V+V^*)$. □

This pairing is useful to describe the condition $L\cap\overline{L}$ in terms of spinors. We will show this in general for $L = \mathrm{Ann}(\varphi)$ and $L' = \mathrm{Ann}(\varphi')$ two maximally isotropic subspaces. We start with the simplest case.

Lemma 2.14. *Let $L = \mathrm{Ann}(\varphi)$ a maximally isotropic subspace. We have $L\cap V = \{0\}$ if and only if $\varphi^{top} \neq 0$.*

Proof. If $\varphi_{top} \neq 0$, we have $i_X\varphi^{top} \neq 0$ and hence $i_X\varphi \neq 0$, so $L\cap V = \{0\}$. Conversely, write $L = L(E, i^*B)$. For $E = \{0\}$, we have $L(0,0) = V^*$ and is trivially satisfied, so we assume $E \neq \{0\}$. From $L(E,i^*B)\cap V = \{0\}$, we have that i^*B is non-degenerate (which implies r is even), that is, there is no $X \in E$ such that $i_X B = \{0\}$. For $X \in E$, we have

$$i_X(B^j \wedge\theta_1\wedge\ldots\wedge\theta_r) = j i_X B\wedge B^{j-1}\wedge\theta_1\wedge\ldots\wedge\theta_r,$$

which implies that $B^j\wedge\theta_1\wedge\ldots\wedge\theta_r \neq 0$ for $j=1$, then for $j=2$, etc., and inductively we arrive to $j = m - r/2$, which corresponds to $\varphi_{top} \neq 0$. □

Lemma 2.15. *Let $L = \mathrm{Ann}(\varphi)$ be a maximally isotropic subspace, we have $L \cap L(E', 0) = \{0\}$ if and only if $(\varphi, vol_{\mathrm{Ann}\, E'}) \neq 0$.*

Proof. Write $L = L(E, i^*B)$, let $r = \dim E$, $r' = \dim E'$. When $r + r'$ is odd, we have

$$(\varphi, vol_{\mathrm{Ann}\, E'}) = 0,$$

and otherwise we have

$$(\varphi, vol_{\mathrm{Ann}\, E'}) = \pm B^{m-(r+r')/2} \wedge \theta_1 \wedge \ldots \wedge \theta_r \wedge \theta'_1 \wedge \ldots \wedge \theta'_{r'}.$$

If $L \cap L(E', 0) \neq \{0\}$, we have two cases

- either $X \in E'$ belongs to $E \cap \ker B$, and $i_X(\varphi, vol_{\mathrm{Ann}\, E'}) = 0$.
- or $\alpha \in \mathrm{Ann}(E')$ belongs to $\mathrm{Ann}(E)$, so $\theta_1 \wedge \ldots \wedge \theta_r \wedge \theta'_1 \wedge \ldots \wedge \theta'_{r'} = 0$.

In both cases, $(\varphi, vol_{\mathrm{Ann}\, E'}) = 0$.

The condition $L \cap L(E', 0) = \{0\}$ is equivalent to i^*B being non-degenerate on $E \cap E'$ and $\mathrm{Ann}\, E \cap \mathrm{Ann}\, E' = \{0\}$. If $E \cap E' \neq 0$ we have

$$i_X(B^j \wedge \theta_1 \wedge \ldots \wedge \theta_r \wedge \theta'_1 \wedge \ldots \wedge \theta'_{r'}) = j i_X B \wedge B^{j-1} \wedge \theta_1 \wedge \ldots \wedge \theta_r \wedge \theta'_1 \wedge \ldots \wedge \theta'_{r'}$$

for $j = 1$, then $j = 2$, until $j = m - (r + r')/2$, which is $(\varphi, vol_{\mathrm{Ann}\, E'}) \neq 0$. Otherwise, $E \cap E' = \{0\}$ and $\mathrm{Ann}\, E \cap \mathrm{Ann}\, E' = \{0\}$ imply that

$$(\varphi, vol_{\mathrm{Ann}\, E'}) = \pm B^{m-(r+r')/2} \wedge \theta_1 \wedge \ldots \wedge \theta_r \wedge \theta'_1 \wedge \ldots \wedge \theta'_r \neq 0.$$

□

Lemma 2.16. *Let $L = \mathrm{Ann}(\varphi)$ and $L' = \mathrm{Ann}(\psi)$ be maximally isotropic subspaces, we have $L \cap L' = \{0\}$ if and only if $(\varphi, \psi) \neq 0$.*

Proof. We write $L' = e^{-B}L(E', 0)$, so that $\psi = e^B \wedge \psi'$. We then have

$$e^B L \cap L(E', 0) = \{0\}.$$

By Lemma 2.15, $(e^{-B} \wedge \varphi, \psi') \neq 0$. By the invariance up to sign (2.14), we get $(\varphi, e^B \wedge \psi') \neq 0$, that is, $(\varphi, \psi) \neq 0$. □

As a consequence, we have the following.

Proposition 2.17. *A linear generalized complex structure is given by a pure form $\varphi = e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r \in \wedge^\bullet V^*_{\mathbb{C}}$ such that*

$$(\varphi, \overline{\varphi}) \neq 0.$$

We finish this section by answering a question about linear complex structures which was posed at the end of Section 1.5. Recall the notation $\dim V = n = 2m$.

Lemma 2.18. *Linear complex structures are in one to one correspondence to linear generalized complex structures of diagonal form* $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$.

Proof. This follows from the fact that the upper left block has to be a linear complex structure (say, $-J$), and the bottom right block has to be minus its dual (J^*). So we always have

$$\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

□

Proposition 2.19. *The forms in* $\wedge^\bullet V_{\mathbb{C}}^*$ *whose annihilator gives a linear complex structure are those decomposable forms* $\Omega = \theta_1 \wedge \ldots \wedge \theta_m$ *such that* $\Omega \wedge \overline{\Omega} \neq 0$. *Moreover, two forms give the same structure if and only if they are multiples of each other.*

Proof. First note that linear generalized complex structures of the form $\left(\begin{smallmatrix} -J & 0 \\ 0 & J^* \end{smallmatrix}\right)$ are given by forms of the form $\varphi = \theta_1 \wedge \ldots \wedge \theta_r$. From the condition

$$(\varphi, \overline{\varphi}) \neq 0$$

we must have $r = m$ and $\varphi \wedge \overline{\varphi} \neq 0$.

By Lemma 2.18, the $-i$-eigenspace of a usual complex structure is the projection to V of the annihilator on $V + V^*$ of φ as above, so the result follows.

□

2.8 The type

Linear symplectic and complex are particular cases of linear generalized complex structures, but there are many other. The type tells us how far we are from being symplectic or complex.

Definition 2.20. The **type** of a linear generalized complex structure is defined as follows:

- For an automorphism $\mathcal{J}$,

$$type(\mathcal{J}) = \frac{1}{2} \dim_{\mathbb{R}} V^* \cap \mathcal{J} V^*.$$

- For a subspace $L = L(E, \varepsilon)$,

$$type(L) = \dim_{\mathbb{C}} V_C - \dim_{\mathbb{C}} E = \dim_{\mathbb{C}} \mathrm{Ann}_{V_{\mathbb{C}}^*} E = \dim_{\mathbb{C}}(V_{\mathbb{C}}^* \cap L).$$

- For a form $\varphi = \varphi_0 + \ldots + \varphi_n$,

$$type(\varphi) = \min\{k \mid \varphi_k \neq 0\},$$

  that is, the degree of the first non-vanishing component of φ.

Lemma 2.21. *The three definitions are equivalent.*

Proof. We start with the equivalence of the latter two. If $L(E, \varepsilon) = \mathrm{Ann}(\varphi)$ and φ has type r, we can write, as in (2.13),

$$\varphi = \lambda e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r,$$

and we have

$$type(\varphi) = r = \dim \mathrm{Ann}\, E = \dim_{\mathbb{C}} V_{\mathbb{C}} - \dim_{\mathbb{C}} E = type(L).$$

We see now the equivalence between the first and the second one. As $V^* \cap \mathcal{J}V^*$ is a complex vector space, we can take a basis $\{\alpha_1, \mathcal{J}\alpha_1, \ldots, \alpha_r, \mathcal{J}\alpha_r\}$. The elements $\{\alpha_1 - i\mathcal{J}\alpha_1\}$ are linearly independent in $V_{\mathbb{C}}^* \cap L$, so

$$\frac{1}{2} \dim_{\mathbb{R}} V^* \cap \mathcal{J}V^* \leq \dim_{\mathbb{C}}(V_{\mathbb{C}}^* \cap L).$$

Conversely, for a basis of $V_{\mathbb{C}}^* \cap L$, say $\{\gamma_1, \ldots, \gamma_r\}$ we see that

$$\{\mathrm{Re}\gamma_1, \mathrm{Im}\gamma_1, \ldots, \mathrm{Re}\gamma_r, \mathrm{Im}\gamma_r\}$$

belongs to $V^* \cap \mathcal{J}V^*$ and is linearly independent, so

$$\frac{1}{2} \dim_{\mathbb{R}} V^* \cap \mathcal{J}V^* \geq \dim_{\mathbb{C}}(V_{\mathbb{C}}^* \cap L),$$

and the equivalence $type(\mathcal{J}) = type(L)$ follows. □

Fine print 2.7. For Dirac structures, the definition of type is the real analogue for the real subspace L and the real form φ.

We describe now an arbitrary linear generalized complex structures in terms of known structures. We use the representation

$$\varphi = e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r \in \wedge^\bullet V_{\mathbb{C}}^*.$$

We start with the extremal cases: type 0 and type m.

Proposition 2.22. *Linear generalized complex structures of type* 0 *and* m *are* B*-field transforms of, respectively, linear symplectic and complex structures.*

Proof. To start with, by commutativity of 2-forms, we have $e^{B+i\omega} = e^B \wedge e^{i\omega}$. As $e^B \wedge \varphi$ corresponds to the transformation $e^{-B}\operatorname{Ann}(\varphi)$ and e^{-B} is a symmetry of $V + V^*$, that is, an orthogonal transformation, we study structures up to the action by real B-fields e^B.

Type 0 structures are given by $\varphi = e^{B+i\omega} = e^B e^{i\omega}$ such that $(\varphi, \overline{\varphi}) \neq 0$. We have

$$(e^B \wedge e^{i\omega}, \overline{e^B \wedge e^{i\omega}}) = (e^B \wedge e^{i\omega}, e^B \wedge e^{-i\omega}) = (e^{2i\omega}, 1),$$

so $(\varphi, \overline{\varphi}) \neq 0$ if and only if $\omega^m \neq 0$. This means that type 0 structures are B-field transforms of linear symplectic structures ω (seen as a linear generalized complex structure).

Analogously, type m structures are transforms, by a complex B-field, of a complex structure Ω seen as a linear generalized complex structure, that is, $\varphi = e^{B+i\omega}\Omega$. The issue now is that $e^{B+i\omega}$ is a complex B-field, and we regard only real B-fields as symmetries. However, if we consider the complex structure on V coming from Ω, we can decompose $V^*_{\mathbb{C}}$ in terms of (p,q)-forms. The form Ω is of type $(m, 0)$ and we can decompose

$$B + i\omega = C^{2,0} + C^{1,1} + C^{0,2}$$

into different types. As there are no $(m + 2, 0)$ or $(m + 1, 1)$-forms, the only component acting is $C^{0,2}$. By defining

$$B' = C^{0,2} + \overline{C^{0,2}} \in \wedge^{\bullet} V^*,$$

we get a real form such that

$$\varphi = e^{B+i\omega} \wedge \Omega = e^{B'} \wedge \Omega,$$

that is, φ is the real B-field transform of a linear complex structure. □

We now describe linear generalized complex structures of type r.

Proposition 2.23. *A linear generalized complex structures of type* r *is equivalent, up to* B*-field transform, to a linear symplectic structure on a* $n - 2r$*-dimensional subspace* $\Delta \subset V$ *together with a linear complex structure on the* $2r$*-dimensional quotient* V/Δ.

Proof. We have already seen how For $\varphi = e^B e^{i\omega} \wedge \Omega \in \wedge^\bullet V^*_{\mathbb{C}}$, the condition $(\varphi, \overline{\varphi}) \neq 0$ means

$$\omega^{m-r} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

Define $\Delta = \ker_V(\Omega \wedge \overline{\Omega}) \subset V$. As we have

$$\dim_{\mathbb{C}} \ker_{V_{\mathbb{C}}}(\Omega \wedge \overline{\Omega}) = n - 2r, \qquad \ker_{V_{\mathbb{C}}}(\Omega \wedge \overline{\Omega}) = (\ker_V(\Omega \wedge \overline{\Omega}))_{\mathbb{C}},$$

it follows that $\dim_{\mathbb{R}} \Delta = n - 2r$.

Moreover, as Ω is decomposable and $\Omega \wedge \overline{\Omega} \neq 0$, we have

$$\ker_{V_{\mathbb{C}}}(\Omega \wedge \overline{\Omega}) = \ker_{V_{\mathbb{C}}} \Omega \cap \ker_{V_{\mathbb{C}}} \overline{\Omega}.$$

This implies that Ω is well defined as a form on $V_{\mathbb{C}}/\Delta_{\mathbb{C}}$. As such, it is a decomposable form of degree equal to half of the dimension and satisfies $\Omega \wedge \overline{\Omega}$, so it defines a linear complex structure by Proposition 2.19. □

We can thus say that linear generalized complex structures are B-field transforms of a symplectic structure on a subspace $\Delta \subset V$ together with a transversal complex structure on V/Δ.

2.9 A final example

Linear version of Poisson structures appeared in Dirac structures. Do they appear somehow in linear generalized complex geometry? A naive answer would be that if they are invertible, they correspond to a linear symplectic structure, and then they do appear. But the truth is that there is always one.

Lemma 2.24. *For a linear generalized complex structure, the map*

$$P := \pi_V \circ \mathcal{J}_{V^*} : V^* \to V$$

is a linear version of a Poisson structure.

Proof. For $\alpha \in V^*$, we have $P(\alpha) = \pi_V(\mathcal{J}\alpha)$. We can use the pairing to write, for $\beta \in V^*$,

$$P(\alpha, \beta) = \beta(\pi_V(\mathcal{J}\alpha)) = \langle \beta, \mathcal{J}\alpha \rangle = \langle \mathcal{J}\beta, -\alpha \rangle = -\langle \alpha, \mathcal{J}\beta \rangle = -P(\beta, \alpha),$$

where we are using that $\mathcal{J}$ is orthogonal. Thus, $P \in \wedge^2 V$. □

This lemma motivates the following example.

Example 2.25. Let J be a linear complex structure and $P \in \wedge^2 T$ be a linear Poisson structure. The endomorphism

$$\mathcal{J} = \begin{pmatrix} J & P \\ 0 & -J^* \end{pmatrix} \tag{2.15}$$

satisfies $\mathcal{J}^2 = -\operatorname{Id} + JP - PJ^*$, so we need to have

$$JP = PJ^*, \tag{2.16}$$

that is, P commutes with the complex structure J. We also need $\mathcal{J}$ to be orthogonal. We have

$$\langle JX + P\alpha - J^*\alpha, JX + P\alpha - J^*\alpha \rangle = i_X\alpha - (J^*\alpha)(P\alpha),$$

so $\mathcal{J}$ is orthogonal if and only if $(J^*\alpha)(P\alpha) = 0$, but, by using the condition (2.16), we get

$$(J^*\alpha)(P\alpha) = \alpha(JP\alpha) = \alpha(PJ^*\alpha) = P(J^*\alpha, \alpha) = -(J^*\alpha)(P\alpha),$$

so $\mathcal{J}$ is indeed orthogonal. The type of this structure is easily computed by using Definition 2.20. By skew-symmetry, the rank of $P : V^* \to V$ (the dimension of the image) is even, and

$$\dim_{\mathbb{R}}(V^* \cap \mathcal{J}V^*) = n - \operatorname{rk} P,$$

so the type is $\frac{n - \operatorname{rk} P}{2}$. Indeed, when $P = 0$, we get type m, and when P is a symplectic structure we would get type 0.

Chapter 3

Geometry

This chapter is purposely very different to the previous one, as we will assume some familiarity with manifolds. Manifolds are much more complicated than vector spaces, but we will not aspire to be on control of everything (as we did for linear algebra), but to approach and work on the most relevant geometric notions within generalized geometry. A straightforward introduction to manifolds can be found in [Hit12], Chapters 1-6. More details can be found in [Tu08].

3.1 The new operations d, L_X and $[\cdot,\cdot]$

Our starting point is a manifold M, its smooth functions $\mathcal{C}^\infty(M)$ and its tangent and cotangent bundles, which we will denote simply by T and T^*. Sections of T are denoted by $\Gamma(T)$ (or $\mathfrak{X}(M)$) and called vector fields, whereas sections of T^* are denoted by $\Omega^1(M)$ and called differential 1-forms. We will express everything in terms of M by convenience, but it is possible to consider all these constructions over an open set U of M.

The operations we performed on vector spaces, as the wedge product, can be performed also on vector bundles. As a result we obtain the exterior bundle

$$\wedge^\bullet(TM) = \bigoplus_{k=0}^{\dim M} \wedge^k(TM),$$

whose sections are called differential forms

$$\Omega^\bullet(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M),$$

and analogously we could talk about multivector fields.

Two of the operations we defined for vector spaces, the insertion operator i_X and the wedge product $\wedge$ carry over to $\wedge^\bullet(TM)$. On a point $p \in M$, for $X \in T_p$ we have

$$i_X : \wedge^k T_p^* \to \wedge^{k-1} T_p^*,$$

and for $\alpha \in \wedge^k T_p^*$, $\beta \in \wedge^l T_p^*$ we have

$$\alpha \wedge \beta \in \wedge^{k+l} T_p^*.$$

These are operations of a linear nature: they can be defined at a point. When extended to vector fields and differential forms we require smoothness of the bundle maps, say, for $X \in \Gamma(T)$ we have

$$i_X : \wedge^k T_p^* \to \wedge^{k-1} T_p^*.$$

Still, the value of $i_X\alpha$ at p only depends on X_p and α_p.

This linear nature changes when doing geometry. Tangent vectors in T_pM are already an example of that. We can see it already if we take the definition as equivalence classes of curves that have the same derivative, through a chart, at p. Even if we try to hide this, by defining the tangent space as derivations at p of functions around p, we get the same thing. For $X \in \Gamma(T)$ and $f \in \mathcal{C}^\infty(M)$, the value of $X(f)$ at p is $X_p(f)$, and does not depend only on $f(p)$, it is not a linear algebra operation.

Vector fields are derivations of functions, so given a vector field $X \in \Gamma(T)$ and a function $f \in \mathcal{C}^\infty(M)$, we produce a new function $X(f) \in \mathcal{C}^\infty(M)$. These suggest some kind of duality (but in the sense of a derivation) of vector fields and functions. The right way to put this is the **exterior derivative** (or exterior differential), which associates to each function a 1-form

$$\begin{aligned} d : \mathcal{C}^\infty(M) &\to \Omega^1(M) \\ f &\mapsto df \end{aligned}$$

defined by

$$df(X) = X(f).$$

Of course, not every element of $\Omega^1(M)$ is of this form, but a linear combination of elements of the form gdf, where $gdf(X) = gX(f)$.

The exterior derivative is then extended to $\Omega^\bullet(M)$ by requiring linearity, $d^2 = 0$ and the property

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta.$$

Note that this is the way we can extend our definition to higher degree differential forms from 1-forms. To start with

$$d(gdf) = dg \wedge df.$$

Fine print 3.1. In many places, the exterior derivative is defined first for an arbitrary differential form in coordinates, the independence of coordinates is checked and then the properties are proved.

Together with the Lie bracket of vector fields, there is another very important operations we want to consider, the Lie derivative

$$L_X : \wedge^k T^* \to \wedge^k T^*.$$

The exterior derivative can be used to define both the Lie bracket and the Lie derivative. For $X, Y \in \Gamma(T)$,

- the Lie derivative L_X can be defined by Cartan's magic formula

$$L_X = di_X + i_X d.$$

- the Lie bracket can be defined by

$$i_{[X,Y]} = L_X i_Y - i_Y L_X =: [L_X, i_Y],$$

  where the bracket on the right-hand side denotes the commutator.

Fine print 3.2. Note that also $L_X = [d, i_X]$ for a graded commutator, as d and i_X have both odd degree ($+1$ and -1 respectively), unlike L_X, which has even degree, 0. With this in mind we have

$$i_{[X,Y]} = [[d, i_X], i_Y]$$

and this is called a derived bracket.

As a general rule for computations, the expression $L_X\alpha$ is linear in X and a derivation on α.

These are not the usual ways to define the Lie derivative and bracket, so we say a word about other approaches.

The most intuitive of geometrical way of introducing the Lie derivative is via the flow of the vector field. The flow is a 1-parameter subgroup of diffeomorphisms $\{\varphi_t\}$. This has a very much dynamical interpretation. The diffeomorphism φ_t tells us how the manifold has changed after time t. We said 1-parameter because we are considering the evolution with respect with one variable, the time t, and it is a subgroup because we want the evolution after $s + t$ seconds to be the evolution after s seconds of the evolution after t seconds, $\varphi_{s+t} = \varphi_s \circ \varphi_t$. This flow can act both in differential forms, and we get what we got above, or vector fields, and we can recover, up to the convention of a sign, the Lie bracket $L_X Y = \pm[X, Y]$. Finally, the exterior derivative can be recovered by Koszul's formula. For $\theta \in \Omega^k(M)$, define, for $X_j \in \Gamma(T)$,

$$d\theta(X_0, \ldots, X_k) = \sum_{j \leq k} (-1)^j L_{X_j} \theta(X_0, \ldots, \widehat{X}_j, \ldots, X_k) + \sum_{j<l} (-1)^{j+l} \theta([X_j, X_l], \ldots, \widehat{X}_j, \ldots, \widehat{X}_l, \ldots, X_k), \quad (3.1)$$

where the notation $\widehat{X}_j$ denotes that X_j is missing.

3.2 Complex and symplectic structures

If we only knew the linear algebra of complex and symplectic vector spaces and now we wanted to define complex and symplectic structures on a manifold M, we would just take the vector-space analogue of our manifold, namely, the tangent bundle, and define a smooth bundle automorphism $J : TM \to TM$ such that $J^2 = -\operatorname{Id}$ and a non-degenerate differential 2-form $\omega \in \Omega^2(M)$. These are on the good path for a definition of complex and symplectic structures on a manifold, and they are actually almost complex and symplectic structures (almost symplectic is usually referred as non-degenerate 2-form). But we are missing the geometrical information.

The key point is that complex and symplectic manifolds are manifolds modelled, via charts, on complex and symplectic vector spaces such that the changes of chart are holomorphic of symplectomorphic (elements of the symplectic group) maps. If we have an atlas consisting of these charts, we can pass to the vector bundle (differentiate) and define J and ω, respectively, as above. But the opposite is not true. There is a constraint to integrate J and ω to an actual atlas. And this is called the integrability condition. An intuitive but detailed approach about what it means for an almost complex structure to be integrable can be found in [Wel08, p. 28-29].

We give the definition of complex, symplectic and presymplectic structures.

Definition 3.1. A complex structure on a manifold M is a bundle map $J : TM \to TM$ such that $J^2 = -\operatorname{Id}$ and the $+i$-eigenbundle L of J is involutive with respect to the Lie bracket, that is,

$$[\Gamma(L), \Gamma(L)] \subset \Gamma(L).$$

Definition 3.2. A symplectic structure on a manifold M is a non-degenerate 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$.

Definition 3.3. A presymplectic structure on a manifold M is a form $\omega \in \Omega^2(M)$ such that $d\omega = 0$.

Remark 3.4. We mention shortly the global version of a hermitian structure, as discussed in Section 1.7. An **almost Kähler** structure on a manifold M is given by a metric g and a complex structure J such that $g(J\cdot,\cdot)$ is a non-degenerate 2-form. We say that this structure is integrable, and hence gives a **Kähler structure**, when $d\omega = 0$.

3.3 Poisson structures

Again, if you look at the linear version of Poisson structures, we would define a Poisson structure on a manifold as

$$\pi \in \Gamma(\wedge^2 T).$$

Just as before, this is on the right direction, but it is missing the crucial geometric property.

Poisson structures on a manifold or Poisson manifolds, were not introduced and are not usually presented as manifolds having a given local model, but as manifolds together with a Poisson bracket: a Lie bracket on $\mathcal{C}^\infty(M)$, a linear and skew-symmetric map

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ (f,g) &\mapsto \{f,g\}. \end{aligned}$$

This bracket determines a bivector by

$$\pi(df, dg) := \{f,g\},$$

and linearity (here we are using again that differential 1-forms are linear combinations of gdf). Actually, it seems easier to define the Poisson bracket from the bivector π and it would seem that they could be equivalent. The key is that the Poisson bracket is not only a skew-symmetric operation, but a Lie bracket, so it satisfies, for $f, g, h \in \mathcal{C}^\infty(M)$,

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\},$$

and satisfies moreover Leibniz's identity

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

Note that the Jacobi identity means that $\{f, \cdot\}$ defines a vector field. We call the vector fields coming from $\mathcal{C}^\infty(M)$ Hamiltonian vector fields and denote them by

$$X_f := \{f, \cdot\}.$$

This property can be expressed in terms of π, but for that we need the Schouten bracket. The Schouten bracket is the only bracket

$$[\ ,\ ] : \Gamma(\wedge^k T) \times \Gamma(\wedge^m T) \to \Gamma(\wedge^{k+m-1} T),$$

extending the Lie bracket (when $k = m = 1$), acting on functions $f \in \mathcal{C}^\infty(M)$ by $[X, f] = \pi(X)(f)$ for $X \in \Gamma(T)$, and satisfying the following properties, for $Z \in \Gamma(\wedge^a T)$, $Z' \in \Gamma(\wedge^b T)$ and $Z'' \in \Gamma(\wedge^c T)$:

- $[Z, Z'] = -(-1)^{(a-1)(b-1)}[Z', Z]$,
- $[Z, [Z', Z'']] = [[Z, Z'], Z''] + (-1)^{(a-1)(b-1)}[Z', [Z, Z'']]$,
- $[Z, Z' \wedge Z''] = [Z, Z'] \wedge Z'' + (-1)^{(a-1)b} Z' \wedge [Z, Z'']$.

One can then show that the definition of a Poisson bracket is equivalent to

Definition 3.5. A **Poisson structure** is a bivector $\pi \in \wedge^2 T$ such that $[\pi, \pi] = 0$ for the Schouten bracket.

Fine print 3.3. The Poisson bracket has a dynamical interpretation. Any function $H \in \mathcal{C}^\infty(M)$, called Hamiltonian, gives a vector field by $\{H, \cdot\}$ and hence determines some dynamics.

By using the properties of the Schouten bracket, one can write a unique formula for this bracket. This is actually the proof that such a bracket exists. For two multivector fields, $Z \in \Gamma(\wedge^k T)$ and $Z' \in \Gamma(\wedge^m T)$, we set

$$\begin{aligned} & Z \circ Z'(df_1, \ldots, df_{k+m-1}) \\ & \quad := \sum_{\sigma \in \Sigma_{k+m-1}} \frac{(-1)^\sigma}{k!(m-1)!} Z(dZ'(df_{\sigma 1}, \ldots, df_{\sigma k}), df_{\sigma(k+1)}, \ldots, df_{\sigma(k+m-1)}), \end{aligned}$$

so that $Z \circ Z'$ is extended by linearity, and define

$$[Z, Z'] = [Z, Z'] - (-1)^{(k-1)(m-1)} Z' \circ Z.$$

Now we can see that the condition $[\pi, \pi] = 0$ corresponds to the Jacobi identity for $\{\cdot, \cdot\}$. Recall that $\pi(df, dg) = \{f, g\}$, so we have

$$\begin{aligned} \frac{1}{2}[\pi, \pi](df, dg, dh) &= \pi(\pi(df, dg), dh) + \pi(\pi(dg, dh), df) + \pi(\pi(dh, df), dg) \\ &= \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}. \end{aligned}$$

Finally, we have a consequence of the integrability condition,

$$X_{\{f,g\}}(h) = \{\{f,g\},h\} = \{f,\{g,h\}\} - \{g,\{f,h\}\} = [X_f, X_g](h), \qquad (3.2)$$

so $X_{\{f,g\}} = [X_f, X_g]$. We can see this as a Lie algebra homomorphism

$$\begin{aligned}(\mathcal{C}^\infty(M), \{\cdot,\cdot\}) &\to (\Gamma(T), [\cdot,\cdot]) \\ f &\mapsto X_f.\end{aligned}$$

3.4 The symplectic foliation

The formula $X_{\{f,g\}} = [X_f, X_g]$ obtained in (3.2) is important in order to describe geometrically a Poisson structure. First regard π as a map $T^*M \to TM$ and consider the image of π. This is giving us a vector subspace of $R_x := \pi(T_x^*M) \subset T_xM$ at each point $x \in M$. This assignment is called a **distribution**. Note that the dimension of R_x is not necessarily the same. This image is $\mathcal{C}^\infty(M)$-generated by Hamiltonian vector fields, as $\pi(df) = X_f$. Property (3.2) is telling us that the distribution R is involutive with respect to the Lie bracket.

Being involutive is a very special property. If we had an immersed manifold $N \subset M$ and consider the distribution $TN \subset TM$, we have that $\Gamma(TN)$ is involutive with respect to the Lie bracket. The surprising fact is that the converse is true when the dimension of R_x is always the same (the distribution is then called **regular**). Given a regular distribution R, there exists a foliation of M, that is, a collection of disjoint immersed submanifolds such that $M = \cup_i N_i$, for $x \in N_i$ we have $R_x = T_xN_i$, and there are local charts $(x_1, \ldots, x_n)$ of M, such that the leaves correspond to the vanishing of the last $n - \dim R_x$ coordinates.

A subtle point is that we want to consider distributions that are not regular. The definition is kind of the same, but in this case we have to say that we can locally find vector fields generating our distribution at each point. A usual condition to check involutivity is the following.

Definition 3.6. A foliation $D \subset T$ is called of **finite type** when around each point $p \in M$ we can find finitely many local vector fields $\{X_i\}$ generating D such that for any smooth $X \in \Gamma(D)$, we have, for some $c_i^j \in \mathcal{C}^\infty(M)$,

$$[X, X_i] = \sum_{i,j} c_i^j X_j.$$

Integrability is then a consequence of finite type (for the statement see [Sus73, Thm. 81], and to read more about this look at [DZ05, Sec. 1.5]).

We had to talk about a few new concepts, but the bottom line is that the image of a Poisson structure can be seen as the tangent spaces of a foliation on our manifold. What structure do we have in our foliation? Well, we are just starting with a Poisson structure π, so we may want to know how π behaves on the leaves. Take a leave S, can we restrict π to S? For this we would need to define a map $T^*S \to TS$. Start with $\beta \in T_x^*S$, in order to use π, extend β to $\alpha \in T_x^*M$ (by this we mean a 1-form defined only on S; we can always do that but not uniquely). Apply now π to get $\pi(\alpha)$. As $T_xS = \operatorname{Im}\pi_x$, we have that $\pi(\alpha)$ is in T_xS. It only remains to check that this definition does not depend on the choice of α. Say we have α, α' extending β. We then have that $\alpha - \alpha'$ is zero on TS. By skew-symmetry, for any $\gamma \in T_S^*M$,

$$\pi(\alpha - \alpha', \gamma) = -\pi(\gamma, \alpha - \alpha') = (\alpha - \alpha')(\pi(\gamma)) = 0. \tag{3.3}$$

In other words, we are defining $\pi_S(\alpha_{|S}) = \pi(\alpha)$, and we have that the bivector restricts to S,

$$\pi_S : T^*S \to TS,$$

but this is a very special bivector, and not only because one can prove that it is a Poisson bivector, but because it is a bijection, as we have seen it is injective by (3.3), or because TS is the image of π. Thus, each of the leaves has a Poisson structure that is non-degenerate, this is the same as saying that we have a symplectic foliation of the manifold M.

Proposition 3.7. *A Poisson structure on M describes a symplectic foliation on M.*

Chapter 4

Generalized geometry

We are arriving to our final destination: generalized geometry. It is time to combine the generalized linear algebra we developed in Chapter 2 with the differential geometry we recalled in Chapter 3.

Let us sum up: in differential geometry we have the tangent bundle T, whose sections are endowed with a Lie bracket satisfying the Leibniz rule

$$[X, fY] = X(f)Y + f[X, Y].$$

Generalized algebra replaced a vector space V with $V + V^*$, so we would like to consider $T + T^*$ as a generalized version, or analogue, of T.

What does it mean for $T + T^*$ an analogue of T? Well, a good start in order to do geometry would be to define a bracket on its sections. There is a notion that describes precisely this situation: a vector bundle with a nicely behaved bracket.

Definition 4.1. A **Lie algebroid** is a vector bundle $E \to M$ together with a bundle map $\pi : E \to T$, called the anchor map, and a Lie bracket on $\Gamma(E)$, such that, for $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$,

$$[X, fY] = \pi(X)(f)Y + f[X, Y].$$

As a consequence, we also have $\pi([X, Y]) = [\pi(X), \pi(Y)]$.

Examples of Lie algebroids are any Lie algebra $\mathfrak{g}$, regarded as a vector bundle over a point $\mathfrak{g} \to \{*\}$, or the tangent bundle T with an identity anchor map. One is therefore tempted to turn $T + T^*$ into a Lie algebroid, and we could do that in many ways. For instance, the bracket $[X + \alpha, Y + \beta] = [X, Y]$, where the right-hand side is the Lie bracket of vector fields. However, this is not very helpful, as we just ignoring the differential 1-forms. Other brackets are possible, but we want one that interacts with Dirac structures and generalized complex structures.

4.1 The Dorfman bracket

Instead of displaying a God-given bracket, we will look for it. The main idea is that the various integrability conditions we have (involutivity for complex structures, closedness for presymplectic and symplectic forms, $[\pi, \pi] = 0$ for Poisson structures) should all be defined in the same way: as involutivity of the corresponding Dirac or generalized complex structures.

Just as in generalized algebra, an almost complex structure J on M determines the subbundle

$$L_J = T^{0,1} + (T^{1,0})^*.$$

We think now about the complexified version of the bracket. Say the $T_{\mathbb{C}}$-component of the bracket $[X + \alpha, Y + \beta]$ is $[X, Y]$. We then have that the involutivity of $L_{\mathcal{J}}$ would imply the involutivity of $T^{0,1}$ and hence the integrability of $\mathcal{J}$, so it seems a good idea for our bracket to be

$$[X + \alpha, Y + \beta] = [X, Y] + P,$$

where P is some 1-form defined in terms of X, α, Y, β.

Let us look now at Dirac structures. For a form $\omega \in \Omega^2(M)$, we have the subbundle

$$L_\omega = \{X + \omega(X) \mid X \in T\}.$$

We would like the involutivity of L_ω to be equivalent to $d\omega = 0$. As we set

$$[X + \omega(X), Y + \omega(Y)] = [X, Y] + P,$$

involutivity means $\omega([X, Y]) = P$. This has to be equivalent to $d\omega = 0$. By the formula $i_{[X,Y]} = [L_X, i_Y]$ we have

$$P = \omega([X, Y]) = L_X\omega(Y) - i_Y L_X\omega = L_X\omega(Y) - i_Y d\omega(X) - i_Y i_X d\omega.$$

Thus, $d\omega = 0$ if and only if $P = L_X\omega(Y) - i_Y d\omega(X)$. This suggests defining the bracket in general as follows.

Definition 4.2. The **Dorfman bracket** of $X + \alpha$, $Y + \beta \in \Gamma(T + T^*)$ is given by

$$[X + \alpha, Y + \beta] = [X, Y] + L_X\beta - i_Y d\alpha.$$

This bracket gives us a good service for symplectic structures, and we will actually check that all the intebrability conditions we know correspond to the same thing: involutivity with respect to this bracket.

Is this a Lie bracket? It is certainly linear to start with, but when it comes to check skewsymmetry...

$$[X + \alpha, X + \alpha] = di_X\alpha = d\langle X + \alpha, X + \alpha\rangle,$$

we see that this is not the case (as we can always find $X + \alpha$ such that $i_X\alpha$ is not constant). This may seem a bit discouraging, but we should not give up so soon. There are also some good news.

Lemma 4.3. *The Dorfman bracket satisfies, for $u, v, w \in \Gamma(T + T^*)$,*

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

Proof. Direct computation, a bit tedious. □

This is to say that $[u, \cdot]$ is a derivation of the bracket. And we have more.

Lemma 4.4. *For $u, v \in \Gamma(T + T^*)$ and $f \in \mathcal{C}^\infty(M)$, we have*

$$[u, fv] = \pi(u)(f)v + f[u, v].$$

Proof. Direct computation. □

Okay, three out of four, we could say. Not bad! But with this obsession with the bracket, we have forgotten about one of the main features of $V + V^*$: the **canonical pairing**. This pairing generalizes automatically and we get a smooth bundle map

$$\langle \cdot, \cdot \rangle : (T + T^*) \times (T + T^*) \to \mathcal{C}^\infty(M)$$

given by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(i_X\beta + i_Y\alpha).$$

And there are more good news as $[u, \cdot]$ is also a derivation of the pairing.

Lemma 4.5. *The Dorfman bracket satisfies, for $u, v, w \in \Gamma(T + T^*)$,*

$$\pi(u)\langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle.$$

Proof. Direct computation. □

These are a good bunch of properties and they are actually the base for the definition of a new structure.

Definition 4.6. A **Courant algebroid** $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ over a manifold M consists of a vector bundle $E \to M$ together with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E, a linear bracket $[\cdot, \cdot]$ on the sections $\Gamma(E)$ and a bundle map $\pi : E \to TM$ such that the following properties are satisfied for any $u \in \Gamma(E)$:

- we have $[u, u] = D\langle u, u \rangle$,

- the operator $[u,\cdot]$ is a derivation of the bracket,
- the operator $[u,\cdot]$ is a derivation of the pairing,

where $D : \mathcal{C}^\infty(M) \to \Gamma(E)$ is defined, for $f \in \mathcal{C}^\infty(M)$, by $Df = (2\langle\cdot,\cdot\rangle)^{-1}\pi^* df$. As a consequence, we have, for $u, v \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$:

- the anchor map preserves the bracket, $\pi([u,v]) = [\pi(u), \pi(v)]$.
- Leibniz's rule, $[u, fv] = \pi(u)(f)v + f[u,v]$.

Fine print 4.1. One could have forced the bracket to be skew-symmetric, just by defining the Courant bracket

$$[u,v]_{Cou} = \frac{1}{2}([u,v] - [v,u]),$$

and some people prefer to do it this way. However, this would spoil the other nice properties we have, so we do make the choice of working with the Dorfman bracket.

What we have proved in this section is that

$$(T + T^*, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$$

is a Courant algebroid. We will not go into the theory of Courant algebroids, so for us it will be the Courant algebroid. It is a good exercise to see what of the things we will say about $T + T^*$ can be said also about an arbitrary Courant algebroid.

Once we have endowed $T + T^*$ with the Courant algebroid structure, we are ready to do generalized geometry.

4.2 Dirac structures

When we defined the Dorfman bracket, we were inspired by the involutivity of $gr(\omega)$. Based on the fact that ω is closed if and only if $gr(\omega)$ is involutive we give the following definition.

Definition 4.7. A **Dirac structure** is a maximally isotropic subbundle $L \subset T{+}T^*$ whose sections are involutive with respect to the Dorfman bracket.

Would we be able to give a geometrical description of this definition, in the same way we showed that Poisson structures are symplectic foliations? To start with, the projection $\pi(L)$ defines a distribution $E \subset T$. The involutivity of L together with $\pi([u,v]) = [\pi(u), \pi(v)]$ yields that this distribution is of finite type (Definition 3.6), and hence there is a singular foliation $M = \cup_a N_a$ integrating E.

Consider a leaf S of this foliation. Over S we can use a global version of the description of maximally isotropic subspaces as $L(E,\varepsilon)$ of Proposition 2.4. In this case, as $\pi(L)_{|S} = TS$, we have

$$L_{|S} = L(TS, \varepsilon_S),$$

where $\epsilon_S \in \wedge^2 T^*S$. The involutivity of L implies, for $X+\alpha, Y+\beta \in \Gamma(L)$ such that $X_{|S}, Y_{|S} \in \Gamma(TS)$, that

$$(L_X\beta - i_Y d\alpha)_{|S} = \varepsilon_S([X,Y]),$$

which means that

$$d_S\varepsilon_S = 0,$$

that is, we have a presymplectic form on each leaf of the foliation.

Analogously to Proposition 3.7, we have the following.

Proposition 4.8. *A Dirac structure on M describes a presymplectic foliation.*

We took the shortest route to get to this geometrical interpretation. Before passing to the next section, let us review some general facts closely related to what we did. To start with, note that if we have a Dirac structure L, we can restrict the pairing, which becomes just zero by isotropy, the bracket, as it is involutive, and the anchor map. As the pairing is zero, the identity $[u,u] = D\langle u,u\rangle$ becomes $[u,u] = 0$, so the restriction of the bracket becomes a Lie bracket and L becomes a Lie algebroid. Any Lie algebroid L, not only a Dirac structures, determines a foliation. Consider the image $\pi(L) \subset T$. By the property $[\pi(u), \pi(v)] = \pi([u,v])$, we have that, when L is involutive $\pi(L)$ is also involutive and hence defines a foliation.

This Lie algebroid approach is actually helpful when the distribution $\pi(L)$ is regular and hence a subbundle of T. We first need to define the Lie algebroid differential. Analogouly to (3.1), one defines an exterior derivative on L. For $\theta \in \Gamma(\wedge^k L^*)$ and $X_i \in \Gamma(L)$, define

$$\begin{aligned} d_L\theta(X_0, \ldots, X_k) = & \sum_{j\le k} (-1)^j \pi(X_j)\theta(X_0, \ldots, \widehat{X}_j, \ldots, X_k) \\ & + \sum_{j<l} (-1)^{j+l} \theta([X_j, X_l], \ldots, \widehat{X}_j, \ldots, \widehat{X}_l, \ldots, X_k). \end{aligned}$$

The exterior derivative on L satisfies similar properties to the usuale exterior derivative and can also de used to define the Lie derivative on $\Gamma(\wedge^\bullet L^*)$ through the identity $L_X = d_E i_X + i_X d_E$. One can then use the inclusion $i : E \to T$, and how it commutes with $i^* L_X = (d_E i_X + i_X d_E) i^*$ to describe **regular** Dirac structure, those where $\pi(L)$ is a regular distribution.

Proposition 4.9. *A regular Dirac structure on M can be described as $L(E, \epsilon)$ where E is an involutive subbundle of T and $d_E\epsilon = 0$ for the Lie algebroid differential.*

From this, it follows that M has a presymplectic foliation on the leaves given by E, but this only covers the regular case.

4.3 Differential forms and integrability

This section applies for real and complex Dirac structures. We defined a Dirac structure as a global version of a linear Dirac structure (Definition 4.7). We upgraded vector subspaces to vector subbundles, while keeping the maximally isotropic condition, and added an integrability condition, the involutivity with respect to the Dorfman bracket, which we defined in Section 4.1 inspired by presymplectic structures.

We had another way of defining linear Dirac structures, as annihilators of pure differential forms (Proposition 2.7). When trying to give a global version of this, the fact that the differential forms is not uniquely defined is an issue, as we may not have a globally defined differential form. However, since we know that any two spinors differ by a scalar, we can make the following definition.

Definition 4.10. The **canonical bundle** of a Dirac structure L is the smooth line bundle K of $\wedge^\bullet V^*$ given pointwise by

$$K_x = \{\varphi \in \wedge^\bullet V^* \mid \operatorname{Ann}(\varphi) = L_x\} \cup \{0\}.$$

Although φ is defined only locally, we state the results using M and $\varphi \in \Omega^\bullet(M)$. If φ is defined only on an open subset U, by setting $M = U$, the results apply.

In the examples we have, note that the subbundle $gr(\omega)$ for $\omega \in \Omega^2(M)$ is the annihilator of the global form $\varphi = e^{-\omega}$. The integrability condition is $d\omega = 0$, or in other words, $d\varphi = 0$. This may be a good guess, but we cannot just rely on the simplest examples.

We move now to finding the right integrability condition. Before doing that, we need to generalize the usual formula $i_{[X,Y]} = [L_X, i_Y]$ to the action of $\Gamma(T + T^*)$ on $\Omega^\bullet(M)$. Define, for $u \in \Gamma(T + T^*)$,

$$L_u\varphi = d(u \cdot \varphi) + u \cdot (d\varphi).$$

We have the following lemma.

Lemma 4.11. *For $\varphi \in \Omega^\bullet(M)$ and $u, v \in \Gamma(T + T^*)$, we have*

$$[u, v] \cdot \varphi = [L_u, v \cdot]\varphi.$$

Proof. Direct but very tedious computation. □

Proposition 4.12. *The subbundle $L = \mathrm{Ann}(\varphi)$ is involutive if and only if*

$$u \cdot (v \cdot d\varphi) = 0. \tag{4.1}$$

Proof. Involutivity means that, for any u, $v \in \Gamma(L)$, we must have $[u, v] \in \Gamma(L)$, that is,

$$[u, v] \cdot \varphi = 0.$$

By Lemma 4.11, and using that $u \cdot \varphi = v \cdot \varphi = 0$, the previous equation is just

$$u \cdot (v \cdot d\varphi) = 0.$$

□

Condition (4.1) is not satisfactory at all, as we say something about φ by using sections of $\mathrm{Ann}(\varphi)$. However, we can reinterpret it thanks to following lemma.

Lemma 4.13. *Let L be a maximally isotropic subbundle. The canonical subbundle of K is the subbundle annihilated by any section of L. The subbundle $(T + T^*) \cdot K$, that is, the bundle whose sections are exactly $\Gamma(T + T^*) \cdot \Gamma(K)$, is the bundle annihilated by exactly any two sections of L.*

Proof. The statement about K follows from its definition. For the statement about $(T + T^*) \cdot K$, consider $u \in \Gamma(T + T^*)$ and $l \in \Gamma(L)$. We have

$$l \cdot (u \cdot \varphi) = -u \cdot (l \cdot \varphi) + 2\langle u, l\rangle\varphi = 2\langle u, l\rangle\varphi.$$

On one hand, this is not zero for all u and l, so not any section of L annihilates $(T + T^*) \cdot K$. On the other hand, for $l' \in \Gamma(L)$,

$$l' \cdot (l \cdot (u \cdot \varphi)) = l' \cdot (2\langle u, l\rangle\varphi) = 0,$$

so it is exactly annihilated by any two sections of L. □

As a combination of Proposition 4.12 and Lemma 4.13 we get the following proposition.

Proposition 4.14. *A maximally isotropic subbundle L given by $\mathrm{Ann}(\varphi)$ is involutive if and only if there exists $X + \alpha \in \Gamma(T + T^*)$ such that*

$$d\varphi = (X + \alpha) \cdot \varphi.$$

Note that $d\varphi = 0$ is stronger than the integrability condition, and it actually appears in the definition of generalized Calabi-Yau structures (a global closed complex form φ such that $(\varphi, \bar{\varphi}) \neq 0$), a special class, and actually the starting point of, generalized complex structures.

Fine print 4.2. The subbundles K and $(T+T^*)\cdot K$ can be seen as terms of a general filtration of the differential forms. This filtration actually becomes a grading when dealing with generalized complex structures. For more details see Sections 3.6 and 4.2 of [Gua04].

4.4 Generalized diffeomorphisms

A critic of generalized geometry would say that generalized complex geometry is just the study of a certain class (those of real index zero) of complex Dirac structures. An enthusiast of generalized geometry would say that generalized geometry is more than just fitting symplectic and complex structures into generalized complex structures (as we will do), but it is a change of mindset: T becomes $T+T^*$, as we have a pairing the linear transformations become orthogonal transformations, the Lie bracket becomes the Dorfman bracket, the Lie algebra of sections of T becomes the Courant algebroid $T+T^*$, and the diffeomorphisms become... this is the question we want to answer now.

Let us see first a way to redefine the usual diffeomorphisms. Note that any diffeomorphism $F: T \to T$ that is a bundle map induces a diffeomorphism $f: M \to M$. The same is valid if we replace T by any vector bundle.

Lemma 4.15. *The group of diffeomorphisms $F: T \to T$ that are bundle maps, act linearly, and satisfy, for $X, Y \in \Gamma(T)$,*

$$[F(X), F(Y)] = F[X, Y]$$

are exactly the differentials f_* of diffeomorphisms $f: M \to M$.

Proof. First, note that f_* satisfies all the hypotheses. Given F as in the statement inducing $f \in \mathcal{C}^\infty(M)$, consider $G = f_*^{-1} \circ F$, so that G induces the identity on M.

Consider $h \in \mathcal{C}^\infty(M)$, and apply Leibniz's rule to both sides of

$$[G(hX), G(Y)] = G([hX, Y]).$$

We get $G(Y)(h) = Y(h)$ so, as G induces the identity on M, we have that G must be the identity. □

Definition 4.16. A **generalized diffeomorphism** is a diffeomorphism $F: T+T^* \to T+T^*$ that is a bundle map, preserves the pairing, and satisfy, for $u, v \in \Gamma(T+T^*)$,

$$[F(u), F(v)] = F[u, v].$$

Example 4.17. Consider a diffeomorphism $f : M \to M$. We use the notation $f_* : T \to T$ for the differential and $f_* = (f^*)^{-1}$ for the inverse of the pullback on T^*. The orthogonal bundle map

$$f_* := \begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix} : T + T^* \to T + T^*$$

is a generalized diffeomorphism.

Lemma 4.18. *For $B \in \Omega^2(M)$, define $e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : T + T^* \to T + T^*$. We have that $[e^B u, e^B v] = e^B[u, v]$ for any $u, v \in \Gamma(T + T^*)$ if and only if $dB = 0$.*

Proof. Direct computation. It is essentially the same one we did in Section 4.1 to define the Dorfman bracket. □

Theorem 4.19. *The group of generalized diffeomorphisms is*

$$\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M).$$

Proof. We follow the proof of Lemma 4.15. For $F : T+T^* \to T+T^*$ covering $f \in \mathcal{C}^\infty(M)$ consider the map $G = f_*^{-1}F$, where f_* is now like in Example 4.17. As G covers the identity, we apply, for $h \in \mathcal{C}^\infty(M)$, Leibniz's rule to

$$[G(hu), v] = G([u, v]),$$

to deduce $\pi(G(Y)) = Y$. This means that G has the form

$$\begin{pmatrix} 1 & 0 \\ B & D \end{pmatrix}.$$

By orthogonality, that is, $\langle G(u), G(u)\rangle = \langle u, u\rangle$, we get, for $u = X$, that $B \in \Omega^2(M)$ and for $u = X + \alpha$, that $(D\alpha)(X) = \alpha(X)$, i.e., $D = \mathrm{Id}$. Thus $G = e^B$, and by Lemma 4.18, we get that B must be closed.

Any generalized diffeomorphism can be written as $f_* e^B$ where $f \in \mathrm{Diff}(M)$ and $B \in \Omega^2_{cl}(M)$.

Finally, note that $e^B \circ f_* = f_* \circ e^{f^*B}$, as we have

$$f_*(f^*B(X, \cdot)) = f_*(B(f_*X, f_*\cdot)) = B(f_*X, \cdot) = i_{f_*X}B.$$

This gives the semidirect product structure, where the action of $f \in \mathrm{Diff}(M)$ on $B \in \Omega^2_{cl}(M)$ is by pullback. □

Fine print 4.3. Formally, we would write $F \in \Gamma(\mathrm{GL}(T))$ or $F \in \Gamma(\mathrm{O}(T+T^*))$, as sections of bundle of groups (not principal bundles!). These are the bundles whose fibre at x are the linear transformations of T_x or the orthogonal transformations of $(T+T^*)_x$.

4.5 Generalized complex structures

We have looked at generalized complex structures in three different ways. There is only one thing missing in order to have a complete picture of those: expressing integrability in terms of $\mathcal{J}$ and not its $+i$-eigenspace. In order to do that, we translate the involutivity of L to $\mathcal{J}$. As

$$L = \{u - i\mathcal{J}u \mid u \in T + T^*\},$$

we have that

$$[u - i\mathcal{J}u, v - i\mathcal{J}v] = [u, v] - [\mathcal{J}u, \mathcal{J}v] - i([\mathcal{J}u, v] + [u, \mathcal{J}v]),$$

and this is a section of L if and only if

$$\mathcal{J}([\mathcal{J}u, v] + [u, \mathcal{J}v]) + ([u, v] - [\mathcal{J}u, \mathcal{J}v]) = 0.$$

Definition 4.20. We define the **Nijenhuis tensor** of $\mathcal{J}$ as

$$N_{\mathcal{J}}(u, v) = [\mathcal{J}u, \mathcal{J}v] - \mathcal{J}[\mathcal{J}u, v] - \mathcal{J}[u, \mathcal{J}v] - [u, v].$$

It is easy to check that this expression is a tensor (that is, $\mathcal{C}^\infty(M)$-linear) and it follows from above that $\mathcal{J}$ is integrable if and only if $N_{\mathcal{J}}$ vanishes. The definition and the proof are actually the same as for usual complex structures.

We thus have three ways of defining a generalized complex structure:

- A map $\mathcal{J} : T + T^* \to T + T^*$ that is orthogonal, satisfies $\mathcal{J}^2 = -\operatorname{Id}$ and $N_{\mathcal{J}} = 0$.

- A maximally isotropic subbundle $L \subset (T + T^*)_{\mathbb{C}}$ such that $L \cap \bar{L} = \{0\}$ and L is involutive with respect to the Dorfman bracket.

- A line subbundle $K \subset \wedge^k T^*_{\mathbb{C}}$ that is locally given by a pure differental form φ such that $(\varphi, \bar{\varphi}) \neq 0$ for the Chevalley pairing, and $d\varphi = (X + \alpha) \cdot \varphi$ for some $X + \alpha \in \Gamma(T + T^*)$.

We look at type 0 and type m structures as we did in Proposition 2.22.

Proposition 4.21. *A type* 0 *generalized complex structure is the* B*-field transform of a symplectic structure.*

Proof. Type 0 corresponds to a subbundle $L = L(T_{\mathbb{C}}, B + i\omega)$, so there is a globally defined differential form, $e^{B+i\omega}$, whose annihilator is L. The subbundle L is integrable if and only if

$$de^{B+i\omega} = (X + \alpha) \cdot e^{B+i\omega}. \tag{4.2}$$

which is equivalent to

$$(dB + id\omega) \wedge e^{B+i\omega} = (i_X(B + i\omega) + \alpha) \wedge e^{B+i\omega}.$$

It follows that $i_X(B + i\omega) + \alpha = 0$ as there are no 1-forms on the left-hand side, so condition (4.2) is $de^{B+i\omega} = 0$, that is, $dB = d\omega = 0$, so ω is a symplectic structure, and B a closed 2-form, a B-field. □

In the case of type m, one proves that, in general, a transformation by a ∂-closed $(2,0)$-form B of a complex structure is recovered [Gua04, Prop. 4.22].

Structures of other type come from Example 2.25, after checking the integrability condition, which comes from the integrability of the complex and Poisson structure involved.

4.6 Type change in Dirac and generalized geometry

We looked at type in Section 2.8, both for Dirac and generalized complex structures, as it is a property of maximally isotropic subspaces. Any maximally isotropic subspace has associated an invariant called the type, which is an integer. When we consider manifolds, we can say that any maximally isotropic subbundle L has associated an invariant called the type, which is a... function with integer values. In principle this integer could change from point to point, while preserving the parity, but is it really possible on a connected manifold?

To answer this question, we can look locally, on a sufficiently small neighbourhood U, where the subbundle L is given by $\varphi \in \Omega^\bullet(U)$. Write

$$\varphi = \varphi_0 + \ldots + \varphi_n.$$

If φ_0 does not vanish, the type is zero everywhere. If φ_0 vanishes at some point, the type will be higher at that point. As the vanishing set of a map is a closed set, we get that the type is an upper-semicontinuous function, that is, it can jump at closed subsets.

This still does not mean the type can jump, so let us see an example where it happens.

Example 4.22. Let $M = \mathbb{R}^3$ with coordinates (x, y, z) and consider the coordinate vector fields $\{\partial_x, \partial_y, \partial_z\}$, which generate T at every point. Consider the 1-forms $\{dx, dy, dz\}$, which are dual to the coordinate vector fields and generate T^* at every point. Define the subbundle

$$L := span(z\partial_y + dx, z\partial_x - dy, dz) \subset T + T^*.$$

To start with, L is a maximally isotropic subbundle, as

$$\langle z\partial_y + dx, z\partial_y + dx\rangle = \langle z\partial_x - dy, z\partial_x - dy\rangle = \langle dz, dz\rangle = 0,$$

$$\langle z\partial_y + dx, dz\rangle = \langle z\partial_y + dx, z\partial_x - dy\rangle = \langle z\partial_x - dy, dz\rangle = 0.$$

Secondly, L is involutive with respect to the Dorfman bracket. As the Leibniz rule is satisfied, it is enough to check the brackets of the generators

$$[z\partial_y + dx, dz] = [z\partial_x - dy, dz] = 0 \in \Gamma(L),$$

$$[z\partial_y + dx, z\partial_x - dy] = L_{z\partial_y}(-dy) - i_{z\partial_x} d(dx) = -dz \in \Gamma(L).$$

Note that $\pi_T(L) \subseteq T$ is $span(\partial_x, \partial_y)$ when $z \neq 0$, but just zero when $z = 0$. Thus, the type of L is 1 when $z \neq 0$ and 3 when $z = 0$. We see that the type is indeed an upper-semicontinuous function and the parity is preserved.

From the argument before the example, we see that generically the type of a maximally isotropic subbundle is zero, so we define the **type-change** locus as

$$\{x \in M \mid type(L_x) \neq 0\}.$$

As this set is locally the zero set of a function φ_0, we see that it is a closed subset of codimension 1 in M for Dirac structures, or of codimension 2 for generalized complex structures.

Example 4.23. Consider the differential form on $\mathbb{C}^2$ given by

$$\varphi = z_1 + dz_1 \wedge dz_2.$$

This form is pure as for $z_1 = 0$, $\varphi = z_1 \wedge z_2$, a decomposable 2-form, whereas for $z_1 \neq 0$,

$$\varphi = z_1\left(1 + \frac{dz_1 \wedge dz_2}{z_1}\right) = z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}, \tag{4.3}$$

a multiple of the exponential of a complex 2-form. Moreover, φ has real index zero, as

$$(\varphi, \bar{\varphi}) = -dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \neq 0.$$

Thus, φ defines a generalized almost complex structure. We finally check integrability, by using Proposition 4.14. Indeed, consider the generalized vector field $-\partial_{z_2} \in \Gamma(T)$, it satisfies

$$d\varphi = dz_1 = (-\partial_{z_2}) \cdot (z_1 + dz_1 \wedge dz_2) = (-\partial_{z_2}) \cdot \varphi.$$

So φ actually defines a generalized complex structure. The type of this structure is 0 when $z_1 \neq 0$, by (4.3), and 2 when $z_1 = 0$. We can say that the structure given on $\mathbb{C}^2$ by φ is the B-field transform of a symplectic structure on the open subset $z_1 \neq 0$ and jumps to a complex structure on $z_1 = 0$.

4.7 Topological obstruction

For vector spaces, the only obstruction for the existence of a linear complex structures was that the dimension must be even. This means that a manifold admitting an almost complex structure must be even-dimensional, but this is just a necessary condition, possibly and actually not sufficient. Note that we are not even talking about integrability...

The thing is that a necessary and sufficient condition is not easy to find. You can read more about this in the short informal note [Mil18] and references therein, where low-dimensional cases are surveyed, or [Gua04, Prop. 4.16], where a finer necessary condition is stated.

Our aim is not dealing with this highly non-trivial issue, but just show the following.

Proposition 4.24. *A generalized almost complex structure exists on M if and only if an almost complex structure exists on M.*

Proof. Let $\mathcal{J}$ be a generalized almost complex structure. We first show the existence of a $\mathcal{J}$-stable positive-definite subbundle $C_+ \subset T$. It is easy to find a positive-definite subbundle: choose any metric g on M (which always exists by partitions of unity) and define $C_+ = \{X + g(X)\}$. This is not necessarily $\mathcal{J}$-stable, but we can find one by starting with $u \in T + T^*$ such that $\langle u, u \rangle = 1$, adding $\mathcal{J}u$, which satisfies $\langle \mathcal{J}u, \mathcal{J}u \rangle = 1$, $\langle u, \mathcal{J}u \rangle = 0$. The orthogonal complement of $span(u, \mathcal{J}u)$ is not a null subspace, as the pairing has signature $(n-2, n)$. We take a positive-definite element $v \in span(u, \mathcal{J}u)^\perp$ and repeat the process until we get to a rank n subbundle C_+. The process stops as the pairing in $(C^+)^\perp$ has signature $(0, n)$.

The anchor map restricted to C_+, $\pi_{|C_+} : C_+ \to T$, is an isomorphism, since C_+ and T have the same rank, and

$$\ker \pi_{|C_+} = \ker \pi \cap C_+ = T^* \cap C_+ = \{0\}$$

as no element of T^* is positive-definite.

Thus, $\mathcal{J}$ induces in T an automorphism

$$J = \pi_{|C_+} \circ \mathcal{J} \circ \pi_{|C_+}^{-1}$$

squaring to $-\operatorname{Id}$, that is, an almost complex structure.

For the converse, recall that any almost complex structure J produces a generalized almost complex structure $\mathcal{J}_J$ as in the linear case, see (2.10).

□

4.8 A generalized complex manifold that is neither complex nor symplectic

We proved that linear generalized complex structures are a symplectic structure on a subspace together with a complex structure on the quotient (Proposition 2.23). But we have seen globally that generalized complex structures are not just "products" of symplectic and complex structures, as we can have type change (Section 2.8). However, we have showed that a manifold admits a generalized almost complex structure if and only if it admits an almost complex structure. Does every manifold with a generalized complex structure necessarily admit a usual complex structure?

We first show that this is a very subtle question. To start with, it is not known, to this date, whether every almost complex manifold of even dimension greater or equal that six always admits a complex structure. There is neither a proof of this fact, neither a counter-example. So, being realistic, we should focus on dimension 2 and 4.

Dimension 2 is not an option either, as the parity of a generalized complex structure is preserved, and a generalized complex structure on a surface is just a symplectic (type 0) or a complex (type 1) structure.

We are left with type 4. Here we had a nice example on $\mathbb{C}^2$ of a type-change generalized complex structure. This example is invariant by translations on z_2, in particular by translations on z_2 by $\mathbb{Z}^2$, so we can induce a type-change structure on

$$\mathbb{C} \times \frac{\mathbb{C}}{\mathbb{Z}^2} \cong \mathbb{C} \times T^2.$$

By taking a neighbourhood of the identity of the first $\mathbb{C}$, we have a type-change generalized complex structure on $D \times T^2$, where D is a disk and T^2 is a 2-torus. The structure is the B-transform of a symplectic everywhere, apart from the torus corresponding to $0 \in D$, where the structure jumps to be complex.

We refer now to [CG07] for more details and very broadly sketch the main ideas.

This disk of tori, with a generalized complex structure whose type jumps from 0 to 2 at the central torus, allows us to introduce type changes in symplectic 4-manifolds. The idea is to replace the tubular neighbourhood of a certain torus (it has to satisfy conditions like triviality of its normal bundle) by the type change $D \times T^2$. The gluing process has to be made with a lot of care, to make sure that we a smooth manifold, and that the symplectic structure is well defined around the gluing area. This is attained thanks to a so-called $\mathcal{C}^\infty$-log transformations.

The interesting point is that it was already known how to do such a $\mathcal{C}^\infty$-log transformation to obtain a manifold that does not admit neither a complex nor a symplectic structure. If we start with a so-called elliptically fibred K3 complex surface, the resulting manifold is

$$3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2},$$

which can be shown not to admit neither symplectic nor complex structures by using Seiberg-Witten invariants.

Yet, the construction above gives a type change generalized complex structure, so the category of generalized complex manifolds is strictly bigger than the one of complex or symplectic manifolds.

4.9 Frame bundles and generalized metrics

We finish this section by making some comments on structure groups. We purposely do this in a sloppy way, omitting many definitions. This section just wants to mention the global version of the homogeneous spaces that appeared at the end of Section 1.6.

The frame bundle of a manifold M is the fibre bundle whose fibre is the $\mathrm{GL}(n, \mathbb{R})$-torsor

$$FM_x = \{\text{bases of } T_xM\}.$$

Thus, $FM = \cup_{x\in M} FM_x$ can be given the structure of a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

If we have a riemannian metric g on M, this gives a linear riemannian metric g_x at T_xM and one can consider

$$OFM_x = \{g\text{-orthogonal bases of } T_xM\},$$

and $OFM = \cup_{x\in M} OFM_x$ is given the structure of a principal $\emptyset(n, \mathbb{R})$-bundle.

We have that $OFM \subset FM$ as a subbundle. This is called a reduction of the structure group of FM from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n, \mathbb{R})$. A riemannian metric is actually equivalent to giving such a reduction. Equivalently, a complex structure is a reduction from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{C})$. These reductions correspond to almost structures (not necessarily integrable) and are referred to as G-structures.

In generalized geometry, the structure group $\mathrm{GL}(n, \mathbb{R})$ is replaced by $\mathrm{O}(n, n)$, as we have a canonical pairing of signature (n, n). In other words, the frame bundle consists of bases orthogonal with respect to the canonical pairing. Just as an almost complex structure is a reduction from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{C})$, a generalized almost complex structure is a reduction from $\mathrm{O}(2m, 2m)$ to $\mathrm{U}(m, m)$. This makes much sense if you remember the identity $\mathrm{U}(n) = \mathrm{O}(2m, \mathbb{R}) \cap \mathrm{GL}(m, \mathbb{C})$.

What about a generalized metric? In the usual case, we said that a metric is a reduction from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n, \mathbb{R})$, a maximal compact subgroup. It is a general principle that a metric can be interpreted as a reduction to the maximal compact subgroup. Thus, a generalized metric should correspond to a reduction from $\mathrm{O}(n, n)$ to $\mathrm{O}(n) \times \mathrm{O}(n)$. This is attained by specifying a rank n subbundle where the pairing is positive definite (the first $\mathrm{O}(n)$, say), and a rank n subbundle where the pairing is negative definite (the second $\mathrm{O}(n)$). Actually, since rank n is the maximal possible rank, such a positive-definite subbundle determines, by its orthogonal complement, the negative-definite one and we can make the following definition.

Proposition 4.25. *A* ***generalized metric*** *is a maximal positive-definite subbundle $C_+ \subset T + T^*$.*

We already saw an example of a generalized metric: C_+ in Section 4.7. Generalized metrics in $T + T^*$ are easy to describe. As $C_+ \cap T^* = \{0\}$, they can be seen as graphs of maps $T \to T^*$. By decomposing such a map into its symmetric g and skew-symmetric part B, we have

$$C_+ = \{X + g(X) + B(X) \mid X \in T\}.$$

Note that B is globally a 2-form, but is not necessarily closed, so it is not a B-field.

Generalized metrics play a fundamental role when defining generalized Kähler manifolds, as we do to finish this section.

Recall from Remark 3.4 that a Kähler manifold is a complex manifold (M, J) together with a riemannian metric such that $\omega := g(J\cdot, \cdot)$ is a closed 2-form (see also Section 1.7 to see the linear version of this).

When it comes to define a generalized Kähler manifold, symplectic and complex structures have become particular cases of a generalized complex structure, so we should have two generalized complex structures, and we have to generalize the property that $-\omega(Ju, v)$ is a riemannian metric.

Definition 4.26. A **generalized Kähler** structure on M is a pair of commuting generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ such that the $+1$-eigenspace of $-\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

Generalized Kähler manifolds were proved to be equivalent to the so-called bihermitian manifolds, which were defined with the suitable hypothesis to establish some field theories equivalences known as mirror symmetry. This started a fruitful interaction between generalized geometry and mirror symmetry, T-duality, etc.

Fine print 4.4. In the language of structure groups, a generalized Kähler structure gives a reduction from $\mathrm{O}(2n, 2n)$ to $\mathrm{U}(n) \times \mathrm{U}(n)$.

For more on generalized Kähler manifolds, look at [Gua04, Ch. 6].

4.10 Interpolation between complex and symplectic structures

We finally see an example of an interpolation between complex and symplectic structures directly taken from [Gua04, Sec. 4.6].

A hyperKähler manifold is a manifold M together with three anticommuting usual complex structures $\{I, J, K\}$ (that is, they satisfy the relations of quaternions, $IJ = -JI$, etc.) and a riemannian metric such that

$$\omega_I := g(I\cdot, \cdot), \qquad \omega_J := g(J\cdot, \cdot), \qquad \omega_K := g(K\cdot, \cdot)$$

are closed two forms.

Note that when we regard $\omega_I, \omega_J, \omega_K$ as maps $T \to T^*$, we have

$$\omega_I I = -I^*\omega_I, \qquad \omega_J I = I^*\omega_J, \tag{4.4}$$

as

$$\begin{aligned}\omega_I(Iu, v) &= -\omega_I(v, Iu) = -g(Iv, Iu) = -g(Iu, Iv) = -\omega_I(u, Iv),\\ \omega_J(Iu, v) &= g(JIu, v) = -g(IJu, v) = g(Ju, Iv) = \omega_J(u, Iv).\end{aligned}$$

Consider the generalized complex structures

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_{\omega_J} := \begin{pmatrix} 0 & -\omega_J^{-1} \\ \omega_J & 0 \end{pmatrix},$$

which clearly anticommute by (4.4).

Consider, for $t \in [0, \frac{\pi}{2}]$,

$$\mathcal{J}_t = \sin t \mathcal{J}_1 + \cos t \mathcal{J}_2.$$

We have that $(\mathcal{J}_t)^2 = -\operatorname{Id}$ by the anticommutativity and thus define generalized almost complex structures. Are they integrable?

In order to answer this question, note that if L be the $+i$-eigenspace of the generalized complex structure $\mathcal{J}$ and $B \in \Omega^2_{cl}$, the operator

$$e^B \mathcal{J} e^{-B}$$

is the $\mathcal{J}$-operator corresponding to $e^B L$, and we know that L is integrable if and only if $e^B L$ is.

For $t \in [0, \frac{\pi}{2})$, set $B = \tan t\omega_K$, we then have

$$e^B \mathcal{J}_t e^{-b} = \begin{pmatrix} 0 & -(\sec t\, \omega_J)^{-1} \\ \sec t\, \omega_J & 0 \end{pmatrix},$$

which is integrable as ω_J is closed. For $t = \frac{\pi}{2}$, the structure $\mathcal{J}_t$ is clearly integrable, so $\mathcal{J}_t$ gives a curve of generalized complex structures connecting a complex and a symplectic structures.

Background material

[CG07] Gil R. Cavalcanti and Marco Gualtieri. A surgery for generalized complex structures on 4-manifolds. *J. Differential Geom.*, 76(1):35–43, 2007.

[DZ05] Jean-Paul Dufour and Nguyen Tien Zung. *Poisson structures and their normal forms*, volume 242 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2005.

[FO17] José Figueroa-O'Farrill. *Spin geometry.* Lecture notes, 2017. Available at `https://empg.maths.ed.ac.uk/Activities/Spin/SpinNotes.pdf`.

[Gar11] D. J. H. Garling. *Clifford algebras: an introduction*, volume 78 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2011.

[Gre78] Werner Greub. *Multilinear algebra.* Springer-Verlag, New York-Heidelberg, second edition, 1978. Available at `https://link.springer.com/book/10.1007%2F978-1-4613-9425-9`.

[Gua04] Marco Gualtieri. Generalized complex geometry. *ArXiv e-prints*, arXiv:math/0401221, January 2004. Available at `https://arxiv.org/abs/math/0401221`.

[Gua11] Marco Gualtieri. Generalized complex geometry. *Ann. of Math. (2)*, 174(1):75–123, 2011. Available at `http://annals.math.princeton.edu/2011/174-1/p03`.

[Hit12] Nigel Hitchin. *Differentiable manifolds.* Lecture notes, 2012. Available at `https://people.maths.ox.ac.uk/hitchin/hitchinnotes/manifolds2012.pdf`.

[Mil18] Aleksandar Milivojevic. *The existence of (stable) almost complex structures on low-dimensional manifolds.* Informal notes, 2018. Available at
https://www.math.stonybrook.edu/~milivojevic/acsLowDim.pdf.

[Rom08] Steven Roman. *Advanced linear algebra*, volume 135 of *Graduate Texts in Mathematics.* Springer, New York, third edition, 2008. Available at
https://link.springer.com/book/10.1007%2F978-0-387-72831-5.

[Sus73] Héctor J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973. Available at
https://www.jstor.org/stable/1996660.

[Tu08] Loring W. Tu. *An introduction to manifolds.* Universitext. Springer, New York, 2008. Available at
https://link.springer.com/book/10.1007%2F978-1-4419-7400-6.

[Wel08] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics.* Springer, New York, third edition, 2008. Available at
https://link-springer.com/book/10.1007%2F978-1-4757-3946-6.