Generalized Geometry, an introduction
Assignment 10

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. Let $\mathcal{J}$ be an almost genearalized complex structure (that is, a bundle map $\mathcal{J} \in \mathrm{O}(T+T^*)$ such that $\mathcal{J}^2 = -1$) and L the $+i$-eigenspace of $\mathcal{J}$ in $(T+T^*)_{\mathbb{C}}$.

• Prove that the following expression defines a tensor (the Nijenhuis tensor), that is, it is $\mathcal{C}^\infty(M)$-linear, where $u, v \in \Gamma(T+T^*)$,

$$N_{\mathcal{J}}(u,v) = [\mathcal{J}u, \mathcal{J}v] - \mathcal{J}[\mathcal{J}u, v] - \mathcal{J}[u, \mathcal{J}v] - [u,v].$$

• Prove that $\mathcal{J}$ is a generalized complex structure if and only if $N_{\mathcal{J}}$ vanishes.

• Compare this to the case of usual complex structures.

Problem 2.

Let L be the $+i$-eigenspace of the generalized complex structure $\mathcal{J}$ and $B \in \Omega^2_{cl}$.

• What is the $\mathcal{J}$-operator corresponding to $e^B L$?

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two anticommuting generalized almost complex structures, that is

$$\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_2\mathcal{J}_1.$$

• Show that, for $t \in [0, \frac{\pi}{2}]$, the following is a generalized almost complex structure:

$$\mathcal{J}_t = \sin t \mathcal{J}_1 + \cos t \mathcal{J}_2.$$

A Kähler manifold is a complex manifold (M, J) together with a riemannian metric such that $\omega := g(J\cdot, \cdot)$ is a closed 2-form (see Section 1.7 of the lecture notes to see the linear version of this). A hyperKähler manifold is a manifold M together with three anticommuting usual complex structures $\{I, J, K\}$ (that is, they satisfy the relations of quaternions, $IJ = -JI$, etc.) and a riemannian metric such that

$$\omega_I := g(I\cdot, \cdot), \qquad \omega_J := g(J\cdot, \cdot), \qquad \omega_K := g(K\cdot, \cdot)$$

are closed two forms.

• Show that, when we regard $\omega_I, \omega_J, \omega_K : T \to T^*$, we have

$$\omega_I I = -I^* \omega_I, \qquad \omega_J I = I^* \omega_J.$$

Consider the generalized complex structures

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_{\omega_J} := \begin{pmatrix} 0 & -\omega_J^{-1} \\ \omega_J & 0 \end{pmatrix}.$$

• Prove that $\mathcal{J}_I$ and $\mathcal{J}_{\omega_J}$ anticommute.

Consider $\mathcal{J}_t$ as above for $\mathcal{J}_1 = \mathcal{J}_I$ and $\mathcal{J}_2 = \mathcal{J}_{\omega_J}$.

• Prove that $\mathcal{J}_t$ is a generalized complex structure. (Hint: use a B-field related to ω_K to transform $\mathcal{J}_t$ into a generalized complex structure whose integrability is easy.)