Generalized Geometry, an introduction
Assignment 2

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. The restriction of a symplectic form to a subspace $U \subset V$ is not necessarily symplectic, but we still have a skew-symmetric bilinear map

$$\omega : V \times V \to k,$$

which is possibly degenerate. This is called a **linear presymplectic structure**.

We found a normal form for a symplectic structure, namely, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Find a normal form for a presymplectic structure, that is, show the existence of a basis (v_i) such that the matrix $\omega(v_i, v_j)$ is always of a similar shape.

Problem 2. Given a symplectic form, we have an isomorphism $V \to V^*$. We invert this isomorphism to get a map $V^* \to V$, which we can see as a map

$$\pi : V^* \times V^* \to k.$$

Prove that the map π is bilinear, non-degenerate and skew-symmetric.

Problem 3. Consider a real vector space with a linear complex structure (V, J) and its complexification $V_{\mathbb{C}}$. Prove that $iJ = Ji$. When does the map

$$ai + bJ,$$

for $a, b \in \mathbb{R}$, define a linear complex structure on $V_{\mathbb{C}}$, seen as a real vector space?

Problem 4. Consider a real vector space with a linear complex structure (V, J). Prove that the map

$$J^* : V^* \to V^*,$$

given by

$$J^*\alpha(v) = \alpha(Jv),$$

for $\alpha \in V^*$ and $v \in V$, defines a linear complex structure on V^*. Given a basis (v_i) with dual basis (v^i), prove that

$$J^*v^i = -(Jv_i)^*.$$

Problem 5. * Describe the space of linear complex structures on a given even-dimensional real vector space.

Problem 6. Consider a real vector space with a linear symplectic structure (V, ω). We say that a linear complex structure J is ω-compatible if

$$\omega(\cdot, J\cdot) : V \times V \to \mathbb{R}$$

defines a linear riemannian metric.

- Prove that a linear complex structure J is ω-compatible if and only if $\omega(Ju, Jv) = \omega(u, v)$ for any $u, v \in V$, and $\omega(u, Ju) > 0$ for any non-zero $u \in V$.
- ** Prove that an ω-compatible linear complex structure always exists.
- ** Let J be an ω-compatible linear complex structure and denote by g the linear riemannian metric $\omega(\cdot, J\cdot)$. Prove that

  $$h(u, v) = g(u, v) - i\omega(u, v)$$

  defines a hermitian metric on the complex vector space (V, J).
- ** With the suitable notion of topology, prove that the space of ω-compatible linear complex structures in path connected.