Generalized Geometry, an introduction
Assignment 4

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. We study the maximally isotropic subspaces $V + V^*$ for the canonical pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(i_X \beta + i_Y \alpha).$$

Recall that we proved that the dimension of an isotropic subspace is at most $\dim V$.

- Prove now that the dimension of a maximally isotropic subspace is exactly $\dim V$. Recall that a maximally isotropic subspace is an isotropic subspace that is not strictly contained in an isotropic subspace.

We first prove a couple of elementary results. Let $E \subseteq V$ a vector subspace:

- Prove that $E + \operatorname{Ann} E$ is maximally isotropic.
- Prove that $E^* \cong \frac{V^*}{\operatorname{Ann}(E)}$.

Denote by π_V the projection $\pi_V : V + V^* \to V$. Let $L \subset V + V^*$ be a maximally isotropic subspace. Define $E = \pi_V(L)$.

- Prove that $\operatorname{Ann}(E) \subseteq L$.
- Let $X + \alpha \in L$. Prove that $X + \beta \in L$ if and only if $\beta \in \alpha + \operatorname{Ann}(E)$.
- Check that the correspondence $\varepsilon : E \to \frac{V^*}{\operatorname{Ann}(E)} \to E^*$ given by $\varepsilon : X \mapsto \alpha + \operatorname{Ann}(E) \mapsto \alpha_{|E}$ whenever $X + \alpha \in L$ is a well-defined map.
- Check that $L = L(E, \epsilon)$, where

$$L(E, \epsilon) = \{X + \alpha \mid X \in E, \alpha_{|E} = \varepsilon(X)\}. \tag{1}$$

Conversely, for any $E \subseteq V$ and $\varepsilon \in \wedge^2 E^*$:

- Show that $L(E, \varepsilon)$, as given by (1), is maximally isotropic in $V + V^*$.

Problem 2. For $X \in V$, we defined the contraction map $i_X : \wedge^k V^* \mapsto \wedge^{k-1} V^*$. For $\alpha \in V^*$ define now, for $\varphi \in \wedge^k V^*$,

$$\begin{aligned} \alpha\wedge : \wedge^k V^* &\mapsto \wedge^{k+1} V^* \\ \varphi &\mapsto \alpha \wedge \varphi. \end{aligned}$$

Note that for $\lambda \in k$ we have $i_X \lambda = 0$ and $\alpha \wedge \lambda = \lambda\alpha$.

In generalized linear algebra, for $X + \alpha \in V + V^*$ and $\varphi \in \wedge^\bullet V^*$, define the action

$$(X + \alpha) \cdot \varphi := i_X \varphi + \alpha \wedge \varphi.$$

- Prove that $(X + \alpha) \cdot ((X + \alpha) \cdot \varphi = \langle X + \alpha, X + \alpha \rangle \varphi$.

Define the annihilator of $\varphi \in \wedge^\bullet V^*$ by

$$\mathrm{Ann}(\varphi) = \{X + \alpha \mid (X + \alpha) \cdot \varphi = 0\}.$$

We want to use annihilators to describe maximally isotropic subspaces in a similar way as we described a complex structure as

$$span(\bar{z}_1, \ldots, \bar{z}_m) = \mathrm{Ann}(z^1 \wedge \ldots \wedge z^m).$$

- Prove that $\mathrm{Ann}(\varphi)$ is always an isotropic subspace, for $\varphi \in \wedge^\bullet V^*$.
- Show that $\mathrm{Ann}(1) = V$ and $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\lambda\varphi)$ for $\lambda \neq 0$.
- Let vol_V be a non-zero element of $\wedge^{\dim V} V^*$, show that $\mathrm{Ann}(vol_V) = V^*$.
- What is the relation between $\{\alpha \in \wedge^k V^* \mid i_X \alpha = 0 \text{ for } X \in E\}$ and $\wedge^k \mathrm{Ann}(E)$?
- Let $E \subseteq V$ be a subspace. Find φ such that $\mathrm{Ann}(\varphi) = E + \mathrm{Ann}(E)$.
- Let $\omega \in \wedge^2 V^*$, find φ such that $\mathrm{Ann}(\varphi) = gr(\omega)$.

If you feel adventurous, you can try also:

- * Let $\pi \in \wedge^2 V$, find φ such that $\mathrm{Ann}(\varphi) = gr(\pi)$.
- ** Prove that $\mathrm{Ann}(\varphi) = \mathrm{Ann}(\psi)$ if and only if $\varphi = \lambda\psi$ for $\lambda \in k^*$.
- *** When does φ define a maximally isotropic subspace?