Generalized Geometry, an introduction
Assignment 5

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. A linear generalized complex structure is an endomorphism $\mathcal{J}$ of $V + V^*$ such that $\mathcal{J}^2 = -1$ that moreover is orthogonal for the canonical pairing, that is, $\langle \mathcal{J}u, \mathcal{J}v \rangle = \langle u, v \rangle$, for $u, v \in V + V^*$.

Let J be a linear complex structure and ω a linear symplectic structure.

- Show that the endomorphisms

$$\mathcal{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \qquad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

  define linear generalized complex structures.

- What are the $\pm i$-eigenspaces for $\mathcal{J}_J$ and $\mathcal{J}_\omega$?

- Let L_J and L_ω be the $+i$-eigenspaces for $\mathcal{J}_J$ and $\mathcal{J}_\omega$. Show that they satisfy

$$L_J \cap \overline{L_J} = \{0\}, \qquad L_\omega \cap \overline{L_\omega} = \{0\}.$$

Just as in usual linear complex structures, a linear generalized complex structure can be described by a maximally isotropic subspace $L \subset (V + V^*)_{\mathbb{C}}$ such that $L \cap \overline{L} = \{0\}$. Note that the condition $\dim L = \dim V$ of linear complex structures is replaced by being maximally isotropic.

Apart from the operator $\mathcal{J}$ and the subspace L, we want to describe linear generalized complex structures by using differential forms. In this case they will have to be complex differential forms, unlike the real differential forms we used for linear Dirac structures.

- Find φ, $\psi \in \wedge^\bullet V^*_{\mathbb{C}}$ such that $\mathrm{Ann}(\varphi) = L_J$ and $\mathrm{Ann}(\psi) = L_\omega$.

In order to solve this, you may want find first $\varphi \in \wedge^\bullet V^*$ such that $\mathrm{Ann}(\varphi) = gr(\omega)$, as asked in the previous assignment.

Problem 2. We have defined $\mathrm{GL}(n,\mathbb{C})$ as a subgroup of $\mathrm{GL}(2n,\mathbb{R})$, but there is another possible, and more intuitive, interpretation, as invertible matrices with complex entries. For instance, the elements of $\mathrm{GL}(1,\mathbb{C})$ are just non-zero complex numbers.

- What is the matrix in $\mathrm{GL}(2,\mathbb{R})$ corresponding to $z = a + ib \in \mathrm{GL}(1,\mathbb{C})$?
- What can you say in general for $\mathrm{GL}(n,\mathbb{C})$?

On the other hand, we saw, in term of matrices, that

$$\mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{O}(2n) \cap \mathrm{GL}(n,\mathbb{C}). \tag{1}$$

We shall show that this is actually the unitary group $\mathrm{U}(n)$, whose definition we recall. First, a hermitian metric on a complex vector space V is a map $h : V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear on the first component and satisfies $h(v,u) = \overline{h(u,v)}$, which implies that is anti-linear on the second component. The usual example is $h(u,v) = u^T\overline{v}$. Define

$$\mathrm{U}(n) := \{M \in \mathrm{GL}(n,\mathbb{C}) \mid h(Mu, Mv) = h(u,v), \text{ for } u,v \in \mathbb{C}^n\}.$$

- Prove that this group equals the intersections in (1).
- Try to make statement (1) valid for any linear riemannian metric, complex stucture, etc., not just the ones given by Id, J, etc.

Problem 3. Denote by $\mathrm{O}(n,n)$ the group $\mathrm{O}(V+V^*, \langle\cdot,\cdot\rangle)$. Describe the elements of $\mathrm{GL}(V+V^*)$ as block matrices

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

with $A : V \to V$, $D : V^* \to V^*$, $B : V \to V^*$ and $C : V^* \to V$.

- Find sufficient and necessary conditions on A, B, C, D for M to belong to $\mathrm{SO}(n,n)$.
- Describe the elements such that $B, C = 0$.
- Describe the elements such that $A = D = 1$, and $C = 0$ or $B = 0$.

We see what the effect of these transformations is.

- Let L be a (maximally) isotropic subspace, show that, for $M \in \mathrm{O}(n,n)$, the subspace $M(L)$ is (maximally) isotropic.
- Describe, for $B \in \wedge^2 V^*$, the subspace $\left(\begin{smallmatrix} 1 & 0 \\ B & 1 \end{smallmatrix}\right) L(E,\varepsilon)$.
- Describe, for $A \in \mathrm{GL}(V)$, the subspace $\left(\begin{smallmatrix} A & 0 \\ 0 & A^* \end{smallmatrix}\right) L(E,\epsilon)$.

Finally, in case you are familiar with the concept of Lie algebra:

- ** Describe the Lie algebra of $\mathrm{O}(n,n)$.