Generalized Geometry, an introduction

Assignment 6

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. A linear generalized complex structure on V is an automorphism $\mathcal{J} \in \mathrm{O}(V + V^*)$ such that $\mathcal{J}^2 = -1$.

- Find a necessary and sufficient condition on V to admit a linear generalized complex structure.

Problem 2. For $B \in \wedge^2 V^*$, use the notation $e^B := \left(\begin{smallmatrix} 1 & 0 \\ B & 1 \end{smallmatrix}\right) \in \mathrm{O}(V + V^*)$. (This is actually the exponential of an element of the Lie algebra $\mathfrak{o}(V + V^*)$ to $\mathrm{O}(V + V^*)$ acting on the standard representation.)

On the other hand, consider the action on $\varphi \in \wedge^\bullet V^*$ given by

$$e^{-B} \wedge \varphi = \varphi - B \wedge \varphi + \frac{1}{2!} B^2 \wedge \varphi - \dots$$

(This is the spinorial action of the exponential of the same $B \in \mathfrak{o}(V + V^*)$, yes, B and not $-B$, to $\mathrm{Spin}(V + V^*)$.)

- Let L be a maximally isotropic subspace such that $L = \mathrm{Ann}(\varphi)$ for some $\varphi \in \wedge^\bullet V^*$. Prove that
$$e^B L = \mathrm{Ann}(e^{-B} \wedge \varphi).$$
- Recall how we write $L(E, 0)$ as an annihilator of a spinor and use the previous identity to say which elements $\varphi \in \wedge^\bullet V^*$ give a maximally isotropic subspace by $\mathrm{Ann}(\varphi)$.
- Spell out the complex version of the previous statement: which elements $\varphi \in \wedge^\bullet V^*_{\mathbb{C}}$ give a maximally isotropic subspace by $\mathrm{Ann}(\varphi) \subset (V + V^*)_{\mathbb{C}}$.
- Regarding the last two items: can you say something about the degree of the φ you found?

Problem 3. Let Γ be the Clifford group of the Clifford algebra $\mathrm{Cl}(W, Q)$.

$$\Gamma := \{g \in \mathrm{Cl}(W)^\times \mid \widetilde{\mathrm{Ad}}_g W = W\},$$

where $\widetilde{\mathrm{Ad}}_g(x) = \tilde{g}xg^{-1}$ and $\tilde{}$ denotes the endomorphism extending $-\mathrm{Id}_W$.

- * Prove that the Clifford group is also given by

$$\{g = v_1 \dots v_r \mid v_j \in W, Q(v_j) \neq 0\}.$$

Recall the definition of the Pin group

$$\mathrm{Pin}(V) = \{g = v_1 \dots v_r \mid v_j \in W, Q(v_j) = \pm 1\}.$$

- * Compute the kernel of the projections

$$\Gamma \to \mathrm{O}(W), \qquad \mathrm{Pin}(W) \to \mathrm{O}(W).$$

Problem 4. We talk now about $\mathrm{Cl}(V \oplus V^*)$.

- Check that $\wedge^\bullet V^* = \mathrm{Cl}(V^*) := \mathrm{Cl}(V^*, \langle \cdot, \cdot \rangle_{|V^*})$.
- Check that $\mathrm{Cl}(V^*)$ is a subalgebra of $\mathrm{Cl}(V \oplus V^*)$.
- Consider the union of dual bases $\{e_i\} \cup \{e^i\}$, which is a basis of $V \oplus V^*$. Check:

$$e_i^2 = 0, \qquad (e^i)^2 = 0 \qquad e_i e^i = 1 - e^i e_i, \qquad e_i e^j = -e^j e_i.$$

We see if there is any relation of the action of $\mathrm{Cl}(V \oplus V^*)$ on $\wedge^\bullet V^*$ with the Clifford product of $\mathrm{Cl}(V \oplus V^*)$ with an element $\mathrm{Cl}(V^*) \subset \mathrm{Cl}(V \oplus V^*)$.

- Find an example to show that they are not the same.

Choose a volume form $\det V$ of V, seen as an element in $\mathrm{Cl}(V) \subset \mathrm{Cl}(V \oplus V^*)$, and regard $\wedge^\bullet V^*$ inside $\mathrm{Cl}(V \oplus V^*)$ as

$$\mathrm{Cl}(V^*)\mathrm{det}V \subset \mathrm{Cl}(V \oplus V^*),$$

where the juxtaposition denotes the Clifford product.

- Convince yourself that the action we defined on $\wedge^\bullet V^* = \mathrm{Cl}(V^*)$, corresponds to the Clifford product on $\mathrm{Cl}(V^*)\mathrm{det}V$.

Problem 5.

- ** Is the action of $\mathrm{O}(V + V^*)$ on maximally isotropic subspaces transitive?