Beyond the canonical symmetric pairing in generalized geometry

Roberto Rubio
joint with Filip Moučka
(who typed most of these slides)

UAB
Universitat Autònoma de Barcelona

Symplectic geometry seminar

impa
Instituto de Matemática Pura e Aplicada

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The canonical skew-symmetric pairing

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has always been there:

2.2. Dirac structures on manifolds. In §1 we saw that we could think of a Dirac structure on a vector space V as a subspace $L \subset V \oplus V^*$ which is isotropic under $\langle\ ,\ \rangle_+$. We now wish to extend some of the results of the linear case to manifolds P. We may define natural symmetric and skew-symmetric pairings on $TP \oplus T^*P$:

$$\langle (X, \omega), (Y, \mu)\rangle_+ = \tfrac{1}{2}(\omega(Y)+\mu(X)), \tag{2.2.1}$$

$$\langle (X, \omega), (Y, \mu)\rangle_- = \tfrac{1}{2}(\omega(Y)-\mu(X)). \tag{2.2.2}$$

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Can we do Dirac and generalized complex geometry with the skew-symmetric pairing?

Lagrangian means now symmetric

$$L = L^{\perp}$$

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Examples of real lagrangian are

$gr(g) = \{X + g(X) : X \in TM\}$ for $g \in \Gamma(\mathrm{Sym}^2\, TM)$.

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And examples of complex Lagrangian are

$gr(-ig) = \{X - ig(X) : X \in TM\}$ for $g \in \Gamma(\mathrm{Sym}^2\, TM)$.

$T_{0,1}M \oplus T^*_{1,0}M$ for a complex structure.

We will use the notation

$$\Upsilon^\bullet(M) = \bigoplus_{k=0}^{\infty} \Gamma(\mathrm{Sym}^k\, TM).$$

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We need a new bracket!

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$$(X + \alpha) \cdot \sigma := \iota_X \sigma + \alpha \odot \sigma.$$

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Let ∇ be a torsion-free connection on M, we introduce the **symmetric derivative**:

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Compare with the Dorfman bracket derivation: $[[(X+\alpha)\cdot\,, \mathrm{d}]_{\mathrm{g}}, (Y+\beta)\cdot\,]_{\mathrm{g}}\,\varphi = [X+\alpha, Y+\beta]\cdot\varphi$.

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Is there a **symmetric version of Cartan calculus** $(d,\iota,[\,,\,],L)$ behind this?

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$\Rightarrow$ The theory depends on the choice of a torsion-free affine connection.

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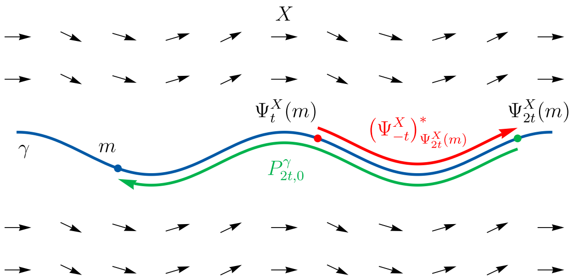

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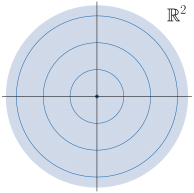

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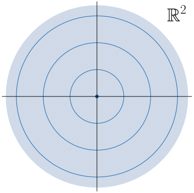

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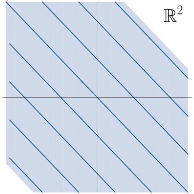

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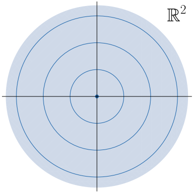

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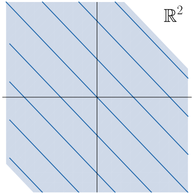

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By [Lewis, 1998], a distribution $\Delta \leq T$ is geodesically inv. iff $\langle\Gamma(\Delta) : \Gamma(\Delta)\rangle_s \subseteq \Gamma(\Delta)$.

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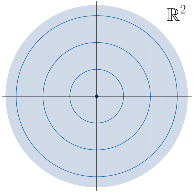
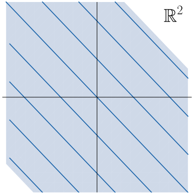
$$\exists t_0 \in I \text{ s.t. } \dot\gamma(t_0) \in \Delta_{\gamma(t_0)} \qquad \Rightarrow \qquad \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ for all } t \in I.$$



a **non** geodesically inv. distribution



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By [Lewis, 1998], a distribution $\Delta \leq T$ is geodesically inv. iff $\langle\Gamma(\Delta) : \Gamma(\Delta)\rangle_s \subseteq \Gamma(\Delta)$.

Let us sum this up:

Classical Cartan calculus	Symmetric Cartan calculus
algebraic features	
graded-commutative algebra $(\Omega^\bullet(M), \wedge)$	commutative algebra $(\Upsilon^\bullet(M), \odot)$
graded derivations $\mathrm{gDer}(\Omega^\bullet(M))$	derivations $\mathrm{Der}(\Upsilon^\bullet(M))$
graded commutator $[\ ,\ ]_{\mathrm{g}}$	commutator $[\ ,\ ]$
$[\iota_X, \iota_Y]_{\mathrm{g}} = 0$	$[\iota_X, \iota_Y] = 0$
differentials	
exterior derivative d	symmetric derivative ∇^s
canonical	depending on the choice of ∇
$(\mathrm{d}f)(X) = Xf$	$(\nabla^s f)(X) = Xf$
$\mathrm{d} \in \mathrm{gDer}_1(\Omega^\bullet(M))$	$\nabla^s \in \mathrm{Der}_1(\Upsilon^\bullet(M))$
$[\mathrm{d}, \mathrm{d}]_{\mathrm{g}} = 2\,\mathrm{d} \circ \mathrm{d} = 0$ (non-trivial)	$[\nabla^s, \nabla^s] = 0$ (trivial)
Lie derivatives	
Lie derivative $L_X := [\iota_X, \mathrm{d}]_{\mathrm{g}}$	symmetric Lie derivative $L^s_X := [\iota_X, \nabla^s]$
$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right\rvert_{t=0} (\Psi^X_t)^*_m \varphi_{\Psi^X_t(m)}$	$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right\rvert_{t=0} P^\gamma_{2t,0} (\Psi^X_{-t})^*_{\Psi^X_{2t}(m)} \varphi_{\Psi^X_t(m)}$
brackets	
Lie bracket $[X, Y] := X \circ Y - Y \circ X$	symmetric $\langle X : Y \rangle_s := \nabla_X Y + \nabla_Y X$
$\iota_{[X,Y]} = [L_X, \iota_Y]_{\mathrm{g}}, \quad [X, Y] = L_X Y$	$\iota_{\langle X:Y \rangle_s} = [L^s_X, \iota_Y], \quad \langle X : Y \rangle_s = L^s_X Y$
foliations	geodesically invariant distributions

More information on:

Moučka, R.
Symmetric Cartan calculus,
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we recover the so-called Patterson-Walker metric

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2. And we can talk about symmetric cohomology.

But that's a different talk...

Back to $TM \oplus T^*M$: symmetries

- The group of symmetries of $\langle\ ,\ \rangle_-$:

$$\mathrm{Sp}(M) := \{(\mathcal{F}, \varphi) \in \mathrm{Aut}(TM \oplus T^*M) \,|\, \varphi^* \langle \mathcal{F}a, \mathcal{F}b \rangle_- = \langle a, b \rangle_- \}$$

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The group of symmetries of C_n-generalized geometry, i.e. the subgroup of automorphisms $\mathcal{F} \in \mathrm{Sp}(M)$ satisfying $[\mathcal{F}a, \mathcal{F}b]_s = \mathcal{F}[a, b]_s$,

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$$\mathrm{Aff}(M, \nabla) \ltimes \mathrm{Kill}^2_\nabla(M).$$

- Affine diffeomorphisms: $\mathrm{Aff}(M, \nabla) := \{\varphi \in \mathrm{Diff}(M) \,|\, \nabla_{\varphi_* X} \varphi_* Y = \varphi_* \nabla_X Y\}$.

- Killing r-tensors: $\mathrm{Kill}^r_\nabla(M) := \{C \in \Upsilon^r(M) \,|\, \nabla^s C = 0\}$.

Lagrangian subbundles

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Example: symmetric Poisson structures

Let $\vartheta \in \mathfrak{X}^2_{sym}(M)$, $\mathrm{gr}(\vartheta)$ is a C_n-Dirac iff (∇, ϑ) is a symmetric Poisson structure. (These are degenerations of inverses of metrics. Integrability is expressed in terms of ∇_s-Schouten bracket. Just as linear Poisson are related to Lie algebras, linear symmetric Poisson are related to Jacobi-Jordan algebras.)

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Definition: C_n-generalized almost complex structures (C_n-gacs)

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Compare with standard gacs: $L \leq (TM \oplus T^*M)_{\mathbb{C}}$ s.t. .

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A C_n-gacs can be **locally** expressed as a C-transform of the direct sum of $\mathcal{J}_J$ and $\mathcal{J}_g$.

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Then we have a curve of C_n-generalized complex structures given, for $t \in [0, \frac{\pi}{2}]$, by

$$\mathcal{J}_t := \sin t \, \mathcal{J}_J + \cos t \, \mathcal{J}_g.$$

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New framework reaching Killing tensors, geodesically invariant distributions, symmetric Poisson structures, pseudo-riemannian metrics and almost complex structures in even dimensions.

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More to come...

Comparison with standard generalized geometry

Standard generalized geometry	C_n**-generalized geometry**
$\frac{1}{2}(\alpha(Y) + \beta(X))$	$\frac{1}{2}(\alpha(Y) - \beta(X))$
Clifford algebra action on $\Omega^\bullet(M)$	Weyl algebra action on $\Upsilon^\bullet(M)$
$B \in \Omega^2(M)$, $X + \alpha \mapsto X + \alpha + \iota_X B$	$C \in \Upsilon^2(M)$, $X + \alpha \mapsto X + \alpha + \iota_X C$
almost complex and almost symplectic	almost complex and (pseudo-)Riemannian
canonical	∇-dependent
$[X, Y] + L_X \beta - \iota_Y \mathrm{d}\alpha$	$\langle X : Y \rangle_s + L^s_X \beta + \iota_Y \nabla^s \alpha$
$\mathrm{Diff}(M) \ltimes \Omega^2_{\text{closed}}(M)$	$\mathrm{Aff}(M, \nabla) \ltimes \mathrm{Kill}^2_\nabla(M)$
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Thank you for your attention!

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