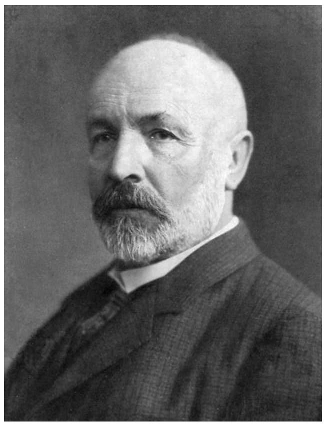
Universitat Autònoma de Barcelona

Final Year Project

Faculty of Sciences

Bachelor's Degree in Mathematics

The Cantor Set Before Cantor



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Abstract

This paper explores the historical and mathematical developments of the Cantor set, focusing on the contributions of H.J.S. Smith, Vito Volterra, and Georg Cantor. Smith independently discovered the Darboux integral before Darboux and constructed nowhere dense sets which anticipate the Cantor set. Volterra, unaware of Smith's work, created a similar set while investigating functions with non-integrable derivatives. Later, when Cantor was working on set theory, he rediscovered the classical ternary Cantor set. This paper clarifies and extends details in their original works and introduces a method to construct sets with nonnatural Cantor-Bendixson ranks that, to the best of our knowledge, has not been previously documented in any mathematical literature. By connecting historical context with rigorous proofs, this study highlights the independent evolution of concepts in analysis and set theory and how they converged to the Cantor set.

Keywords: Cantor set, integration, history, set derivation, measure theory.

Contents

1 Introduction and background

The concept of infinity has puzzled both mathematicians and philosophers for centuries. Ever since Zeno's paradox during the Greek time, the study of objects that are infinitely small or infinitely large and their behaviors has played an important role in the development of mathematics. One of the most significant advances in this area is Cantor's set theory, which is considered the cornerstone of modern mathematics. His studies led to numerous counterintuitive results, challenging and even turning traditional mathematical beliefs. These discoveries include the famous Cantor set, a mathematical object with many pathological properties. Surprisingly, this set wasn't solely created by the hand of Georg Cantor alone. H.J.S. Smith and Vito Volterra independently found an early version of the Cantor set, but their work had little to no impact during their time.

This paper has three main goals:

1. To trace different origins of the Cantor set and explain the mathematical needs that motivated each construction.
2. To provide a detailed reconstruction of Smith's and Volterra's work, highlighting their anticipation of Cantor's ideas.
3. To extend Cantor's theory of set derivation by giving a novel method for constructing sets with nonnatural Cantor-Bendixson rank.

In terms of methodology, this paper combines a historical analysis with a modern mathematical exposition. Working from original sources such as Smith's paper on the Riemann integral and Volterra's article on pathological functions, we not only clarify their original arguments but also present them with contemporary mathematical rigor. This dual approach allows us to properly contextualize their discoveries while making them accessible to modern readers. Key concepts like nowhere dense sets, set derivation, and perfect sets are introduced along with key theorems such as the characterization theorem of Riemann integrability and the Cantor-Bendixson theorem.

Beyond historical analysis, this work makes two original advances. One the one hand, we give complete proofs for several important but often omitted details in classical papers, such as Riemann's claim about his everywhere discontinuous yet integrable function $f(x) = \sum_{n=1}^{\infty} \frac{\mathrm{Exc}(nx)}{n^2}$. On the other hand, and most significantly, we introduce an original constructive method for generating sets with nonnatural Cantor-Bendixson rank, giving an explicit example of what Cantor treated only abstractly.

2 Smith's path to the Cantor set

2.1 The integration problem

In the early 1800s, mathematicians discovered that functions could be decomposed into simpler, more manageable components that, when combined, could reconstruct the initial function. J. Fourier was the first to publish papers on this idea, which is why we now refer to these decompositions as the "Fourier" series. His work was initially aimed at solving partial differential equations. However, it opened a new field in mathematics, which we call functional analysis.

There were many active analysis topics in the 19th century, like the definition of real numbers or the $\varepsilon - \delta$ language for limits. One of the central problems of the early 19th century was the convergence of the Fourier series.

Even today, different convergence problems in the space $\mathcal{L}^1$ still remain as a popular topic. This space includes many pathological functions. For example, Kolmogorov found an integrable function whose Fourier series is divergent almost everywhere [Ul'83].

The PDE that Fourier was trying to solve in his first paper on the topic was the 1-dimensional heat equation. In one of these papers, he claimed that all functions, given their explicit expression $f(x)$ (original text: graphisch), can be represented as an infinite sum of trigonometric functions. He asserted that these formulas could decompose the function:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$

where:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx, \quad n \geq 1.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx, \quad n \geq 1.$$

However, many mathematicians found counterexample functions where this series does not converge pointwise on some subset. Meaning that Fourier's initial claim was incorrect. To have convergence, the function must follow some stronger conditions. Several well-known mathematicians began investigating convergence conditions, including P.G.L. Dirichlet, A.L. Cauchy, and N.H. Abel. Ultimately, Dirichlet (1805-1859) established the first convergence theorem.

Theorem 2.1 ([Dir08] [Rie68])**.** *Given a 2π-periodic real function $f(x)$. If $f(x)$ is:*

1. *Integrable over $[0, 2\pi]$.*

2. *$f(x)$ has a finite number of local extrema over $[0, 2\pi]$.*

3. $f(x)$ has a finite number of discontinuities, and they are all jump discontinuities.

If we denote $f(x^+), f(x^-)$ as right and left side limits, respectively. Then the Fourier series $S_n(f, x)$ converges to:

$$\lim_{n\to\infty} S_n(f, x) = \begin{cases} f(x), & \text{if } f(x) \text{ is continuous at } x, \\ \frac{f(x^+)+f(x^-)}{2}, & \text{if } f(x) \text{ has a jump discontinuity at } x. \end{cases}$$

Even though Dirichlet provided a sufficient condition for convergence, it revealed a deeper issue: what does it mean to be integrable? Computing Fourier coefficients depends on evaluating some integrals, and the convergence theorem also relies on integrability. However, during Fourier's time, they lacked a clear and formal integration theory. Integration was often carried out by intuition or geometric reasoning, leaving room for ambiguity, especially when dealing with irregular functions. This conceptual gap became increasingly evident as mathematicians encountered functions whose Fourier series show unexpected behavior, prompting a more rigorous examination of what it means for a function to be integrable.

Recognizing the lack of a rigorous definition for integrals, G.F.B. Riemann (1822-1866) tried to develop a systematic integration theory. In 1854, he introduced what is now known as the Riemann integral [Rie68].

Its core idea is to divide the integration interval, say $[a, b]$, into smaller ones, obtaining a partition of the interval through separation nodes.

$$P = \{x_i \in [a, b] : a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

$$\Delta x_i = x_i - x_{i-1}, \quad \text{for } i = 1, 2, \ldots, n.$$

Then for each subinterval Δx_i we pick any point $x_i^* \in \Delta x_i$. Then sum up the product of subinterval lengths with their corresponding $f(x_i^*)$. The resulting expression is known as the Riemann Sum. In his paper, Riemann used $x_i^* = x_i + \frac{p}{q}\Delta x_i$ with $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$.

$$S_P = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Riemann defines that the function $f(x)$ is integrable over $[a, b]$ if and only if the limit of its Riemann sum exists when the partitions get finer and finer. If the limit exists and is finite, then we say it is the integral of f over $[a, b]$:

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i, \quad \text{where } \|\Delta x\| = \max_{1\le i\le n} \Delta x_i.$$

This theory is considered the first commonly accepted integration framework, and it has broadened mathematicians' view on integrable functions. To show the generality of his theory, Riemann gave an example of an integrable function with infinitely many discontinuities with the following example.

2.2 An integrable function with infinitely many discontinuities

In [Rie68], Riemann sketched the proof of a Riemann integrable function with infinitely many discontinuities in his work on integration theory. His argument had many gaps in the rigorous justification of its properties. In this section, we provide a complete and detailed proof of the function's integrability and discontinuity region, which fills the technical gaps left in Riemann's original argument.

We start by considering the excess function:

Definition 2.2 ([Rie68])**.** The *excess function* $\mathrm{Exc} : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\mathrm{Exc}(x) := x - \mathrm{round}(x),$$

where $\mathrm{round}(x)$ denotes the nearest integer to x, with $\mathrm{round}(x) := 0$ if $x \in \mathbb{Z}^+ + \frac{1}{2}$.

Using the excess function, Riemann defined the following series:

$$f(x) := \sum_{n=1}^{\infty} \frac{\mathrm{Exc}(nx)}{n^2} = \sum_{n=1}^{\infty} f_n(x), \quad \text{where } f_n(x) := \frac{\mathrm{Exc}(nx)}{n^2}.$$

Since $|\mathrm{Exc}(nx)| < \frac{1}{2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have $|f_n(x)| < \frac{1}{2n^2}$.

Therefore, the series is absolutely convergent and uniformly by the Weierstrass M test:

$$\sum_{n=1}^{\infty} \left| f_n(x) \right| < \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12}.$$

To study the continuity, we split the series into two parts: the first N terms with a convenient N and the terms from $N+1$ to infinity, called the tail.

Fix a point $x_0 \in \mathbb{R}$ and let $\varepsilon_0 > 0$, then there is a natural number N such that:

$$\left| \sum_{n=N'+1}^{\infty} f_n(x_0) \right| \leq \sum_{n=N'+1}^{\infty} \frac{1/2}{n^2} < \varepsilon_0 \quad \forall N' \geq N.$$

In this way, we can dismiss the tail by taking $\varepsilon_0 \to 0$.

Lemma 2.3. *The function f is continuous on the set $A = \{x \in \mathbb{R} \mid nx \notin \mathbb{N} + \frac{1}{2}, \forall n \in \mathbb{Z}^+\}$.*

Proof. If $x_0 \in A$, then for every $n \in \mathbb{Z}^+$, the corresponding term $f_n(x)$ would be continuous at x_0 and, in particular, the first N terms have their own continuity subinterval around x_0. In other words, for all $1 \leq n \leq N$, there exists $h_n > 0$ such that if $x \in (x_0 - h_n, x_0 + h_n)$, then $|\mathrm{Exc}(nx) - \mathrm{Exc}(nx_0)| < \varepsilon_0$. The finite intersection of these open subsets gives an open subinterval I_N where $\mathrm{Exc}(nx)$ is continuous for all $n \leq N$:

$$I_N = \bigcap_{1 \leq n \leq N} (x_0 - h_n, x_0 + h_n)$$

Take $x \in I_N$, we have:

$$|f(x) - f(x_0)| < \sum_{1 \leq n \leq N} \left(\frac{|\operatorname{Exc}(nx) - \operatorname{Exc}(nx_0)|}{n^2} \right) + 2\varepsilon_0$$
$$< \varepsilon_0 \left(\sum_{1 \leq n \leq N} \frac{1}{n^2} \right) + 2\varepsilon_0 < \frac{(\pi^2 + 12)\varepsilon_0}{6}.$$

Since this inequality holds for all $\varepsilon_0 > 0$, $f(x)$ must be continuous at x_0. Which concludes that $f(x)$ is continuous on the set A. □

On the other hand, we claim that f is discontinuous on A^c, the complement of A. The following lemma allows us to express the elements of A^c in a simple form.

Lemma 2.4. *We have* $A^c = \{\frac{p}{2q} \in \mathbb{Q} : p$ *is odd,* $2q$ *relatively prime to* $p\}$.

Proof. $\subseteq$) If $x \notin A$, then there must exist a pair of natural numbers m, k such that $mx = k + \frac{1}{2}$. Equivalently, $x = \frac{2k+1}{2m}$. The numerator $2k + 1$ is odd, but the denominator $2m$ is not necessarily coprime to $2k + 1$. If the $\gcd(2k + 1, 2m) = d \neq 1$, then $2k + 1 = dp$ and $2m = d(2q)$, which yields $x = p/2q$. Since p is a divisor of the odd number $(2k + 1)$, p is also odd, and by construction, $2q$ is relatively prime to p. In the case that $\gcd(2k + 1, 2m) = d = 1$, $2m$ is already coprime to $2k + 1$.

$\supseteq$) If $x = \frac{p}{2q}$, where p is an odd integer and $2q$ is coprime to p, then

$$mx = \frac{mp}{2q} = k + 1/2 \iff \frac{mp}{q} = 2k + 1 \quad \text{an odd integer.}$$

Since p and $2q$ are coprime, p must also be coprime with q. But $\frac{mp}{q} \in \mathbb{Z}$, so q divides m, say $m = qj$ with $j \in \mathbb{Z}^+$. On the other hand, $2k + 1 = \frac{mp}{q} = jp$ is also odd; therefore, j must also be odd because p is odd. In short, given $x = \frac{p}{2q} \in A^c$:

$$mx = \frac{mp}{2q} \in \mathbb{Z}^+ + \tfrac{1}{2} \quad \Leftrightarrow \quad m = qj \text{ where } j \text{ is an odd integer.}$$

□

Now we prove that the side limits of f exist for $x \in A^c$. For all $N \in \mathbb{Z}^+$:

$$\left| \lim_{x \to x_0^\pm} \sum_{n=1}^{N} f_n(x) \right| = \left| \sum_{n=1}^{N} \lim_{x \to x_0^\pm} f_n(x) \right| = T_N < \sum_{n=1}^{N} \frac{1/2}{n^2}.$$

Therefore, $\sum_{n=1}^{N} f_n(x) \xrightarrow{x \to x_0} T_N \in \mathbb{R}$ and $\sum_{n=1}^{N} f_n(x) \xrightarrow{N \to \infty} f(x)$ uniformly on $\mathbb{R}$. By the Moore-Osgood theorem (theorem A.7), the side limits of f exist and:

$$\lim_{x \to x_0^\pm} \lim_{N \to \infty} \sum_{n=1}^{N} f_n(x) = \lim_{N \to \infty} \lim_{x \to x_0^\pm} \sum_{n=1}^{N} f_n(x) \in \mathbb{R}.$$

Finally, we can show that f is discontinuous on A^c

Proposition 2.5. *The function f is discontinuous on the set A^c.*

Proof. Let $x_0 = p/2q \in A^c$, then:

$$\begin{aligned}
\lim_{x\to x_0^\pm} f(x) - f(x_0) &= \left(\sum_{n=1}^{\infty} \lim_{x\to x_0^\pm} \frac{\mathrm{Exc}(nx)}{n^2}\right) - \sum_{n=1}^{\infty} \frac{\mathrm{Exc}(nx_0)}{n^2} \\
&= \left(\sum_{\substack{n\in\mathbb{N}\\ nx_0\in\mathbb{N}+1/2}} \frac{\mp 1/2}{n^2} + \sum_{\substack{n\in\mathbb{N}\\ nx_0\notin\mathbb{N}+1/2}} \frac{\mathrm{Exc}(nx_0)}{n^2}\right) - \sum_{n=1}^{\infty} \frac{\mathrm{Exc}(nx_0)}{n^2} \\
&= \left(\sum_{\substack{n\in\mathbb{N}\\ nx_0\in\mathbb{N}+1/2}} \frac{\mp 1/2 - 0}{n^2} + \sum_{\substack{n\in\mathbb{N}\\ nx_0\notin\mathbb{N}+1/2}} \frac{\mathrm{Exc}(nx_0) - \mathrm{Exc}(nx_0)}{n^2}\right) \\
&= \sum_{\substack{n\in\mathbb{N}\\ nx_0\in\mathbb{N}+1/2}} \frac{\mp 1}{2n^2}.
\end{aligned}$$

We have proved in lemma 2.4 that $nx \in (\mathbb{Z}^+ + 1/2)$ if and only if $n = qj$ where j is an odd integer. Therefore:

$$\lim_{x\to x_0^\pm} f(x) - f(x_0) = \sum_{\substack{n\in\mathbb{N}\\ nx_0\in\mathbb{N}+1/2}} \frac{\mp 1}{2n^2} = \sum_{\text{odd j}} \frac{\mp 1}{2(qj)^2} = \frac{\mp 1}{2q^2}\left(\sum_{\text{odd j}} \frac{1}{j^2}\right) = \frac{\mp \pi^2}{16q^2}.$$

The side limits of f do not match on A^c, meaning that f is discontinuous on A^c. □

This lemma proves that f has infinitely many discontinuities. On the other hand, for any given interval $[a, b]$ and $n \in \mathbb{Z}^+$, f_n has only finitely many discontinuities, so the partial sum $\sum_{n=1}^{N} f_n$ is Riemann integrable. Combined with the uniform convergence $\sum_{n=1}^{N} f_n \xrightarrow{N} f$, we obtain the integrability of f.

Therefore, f is a Riemann integrable function with infinitely many discontinuities.

2.3 Integrability and nowhere dense sets

Riemann could not finish this theory during his lifetime. It was not until 1868, two years after his death, that his colleague Richard Dedekind recognized the importance and published Riemann's manuscript. His incomplete work led to much confusion and incorrect results based on his definition. H.J.S. Smith (1826-1883) criticized this misinterpretation trend and made several clarifications for Riemann's integral.

In his 1868 paper [Smi74], Smith discussed the core idea of the Riemann integral. He addressed a common misconception: that the convergence of Riemann sums along a particular family of partitions is sufficient to ensure Riemann integrability. However, this is not generally true. For a function to be Riemann integrable, convergence must occur over all partitions with sufficiently small diameters, not just over a certain subset. This idea led him to discover the essence of the Darboux integral:

> ... the difference Θ between the greatest and the least values that S [1] can acquire for a given norm ... that S converges to a definite limit, when d is diminished without limit, we must be sure that Θ diminishes without limit.

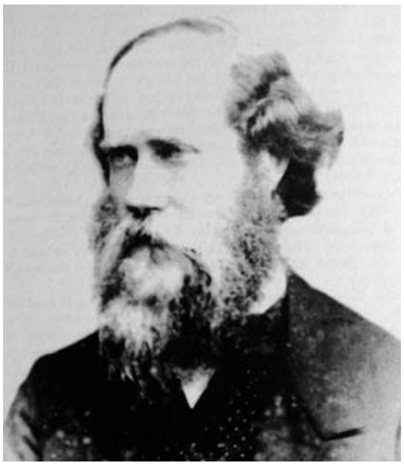

Figure 1: H.J.S. Smith

In the same work, Smith also established that for any given diameter, the upper sum is always bigger than the lower sum, unless the function is a step function and in which case these two sums are equal. On the other hand, He also noticed that as the diameter gets smaller, the upper sum decreases and the lower sum increases. Since they are monotone and bounded by each other, they must converge somewhere when the diameter approaches zero. This is nearly everything about the Darboux integrability. Since Darboux's integral was published in 1875, and Smith is unlikely to have exchanged letters with Darboux, Smith likely discovered the Darboux integral independently before its publication.

Having clarified the integration problem, Smith turned his focus to exploring the boundary of Riemann integrability. Smith aimed to find a Riemann integrable function, but as discontinuous as possible. He followed Dr. H. Hankel's idea that Riemann integrability might be connected to what Hankel referred to as "loose order set". In modern mathematics, it is called nowhere dense.

Definition 2.6. Let X be a topological space. A subset $N \subseteq X$ is said to be **nowhere dense** in X if and only if for every nonempty open subset U, there is another nonempty open subset $V \subseteq U$ such that $V \cap N = \emptyset$, i.e. N is not dense in any open subset of X.

Example 2.7. $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ with the standard topology: Given a non empty open set $U \subseteq \mathbb{R}$ such that $U \cap \mathbb{Z} \neq \emptyset$, and $p \in U \cap \mathbb{Z}$. Take $I = (a, b) \subset U$, an open neighborhood of p in U with interval length $b - a < 1$. Then the only integer that this interval contains is p. We can take $V = (a, b) \setminus \{p\}$.

Example 2.8. A convergent sequence is nowhere dense in a metric space: Let us say that $x_n \to x$ in a metric space M. Take an open set U, then $U \setminus (\{x_n\}_n \cup \{x\})$ is a subset that does not contain any point of the sequence. It is open, since we are removing a closed set $(\{x_n\}_n \cup \{x\})$.

Smith observed that a function with discontinuities over a dense subset of its domain cannot be Riemann integrable. Thus, Riemann integrability requires the set of discontinuities to be at most nowhere-dense. To study the relation between integrability and

[1] S represents the Riemann sum

the nowhere-dense property, he gave a relatively general method to construct a nowhere dense sets as follows:

> "Let m be any given integral number greater than 2. Divide the interval from 0 to 1 into m equal parts, and exempt the last segment from any subsequent division. Divide each of the remaining $m-1$ segments into m equal parts, and exempt the last segment of each from any subsequent division. If this operation is continued ad infinitum, we shall obtain an infinite number of points of division P upon the line from 0 to 1."

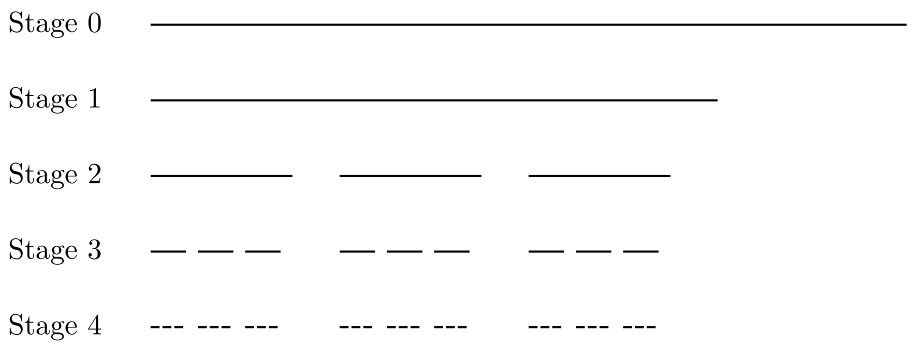

Figure 2: Construction of Smith's set with $m = 4$ over 4 iterations.

Proposition 2.9. *The Smith's set S_m obtained from the previous construction is nowhere dense.*

Proof. Given an open interval $J = (a, b)$, we aim to find an open subinterval $J' \subseteq J$ such that there is no element of S_m inside J'.

Notice that we can interpret the Smith set in the following way: For stage k, we first divide the entire unit interval into m^k equal subintervals. Then, we remove those subintervals that were the last part of some iteration before, i.e, subintervals of the form:

$$\left(\frac{nm^{k-j} + (m-1)}{m^k}, \frac{nm^{k-j} + m}{m^k}\right), \quad 1 \leq j \leq k, \quad 0 \leq n \leq m^{j-1} \tag{1}$$

Knowing this, we may choose a big integer k such that

$$\frac{1}{m^k} < \frac{b-a}{2}.$$

This ensures that at stage k, the interval J contains at least one subinterval from the family (1). Let us say $I = \left(\frac{M}{m^k}, \frac{M+1}{m^k}\right) \subseteq J$ for an integer M.

For our subinterval I, exactly one of these two scenarios is true:

- The subinterval I has already been excluded in a previous iteration, so it is sufficient to take $J' = J$.
- I is not excluded for the first k stages, but the last $1/m$ part of it will be removed for the next stage. Thus, we can take J' to be the removing part, that is

$$J' = \left(\frac{nm + (m-1)}{m^{k+1}}, \frac{(n+1)m}{m^{k+1}}\right)$$

□

This construction of the Smith set can be generalized. For instance, instead of removing the last subintervals, one may remove a subinterval of the same length anywhere inside the interval. It is also possible to remove various subintervals at the same time (removing at most $m-2$) or even eliminate a different proportion each time. The proof of the nowhere dense property of these variations can be found, for instance, in [Val13].

As mentioned, Hankel conjectured in 1870 that integrability is characterized by having nowhere dense discontinuity [Smi74, Han82]. According to the Lebesgue criterion for Riemann integrability, a bounded function f on a closed interval is Riemann integrable if and only if it is continuous almost everywhere.

Theorem 2.10. *(Lebesgue criterion for Riemann integrability [Leb02]) Let $f : [a, b] \to \mathbb{R}$ be a bounded, real-valued function. Then:*

$$f \text{ is Riemann integrable} \iff f \text{ is continuous almost everywhere.}$$

This result is now well-known, but it was not formally proven until Lebesgue's work in 1902. Earlier, mathematicians had partial versions of the criteria for Riemann integrability. For instance, Riemann himself proposed an early form of the condition, though his version lacked the rigor and detail that Lebesgue later provided. We will discuss and prove this integrability condition in detail later through Volterra's proof (theorem 3.3).

Using this condition, Smith noticed that even if a function has discontinuities on a nowhere dense set, it might still not be Riemann integrable. In [Smi74], he constructed examples of nowhere dense sets with positive measure. By applying Riemann's condition, he proved that any function discontinuous on such a set cannot be Riemann integrable.

The following is the set that Smith had constructed:

Definition 2.11. (Generalized Smith Set I_m) Let $m \geq 3$ be a fixed integer. We start from the closed unit interval and denote it as $I_{m,0} = [0, 1]$. We divide this interval into $m^1 = m$ subintervals of equal length and remove the last segment: $(\frac{m-1}{m}, 1)$. Call the remaining set $I_{m,1}$, now for further stages $k \geq 2$ divide each of the remaining subintervals into m^k equal length parts and then remove the last segment to obtain $I_{m,k}$. The limit set is the generalized Smith set:

$$I_m = \bigcap_{k=0}^{\infty} I_{m,k}$$

Removing open or closed subintervals at each step does not affect the nowhere dense property. Here, we take open subintervals because it makes calculations easier.

Proposition 2.12. *Let μ be the Lebesgue measure, $m \geq 3$. Then I_m is measurable and $\mu(I_m) > 0$.*

Proof. First, I_m is closed because it is the countable intersection of closed sets $I_{m,k}$, and in particular, this makes I_m measurable. Secondly, we can apply the monotone convergence theorem (theorem A.6) to get $\mu(I_m) = \lim_{k\to\infty} \mu(I_{m,k})$.

At stage 0, nothing is being eliminated yet. At stage 1 we have removed so far one subinterval of length $\frac{1}{m}$, and then at stage 2 we remove $(m-1)$ segments of length $\frac{1}{m^3}$, at stage 3 there is $(m-1)(m^2-1)$ parts of length $\frac{1}{m^6}$, inductively at stage k we remove $\prod_{i=1}^{k-1}(m^i - 1)$ intervals of length $\frac{1}{m^{\Sigma_k}}$ where Σ_k is the sum of first k positive integers. Thus, we can conclude:

$$\begin{aligned}
\mu(I_{m,k}) &= 1 - \left(\frac{1}{m} + \sum_{j=2}^{k} \frac{\prod_{i=1}^{j-1}(m^i - 1)}{m^{\Sigma_j}}\right) \geq 1 - \left(\frac{1}{m} + \sum_{j=2}^{k} \frac{\prod_{i=1}^{j-1} m^i}{m^{\Sigma_j}}\right) \\
&= 1 - \left(\frac{1}{m} + \sum_{j=2}^{k} \frac{m^{\Sigma_{j-1}}}{m^{\Sigma_j}}\right) = 1 - \left(\frac{1}{m} + \sum_{j=2}^{k} \frac{1}{m^j}\right) \\
&= 1 - \sum_{j=1}^{k} \left(\frac{1}{m}\right)^j = 1 - \frac{1 - \frac{1}{m^{k+1}}}{m-1} \xrightarrow{k\to\infty} 1 - \frac{1}{m-1} = \frac{m-2}{m-1} > 0.
\end{aligned}$$

□

By the Lebesgue criteria of Riemann integrability, a function that is discontinuous on a generalized Smith set is not Riemann integrable because it is discontinuous on a positive measure set.

So far, we have seen that starting with the closed unit interval and inductively removing part of it can give a big family of nowhere dense sets. This method, which is likely the origin of Cantor set's constructions, illustrates how topologically small sets can also be large in terms of measure. As shown in the last proposition, a nowhere dense set can have positive measure, meaning a set that is porous everywhere can occupy space. This subtle distinction between topological sparseness and measure size represents the conceptual breakthrough in integration theory. It also anticipates the idea of the Lebesgue integral and modern measure theory.

We now conclude this historical path to the Cantor set by introducing its first and most common definition. The construction of the Cantor set $\mathcal{C}$ as a slightly modified Smith set with $m = 3$ 2.11. Instead of removing the last segment of a subinterval, the Cantor set removes the middle segment in every iteration.

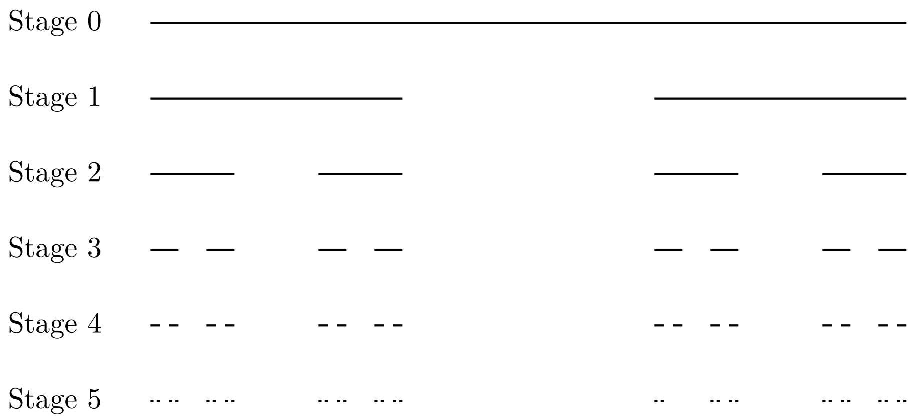

Figure 3: The firsts stages of Cantor set construction: $I_{m,k}$ for $k \leq 5$

Definition 2.13 (Cantor set $\mathcal{C}$)**.** Let m=3 and $D_0 = [0, 1]$. We delete the open middle third open to gain $D_1 = [0, 1/3] \cup [2/3, 1]$ and then inductively remove the open middle third of every remaining subintervals to get D_k, $k \geq 2$. This process can be written in set operations as:

$$D_k := \frac{D_{k-1}}{3} \cup (\frac{2}{3} + \frac{D_{k-1}}{3}).$$

Then we can define the Cantor set as the limit of this nested sequence of sets.

$$\mathcal{C} := \bigcap_{k=0}^{\infty} D_k.$$

The definition through set operation gives another way to define $\mathcal{C}$. It is the invariant set under the iterated function system (IFS) when you substitute the terms D_{k-1} for x. We will not delve into this approach, more details can be found in [Val13].

A final observation is that $\mathcal{C}$ is compact. This is because D_k is closed and bounded for all k, and applying the Heine-Borel theorem (theorem A.2) gives the compactness of $\mathcal{C}$.

3 The Volterra function and the Cantor set

3.1 The fluctuation and continuity of a function

Riemann's formulation of the integral marked a turning point in the history of analysis. For the first time, studying integrals within a rigorous framework became possible. From the era of Newton and Leibniz, mathematicians knew that integration and differentiation are inverse operations, which is known as the fundamental theorem of calculus. However, there was no formal proof since they did not have a commonly accepted integration theory.

Figure 4: Vito Volterra

V. Volterra (1860-1940), known for his studies in dynamical systems, also explored the limits of Riemann's integration theory. His primary interest was the relationship between differentiation and integration under Riemann's framework. As noted earlier, the fundamental theorem of calculus states that any continuous function f over a closed interval $[a, b]$ possesses a uniformly continuous antiderivative $F = \int_a^x f dx$ for $x \in [a, b]$. This means that if you integrate and then differentiate a continuous function, you get the initial function. A natural question to ask is: what happens if we first differentiate a function and then integrate the result? Do these operations cancel each other out, recovering the original function up to an additive constant?

Today, we know that Riemann integrability is characterized by Lebesgue's criterion, which links the Riemann integrability to the set of discontinuities of the function. One would think that this is something discovered after the Lebesgue integral, which is around the early 1900s. However, Riemann explicitly wrote a similar integrability condition in his paper [Rie68] using the concept of fluctuation.

Definition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a real-valued function and a subinterval $[c, d] \subseteq [a, b]$. The fluctuation of f in $[c, d]$ is defined as:

$$W_f[c, d] = \sup\{\, |f(x) - f(x')| : x, x' \in [c, d]\}.$$

We also define the fluctuation at a point $x \in (a, b)$ as

$$\omega_f(x) = \lim_{h \to 0^+} W_f[x - h, x + h].$$

The fluctuation of points in the boundary is defined analogously using their corresponding side limit.

Lemma 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the continuity of a point $x \in [a, b]$ is characterized by its fluctuation:*

$$f \text{ is continuous at } x \text{ if and only if } \omega_f(x) = 0.$$

Proof. $\Rightarrow$) Let $x \in [a, b]$. The function f is continuous at x if and only if for any given $\varepsilon > 0$, there is $\delta > 0$ such that for all $y \in (x-\delta, x+\delta) \cap [a, b]$, we have $|f(x)-f(y)| < \varepsilon/2$. Then:

$$\sup_{y,z\in(x-\delta,x+\delta)\cap[a,b]} |f(y) - f(z)| \leq \sup_{y,z\in(x-\delta,x+\delta)\cap[a,b]} (|f(y) - f(x)| + |f(x) - f(z)|) \leq \varepsilon.$$

Hence, $\omega_f(x) = 0$.

$\Leftarrow$) Let x with $\omega_f(x) = 0$. For any given $\varepsilon > 0$, by definition, there is a $h > 0$ such that $\sup_{y,z\in(x-h,x+h)\cap[a,b]} |f(y) - f(z)| < \varepsilon$. Choose $\delta = h$, for any $y \in (x - \delta, x + \delta) \cap [a, b]$, we have:

$$|f(x) - f(y)| < \sup_{t,s\in(x-h,x+h)\cap[a,b]} |f(t) - f(s)| < \varepsilon.$$

Therefore, f is continuous at x. □

Unfortunately, the notion of fluctuation can not be used to characterize discontinuity in a given interval. The condition $W_f[a, b] > 0$ is necessary for having a discontinuity in an interval $[a, b]$, but not a sufficient condition.

Theorem 3.3 ([Rie68] [Vol81])**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The function f is Riemann integrable on $[a, b]$ if and only for all $\varepsilon, \sigma > 0$, there exists $\delta > 0$ such that for any partition $P = (a = x_0 < x_1 < ... < x_n = b)$ with norm $|P| < \delta$:*

$$s(\sigma, P) = \sum_{\substack{0\leq k\leq n-1 \\ W_f[x_k,x_{k+1}]>\sigma}} x_{k+1} - x_k < \epsilon.$$

Proof. For simplicity, we use the Darboux integral instead of the Riemann integral, since they are equivalent. We denote the upper sum and the lower sum of the function f for a partition P as $U(f, P)$ and $L(f, P)$, respectively.

Let f be a bounded function over $[a, b]$. Note that:

$$s(\sigma, P) = \text{The total length of those subintervals of } P \text{ with fluctuation bigger than } \sigma\ .$$

We can classify the subintervals of any given partition by how big their fluctuation is. The first class is formed by those subintervals with fluctuation bigger than σ, and the intervals with fluctuation smaller than or equal to σ is called the second class.

$\Rightarrow$) By the definition of Darboux integrability, for every $\sigma, \varepsilon > 0$, there exists a $\delta > 0$ such that any partition P with $|P| < \delta$: $U(f,P) - L(f,P) < \varepsilon\sigma$. We have

$$\begin{aligned} s(\sigma,P)\sigma &\leq \sum_{\substack{0\leq k\leq n-1 \\ W_f[x_k,x_{k+1}]>\sigma}} (x_{k+1}-x_k)W_f[x_k,x_{k+1}] \\ &\leq \sum_{k=0}^{n-1}(x_{k+1}-x_k)W_f[x_k,x_{k+1}] \\ &= U(f,P) - L(f,P) < \varepsilon\sigma. \end{aligned}$$

Meaning that $s(\sigma,P) < \varepsilon$.

$\Leftarrow$) Given $\varepsilon > 0$, and let $\sigma = \frac{\varepsilon}{2(b-a)} > 0$. By the hypothesis there exists $\delta > 0$ such that for every partition $P = \{a = x_0, x_1, ..., x_n = b\}$ such that $|P| < \delta$, we have $s(\sigma,P) < \varepsilon/2W_f[a,b]$, here we have assumed that f is not constant, since it is a trivial case. We bound the sum of different classes individually. The first class has total length $s(\sigma,P)$, and the fluctuation of any subinterval can be bounded by the global fluctuation $W_f[a,b]$. On the other hand, the second class, those subintervals with fluctuation smaller than or equal to σ, has length $(b-a-s(\sigma,P)$ and the fluctuation can be bounded by σ. By combining these bounds, we get:

$$U(f,P) - L(f,P) \leq s(\sigma,P)W_f[a,b] + (b-a-s(\sigma,P))\sigma < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

This theorem is a primitive version of Lebesgue's criterion for the Riemann integral. The condition that the total length of subintervals with large fluctuation is small corresponds to requiring that the set of discontinuities has zero Lebesgue outer measure. However, proving the fluctuation condition is technically difficult, so for practical purposes, we will be using the Lebesgue criterion for simplicity.

3.2 The construction of a function with non-integrable derivative

Volterra's goal was to construct a differentiable function whose derivative is not Riemann integrable. According to Lebesgue's criteria, we know it can be done by finding a differentiable function such that its derivative is discontinuous at a positive measure set.

It is not hard to find a function that is differentiable everywhere, and its derivative is discontinuous at one point. A classical example is the function $x^2 sin(1/x)$. It is continuous everywhere but not defined at $x = 0$. However, its side limits match, so we can put the limit value when $x = 0$ to get the continuity.

$$f(x) = \begin{cases} x^2 sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

The derivative of this function can be calculated through the chain rule at every point but $x = 0$. So we plug in the definition to check the differentiability at zero:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 sin(1/h) - 0}{h} = \lim_{h\to 0} hsin(1/h) = 0.$$

Thus, we have that the function f is differentiable everywhere and f' can be expressed explicitly by the following piecewise function:

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

On the other hand, if we calculate the fluctuation of f' at 0:

$$\begin{aligned} \omega_{f'}(0) &= \lim_{h\to 0^+} \sup \{ |f'(s) - f'(t)| : s,t \in [-h,h] \} \\ &= \lim_{h\to 0^+} \sup \left\{ \left|\cos\left(\frac{1}{s}\right) - \cos\left(\frac{1}{t}\right)\right| : s,t \in [-h,h] \right\} \\ &= 2. \end{aligned}$$

By the lemma 3.2, f' is discontinuous at the origin.

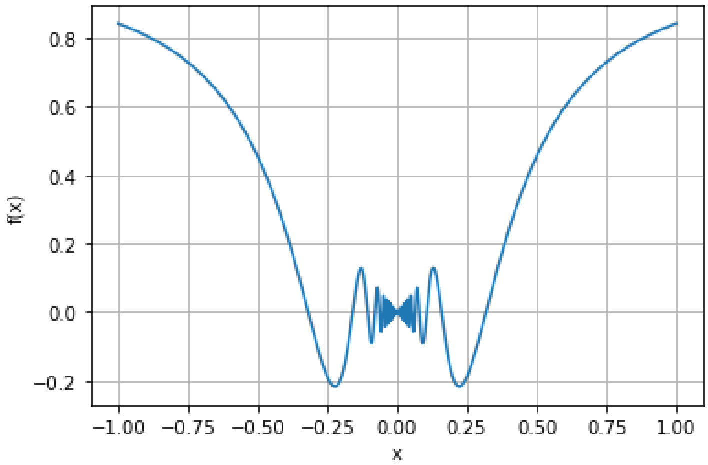

Figure 5: Graph of $f(x) = x^2 sin(1/x)$ near the origin

Now the challenge is whether we can extend or not this behaviour to a positive measure set. Volterra's approach was to reproduce this discontinuous function on a specially constructed set, now known as the Volterra set V. It is a slight modification of the generalized Smith set with parameter $m = 4$. However, he never mentioned Smith's article in his paper [Vol81]. This is a sign that Volterra might have independently rediscovered a Cantor set-like construction from a new perspective.

Volterra's set V is defined iteratively and based on interval removal and subdivision in the same style as the generalized Smith set.

Start with the closed unit interval $[0,1]$. Remove $1/m = 1/4$ portion of the rightmost interval $(0+3(1-0)/4, 1] = (3/4, 1]$. Then, instead of splitting the remaining interval into

$m-1=3$ equal parts, you split it into a countable disjoint union of intervals with length smaller than $1/4$. Furthermore, these intervals must accumulate at the left endpoint. This process is equivalent to taking a partition of the interval through a decreasing sequence $\{x_n\}_{n\geq 1} \subset [0,1]$ such that:

$$\begin{aligned} &1. \quad x_n \to 0 \quad \text{as } n \to \infty \\ &2. \quad x_1 - x_2 = \frac{1}{4} \\ &3. \quad x_n - x_{n+1} < \frac{1}{4} \quad \text{for all } n \geq 2. \end{aligned}$$

Call the remaining points $V_1 = [0, 3/4]$, where there is now the sequence: $\{x_n\}_{n\geq 2} \subset V_1$. Every two consecutive numbers of this sequence define an interval $I_n = (x_{n+1}, x_n)$. We repeat the elimination and subdivision process inductively on each of these subintervals. For every interval I_n, remove its rightmost quarter:

$$\left(x_n - \frac{1}{4}(x_n - x_{n+1}),\, x_n\right).$$

What remains is:

$$\left(x_{n+1},\, x_n - \frac{1}{4}(x_n - x_{n+1})\right).$$

Next, within each interval (x_{n+1}, x_n), take another decreasing sequence $\{x_{n,m}\}_{m\geq 1} \subset (x_{n+1}, x_n)$ such that:

$$\begin{aligned} &1. \quad x_{n,m} \to x_{n+1} \quad \text{as } m \to \infty \\ &2. \quad x_{n,1} - x_{n,2} = \frac{(x_n - x_{n+1})}{4} \\ &3. \quad x_{n,m} - x_{n,m+1} < \frac{(x_n - x_{n+1})}{4} \quad \text{for all } m \geq 2. \end{aligned}$$

This process yields another sequence of subintervals, to which we apply the same construction recursively, ad infinitum. If we call the remaining points in the k-th iteration as V_k, the Volterra set generated by these sequences is $V = \cap_{k\geq 1} V_k$.

Although Volterra sets depend heavily on the specific choice of sequences used at each step of the construction, any choice satisfying the given conditions will still lead to the same results that we aim to prove.

Proposition 3.4. *The Volterra set V is nowhere dense, measurable, and $\mu(V) > 2/3$ where μ is the Lebesgue measure.*

Proof. For the nowhere dense property, it can be proved analogously to the Smith set. Given an interval with length $l > 0$, there exists n such that $1/4^n < l/2$. This means that in iteration n, either this subinterval has already been removed, or a portion of it will get removed in the next iteration.

For measurability, we have that for every $k \geq 1$, V_k^c is a finite union of open subintervals, therefore open. Since $V = \cap_{k\geq 1} V_k = (\cup_k V_k^c)^c$, V is the complement of an open set, hence closed and Lebesgue measurable.

On the other hand, $V_k \supset V_{k+1}$ for all k. Applying the monotone convergence theorem (theorem A.6) for sets yields:

$$\mu(V) = \lim_{k\to\infty} \mu(V_k) = 1 - \lim_{k\to\infty} \mu(V_k^C) > 1 - (\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + ...) = 2/3.$$

□

As mentioned before, when setting the parameter $m = 4$ to the generalized Smith set, it is related to the Volterra sets. Nevertheless, the set I_4 is not a possible Volterra set because in every iteration during the construction, the removal part is a finite union of intervals. In contrast, the elimination part in a Volterra set is a countable union except for the first iteration. If we enable the construction of the Volterra set to remove at most countably many, then the Smith set would be a special case of the Volterra set.

Recall that our final goal was to mimic the discontinuous derivative of the function $f(x) = x^2 sin(x)$ on more points instead of just at the origin. We start by copying the behaviour of $x = 0$ to a second point. An easy way is to take the symmetrical extension of $f(x)$.

Suppose we want to duplicate the discontinuity at the endpoints of an interval $[a, b]$. We can copy the function $f(x)$ for the first half of the interval, and if $x \in [a + \frac{b-a}{2}, b]$, we take its symmetrical extension. Call it $g_{a,b} : [a, b] \to \mathbb{R}$.

$$g_{a,b}(x) = \begin{cases} f(x-a) & x \in [a, a + \frac{b-a}{2}], \\ f(b-(x-a)) & x \in [a + \frac{b-a}{2}, b]. \end{cases}$$

However, this extension does not guarantee differentiability at the middle point, where two parts of the function meet. To ensure smoothness, take f not until the midpoint, but x_1, the maximal value of f in $[a, a + \frac{b-a}{2}]$. Prolong this maximal value until reaching the middle point, and then take the symmetrical extension.

Let x_1 be the abscissa of the maximum of f in the first half interval:

$$x_1 = \max\left\{x \in \left[a, a + \frac{b-a}{2}\right] : f(x) \geq f(y)\ \forall\, y \in \left[a, a + \frac{b-a}{2}\right]\right\}.$$

Then define $f_{a,b} : [a, b] \to \mathbb{R}$ by

$$f_{a,b}(x) = \begin{cases} f(x-a) & \text{for } x \in [a, x_1], \\ \\ f(x_1 - a) & \text{for } x \in [x_1, b - x_1], \\ \\ f(b-x) & \text{for } x \in [b-(x_1-a), b]. \end{cases}$$

This piecewise definition ensures that $f_{a,b}$ is differentiable on $[a,b]$. In the outer region $x \in [a, x_1] \cup [b-(x_1-a), b]$, $f_{a,b}$ is either a copy of f or its mirror version. While for the inner region $x \in [x_1, b-x_1]$, f is constant, so its derivative is identically zero.

The same result could be obtained using any local maxima or minima, and the result would be as good as taking the maximal value. All these possible candidates have derivative 0 at the junction points.

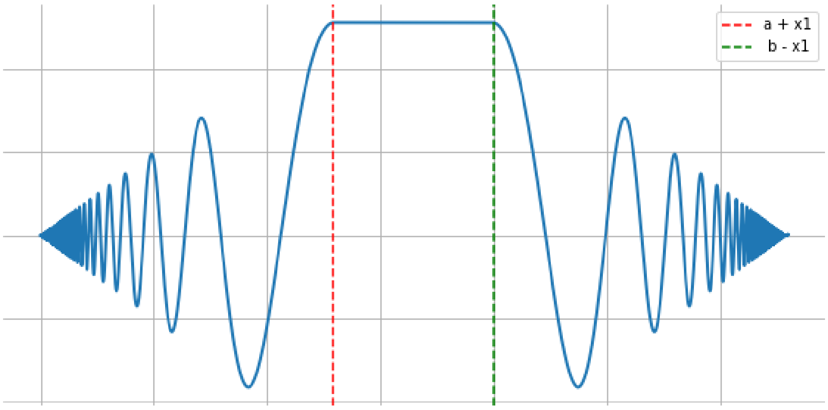

Figure 6: An ilustration of the symmetrical extension of $x\sin(1/x)$.

3.3 The Volterra function

Now, consider the Volterra function defined below [PCMA15]:

$$F(x) = \begin{cases} f_{a,b}(x), & \text{if } x \in (a,b) \text{ for some maximal subinterval } (a,b) \subset [0,1] \setminus V, \\ 0, & \text{if } x \in V. \end{cases}$$

Note that $[0,1] \setminus V = \bigsqcup_{n\in\mathbb{N}} I_n$, where $\{I_n\}_{n\in\mathbb{N}}$ is a enumeration of removed intervals during its construction. Since this is the complement of a closed nowhere dense set, $[0,1] \setminus V$ is open and dense.

Theorem 3.5. *The Volterra function F is continuous in $[0,1]$ and differentiable in $(0,1)$. Its derivative is bounded but not Riemann integrable.*

Proof. If the function is differentiable, then it must be continuous. Thus, we can directly prove differentiability. Given a point $x \in (0,1)$ and $h \in \mathbb{R}$.

1. Both x and $x+h$ are in V:
$$\frac{F(x+h)-F(x)}{h} = \frac{0-0}{h} = 0.$$

2. If $x \in V$ and $x + h \in [0,1] \setminus V$, say $x + h \in (a,b)$:

$$\left|\frac{F(x+h)-F(x)}{h}\right| = \left|\frac{f_{a,b}(x+h)}{h}\right| \leq \left|\frac{(x+h-a)^2}{x+h-a}\right| \leq h \xrightarrow{h\to 0} 0.$$

3. If $x \in [0,1] \setminus V$, by definition there exists a open interval $I_n = (c,d)$ where $x \in I_n$. Then, by construction, F is differentiable at x. We also have:

$$|F'(x)| = |f'_{c,d}(x)| \leq \sup_{x\in(0,1]} 2xsin(1/x) - cos(1/x) \leq 3.$$

If $x \in \{a,b\}$, the bound is also valid, but within their corresponding side limit.

Through these inequalities, we can say that $|F'(x)| \leq 3$. The derivative of F is bounded.

On the other hand, F' is discontinuous on V. Given $x \in V$, if x is an endpoint of a removed subinterval $I_m = (c,d)$, let us say that $x = c$. By construction:

$$\omega_{F'}(x) = \omega_{f_{c,d}}(c) = 2.$$

If x is not an endpoint, then for all $h > 0$, $(x-h, x+h) \cap ([0,1] \smallsetminus V) \neq \emptyset$ because $[0,1] \setminus V$ is dense. By the construction of V, there exists a maximal subinterval $(a_h, b_h) \subset (x-h, x+h) \cap ([0,1] \smallsetminus V)$. In particular, the fluctuation at x is bigger than 2:

$$\begin{aligned} W_{F'}[x-h, x+h] &= \sup\{\, |F'(y) - F'(z)| : y, z \in [x-h, x+h] \,\} \\ &\geq \omega_{F'}(x_h) = 2 \quad \forall h > 0 \\ &\implies \lim_{h\to 0^+} W_{F'}[x-h, x+h] = \omega_{F'}(x) \geq 2. \end{aligned}$$

Therefore, every point $x \in V$ has positive fluctuation. By the lemma 3.2, F' is discontinuous on V, which has positive measure. □

This construction can be generalized to any nowhere-dense set with positive measure. The proof is analogous to the one presented above. However, replicating such construction on the classical Cantor set does not give a function with nonintegrable derivative because the Cantor set has measure zero.

4 Cantor's set derivation and the continuum

4.1 Set derivation and perfect set

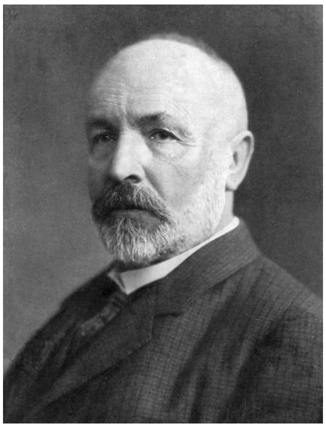

Figure 7: Georg Cantor

In Section 2, we mentioned that Dirichlet provided a sufficient condition for the Fourier series to converge. However, other desirable properties, such as uniqueness or uniform convergence, were also studied throughout the 19th century.

In the late 1860s, E. Heine (1821-1881), a close colleague of Cantor, presented to him the uniqueness problem of trigonometric representation [BBT08]: under what conditions is a trigonometric series that converges to zero necessarily the zero series?

Cantor published his solution to the problem in 1870 with the following theorem:

Theorem 4.1 ([Can70])**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences of complex numbers. Consider the formal trigonometric series*

$$f(x) := \sum_{n\in\mathbb{N}} a_n cos(nx) + b_n sin(nx).$$

If the series converges to zero for all real numbers $x \in \mathbb{R}$, then all the coefficients are zero:

$$a_n, b_n = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Instead of relying on a typical analytic argument, which was common at that time, Cantor used a topological and set-theoretic method to prove this theorem. His approach focuses on the concept of set derivation, which links the set and its accumulation points. Cantor noticed that this method was not limited to this particular problem. It offers a topological way to investigate the properties of sets based on how their elements accumulate. This discovery marked a shift from the classical analysis method based on inequalities to a more abstract, topological understanding of functions and sets.

Definition 4.2. (Accumulation points and derived set) Let X be a topological space and $N \subseteq X$. Given $x \in X$, we say that x is an accumulation point of N if and only if for any subset $U \subseteq X$, neighborhood of x, we have:

$$(U \setminus \{x\}) \cap N \neq \emptyset.$$

The set of all accumulation points of N is called the derived set of N, and it is commonly denoted as N'.

$$N' := \{x \in X \mid x \text{ is an accumulation point of } N\}.$$

Proposition 4.3. *Let (M, d) be a metric space, N is a subset of M, and $x \in M$. Then the following are equivalent:*

1. *x is an accumulation point of N.*
2. *There is a sequence $(x_i)_{i \in \mathbb{N}} \in (N \setminus \{x\})^{\mathbb{N}}$ that converges to x.*
3. *x is a nonisolated limit point of N.*

Proof. (1) ⇒ (2)**:** Suppose x is an accumulation point of N. We construct a sequence $(x_k)_{k \in \mathbb{N}} \in (N \setminus \{x\})^{\mathbb{N}}$ in this way: for each k, consider the open ball with radius $1/k$ centered at x, this is an open neighborhood of x and by definition of accumulation point we can pick a point $x_k \neq x$ in this ball. The resulting sequence $\{x_k\}_k \geq 1$ converges to x.

(2) ⇒ (3)**:** Suppose x is the limit of $(x_i)_{i \in \mathbb{N}} \in (N \setminus \{x\})^{\mathbb{N}}$, it is obvious that x is a limit point of N. Now given U, an open neighborhood of x, by definition there exists a positive number $r > 0$ such that $B(x, r) \subseteq U$. By the definition of a convergent sequence, there exists $m \in \mathbb{N}$ such that for all $m \geq m_0$ we have $d(x_m, x) < r$; thus $x_m \in U \quad \forall m \geq m_0$.

(3) ⇒ (1)**:** Contrapositive proof: Suppose x is not an accumulation point. This means that there is a subset U, neighborhood of x, such that $U \cap N = \{x\}$, thus x is isolated.

□

Example 4.4. Consider the set formed by the elements of the sequence $(x_n)_n = (\frac{1}{n})_n \xrightarrow{n} 0$. This sequence converges to zero, hence, 0 is the only accumulation point.

Example 4.5. Consider the set $A = \{\frac{1}{k_1} + \frac{1}{k_2} \mid k_1, k_2 \in \mathbb{Z}_+\}$. For all $n \in \mathbb{Z}_+$ we can do $\frac{1}{n} + \frac{1}{m} \xrightarrow{m \to \infty} \frac{1}{n}$, so the set $\{\frac{1}{n}\}_n \subseteq A'$. The point 0 is also an accumulation point, since it is the limit of limit points. We claim that there are no more accumulation points of A. Let us say that x is an accumulation point of A and $\frac{1}{k_1(n)} + \frac{1}{k_2(n)} \xrightarrow{n \to \infty} x$, then by the pigeonhole principle, at least one of the denominator goes to infinity as n grows. Without loss of generality, we can assume that $k_2(n) \xrightarrow{n \to \infty} \infty$. Hence, we can only choose freely at most one of the denominators, concluding the claim.

Note that in example 4.5 the set of accumulation points A' has its own accumulation points:

$$\{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq A' \implies 0 \in (A')'.$$

We denote the $A^{(2)}$ as doing set derivations twice on the set A, similarly we can define $A^{(n)}$ for $n \in \mathbb{Z}^+$. This allows us to iterate the set derivation infinitely many times. A natural question would be: Will it converge in some sense? Does some property characterize convergence?

Example 4.6. Take an interval (a, b) where $a < b$. Its derived set is $[a, b]$, and then it remains invariant under the operation of set derivation. Similarly, having a dense subset of an interval, its derived set would be the whole interval, and it would stay fixed under set derivation.

Cantor noticed that there are subsets that remain fixed under set derivation and others that change endlessly. He tried to label sets with their minimal stabilizing iteration, which is now called their Cantor-Bendixson Rank.

Definition 4.7. (Cantor-Bendixson Rank) Let X be a topological space and N a subset of it. We denote $N^{(n)}$ for the set obtained from deriving n times from the set N. Consider the following number called the Cantor-Bendixson rank of N:

$$k = \min\{n \in \mathbb{N} \mid N^{(n)} = N^{(n+1)}\}.$$

If $N' = N$, we say that the set is perfect. If the set N does not stabilize after any finite derivation, then we say it has nonnatural ordinal Cantor-Bendixson rank. For brevity, we refer to this simply as the rank of the set N whenever the context makes the meaning clear.

4.2 A method to construct sets of nonnatural ordinal rank

At first, it seems like all sets will stabilize after enough derivations. This is true only if we extend the definition of set derivation to any ordinal number by transfinite induction. In such case, every subset of real numbers stabilizes with some ordinal number, which is not necessarily a natural number. However, if we only consider the natural numbers, it is possible to find sets that never stabilize in a finite number of set derivations.

In this section, we present a novel constructive method for generating sets whose rank is a nonnatural ordinal. Any set obtained from this method changes with each application of the set derivation. We have searched both Google Scholar and MathSciNet using the keyword "set derivation" and reviewed the most relevant and closely matched results. Even though these works primarily address sets with nonnatural rank, none appear to explicitly construct sets with nonnatural rank via a concrete iterative method. To the best of our knowledge, this construction is original.

First of all, we can construct sets that become empty after exactly n steps for any given positive integer n. In example 4.5, we were in some sense rescuing example 4.4 from being eliminated by set derivation. We did it by adding a sequence that goes to zero to each element of example 4.4. We have also seen that the resulting set becomes empty after exactly 3 derivations. We can reproduce this technique in the example 4.5 again and get a set that vanishes exactly after 4 derivations. We can repeat this process to get a set that stabilizes exactly after n set derivations for any given $n \in \mathbb{N}$.

To make this iterative construction more general, two objects are needed: a point $x \in \mathbb{R}$ and a sequence that converges to zero: $(x_n)_{n\geq 1}$.

We define a family of sets depending on x and the sequence $(x_n)_n$ as follows:

$$A_0 = \{x\},$$

and for each $m \geq 1$,

$$A_m = A_{m-1} + \{x_n\}_{n\geq 1} = \left\{ x + \sum_{i=1}^{m} x_{n_i} : x_{n_i} \in \{x_n\}_{n\geq 1} \right\}. \tag{2}$$

To prove that any member of this family has the rank we want, we have to first prove this lemma that allows us to change the operation of set derivation and finite union.

Lemma 4.8. *Let $N \in \mathbb{Z}^+$ and $X_i \subseteq \mathbb{R}$ for $i = 1, 2, 3, ..., N$, then $\bigcup_{k=1}^{N} X_k' = (\bigcup_{k=1}^{N} X_k)'$.*

Proof. We first prove the case $N = 2$: $X_1' \cup X_2' = (X_1 \cup X_2)'$. Then apply induction.

($\subseteq$) Let $(x_n)_n \subset X_1$ be a convergent sequence with $x_n \to x$, then $\{x_n\}_n \subseteq X_1 \subseteq X_1 \cup X_2$, which means $x \in (X_1 \cup X_2)'$. Therefore, $X_1' \subseteq (X_1 \cup X_2)'$. Analogously, $X_2' \subseteq (X_1 \cup X_2)'$.

($\supseteq$) If $(z_n)_n \subset X_1 \cup X_2$ and $z_n \to z$, then by the pigeonhole principle (theorem A.1), it must happen that $|\{z_n\}_n \cap X_1| = \infty$ or $|\{z_n\}_n \cap X_2| = \infty$. By symmetry, assume $|\{z_n\}_n \cap X_1| = \infty$, which means that there is a subsequence $(z_{n_k})_k \subset X_1$. Since every subsequence of a convergent sequence is also convergent, we have $(z_{n_k})_k \xrightarrow{k} z$, hence $X_1' \cup X_2' \supseteq (X_1 \cup X_2)'$.

By induction, we have:

$$\bigcup_{k=1}^{N} X_k' = \left(\bigcup_{k=1}^{N} X_k \right)' \quad \forall N \in \mathbb{N}.$$

□

Theorem 4.9. *Let A_m be the family of sets defined in (2). For all $m \geq 0$, A_m is countable, has rank $= m + 1$ and $(A_m)' = \bigcup_{i=0}^{m-1} A_i$.*

Proof. For any given $m \in \mathbb{Z}_+$:

1) A_m is countable.

$A_1 = \{x + x_n\}_n$, which is obviously countable and $A_1' = \{x\} = A_0$. Now suppose that A_{m-1} is countable. Since $A_m = A_{m-1} + \{x_n\}_n$, the set A_m has at most the cardinality of $A_{m-1} \times \{x_n\}_n$, which is countable.

2) $(A_m)' = \bigcup_{i=0}^{m-1} A_i$.

$\subseteq$) By the pigeonhole principle, for every convergent sequence $\{x + \sum_{i=1}^{m} x_{n_i(j)}\}_j \subset A_m$, there are some indexes $I \subseteq \{1, 2, ...m\}$ that go to infinity $i \in I : n_i(j) \xrightarrow{j} \infty$. This means that $\lim_j x + \sum_{i=1}^{m} x_{n_i(j)} \in \{x + \sum_{i=1}^{m-|I|} x_{k_i}\} = A_{m-|I|}$. Since I can be any non-empty subset of $\{1, 2, ..., m\}$, $(A_m)' \subseteq A_0 \cup A_1 \cup ... \cup A_{m-1}$.

$\supseteq$) Given a point $y \in \bigcup_{i=0}^{m-1} A_i$, let us say $y = x + \sum_{i=1}^{k} x_{n_i} \in A_k$ where $k \in \{1, 2, ..., m-1\}$. Then the sequence $\{x + \sum_{i=1}^{k} x_{n_i} + \sum_{i=k+1}^{m} x_j\}_j \in (A_m)^{\mathbb{N}}$ and:

$$\{x + \sum_{i=1}^{k} x_{n_i} + \sum_{i=k+1}^{m} x_j\}_j \xrightarrow{j\to\infty} x + \sum_{i=1}^{k} x_{n_i} = y.$$

3) A_m has rank $= m + 1$.

For all $M \leq m$, by lemma 4.8, we have:

$$\begin{aligned} A_m^{(M)} = (A_m')^{(M-1)} &= \left(\bigcup_{k=0}^{m-1} A_k\right)^{(M-1)} = \left(\bigcup_{k=0}^{m-1} A_k'\right)^{(M-2)} = \left(\bigcup_{k=0}^{m-1}\bigcup_{l=0}^{k-1} A_l\right)^{(M-2)} \\ &= \left(\bigcup_{k=0}^{m-2} A_k\right)^{(M-2)} = \quad ... \quad = \left(\bigcup_{k=0}^{m-M} A_k\right)^{(M-M)} = \bigcup_{k=0}^{m-M} A_k. \end{aligned}$$

Hence, $A_m^{(M)} = \bigcup_{k=0}^{m-M} A_k = A_0 = \{x\} \iff M = m$, meaning that the rank is exactly $m + 1$. □

Having defined these sets, which allow us to determine when a set vanishes Having defined these sets, which allow us to decide when a set vanishes, what we want to do now is to make a bigger set from these smaller ones that does not become empty after finitely many derivations. Since all of these sets have finite rank, the easiest idea is to take their union: $A = \bigcup_{m\geq 0} A_m$. This ensures that A does not fade away, but it also makes it a perfect set. Given an element $a \in A$, it must belong to A_m for some natural number m, which is a limit point of A_{m+1}. To avoid this from happening, we can place sets of different ranks disjointedly. This way, there will always be some sets being dropped off while deriving. For instance:

$$\begin{aligned} B_0 &= A_0, \\ B_m &= \max(B_{m-1}) + 1 + \bigcup_{k\leq m} A_k \quad \text{for } m \geq 1, \\ B &= \bigsqcup_{m\in\mathbb{N}} B_m. \end{aligned}$$

This is rather an abstract description, we present an example to see the philosophy.

Consider the sequence $\{x_n\}_{n\geq 1} := \{\frac{1}{2^n}\}_{n\geq 1}$, which converges to zero, and the largest term is $\frac{1}{2}$. Let $A_0 = \{x\} = \{0\}$, and define

$$A_m = A_{m-1} + \left\{\frac{1}{2^n}\right\}_{n\geq 1} \qquad \text{for } m \geq 1.$$

We can rewrite A_m in the following form:

$$A_m = \left\{ \sum_{j=1}^{m} \frac{1}{2^{k_j}} : k_j \in \mathbb{Z}^+ \right\}.$$

Note that the maximal element of A_m is

$$\sup A_m = \sum_{j=1}^{m} \frac{1}{2} = \frac{m}{2} = \sup A_{m+1} - 1/2.$$

We now define $B_0 = A_0$, and for $m \geq 1$, set

$$B_m = \sup B_{m-1} + \frac{1}{2} + \bigcup_{k \leq m} A_k.$$

Described in words: we begin at the origin and place $A_0 = \{0\}$ there. Then, from the rightmost point of the previous set ($\sup B_{m-1}$), we move $\frac{1}{2}$ to the right ($+1/2$) and place the rank-$(m+1)$ set $\bigcup_{k \leq m} A_k$.

In this case, we can compute an explicit formula for $\sup B_m$:

$$\begin{aligned}
\sup B_m &= \sup \bigcup_{k \leq m} A_k + \frac{1}{2} + \sup B_{m-1} \\
&= \frac{m}{2} + \frac{1}{2} + \sup B_{m-1} \\
&= \left(\frac{m}{2} + \frac{1}{2}\right) + \left(\frac{m-1}{2} + \frac{1}{2} + \sup B_{m-2}\right) \\
&= \cdots \\
&= \sum_{j=1}^{m} \left(\frac{j}{2} + \frac{1}{2}\right) + \sup B_0 \\
&= \frac{1}{2} \left(\sum_{j=1}^{m} j + \sum_{j=1}^{m} 1\right) \\
&= \frac{1}{2} \left(\frac{m(m+1)}{2} + m\right) = \frac{m(m+1) + 2m}{4} = \frac{m(m+3)}{4}.
\end{aligned}$$

Hence,

$$B_m = \bigcup_{k \leq m} A_k + \left(\sup B_{m-1} + \frac{1}{2}\right) = \bigcup_{k \leq m} A_k + \left(\frac{(m-1)(m+2)}{4} + \frac{1}{2}\right) = \bigcup_{k \leq m} A_k + c_m.$$

The family $\{B_m : m \in \mathbb{N}\}$ consists of pairwise disjoint sets, each at least $\frac{1}{2}$ apart from the others. Furthermore, B_m has rank $m+1$, so their union

$$B = \bigsqcup_{m \in \mathbb{N}} B_m$$

never stabilizes and for every $m \in \mathbb{N}$, we have $B^{(m)} \neq B^{(m+1)}$. In fact,

$$B^{(m)} = B + \left(\frac{(m-1)(m+2)}{4} + \frac{1}{2}\right) = B + c_m \quad \forall m \geq 2.$$

This can be interpreted as every time we derive, we are removing the rank 0 part. For other members $\cup_{k\leq m} A_k + c_m$, their rank gets lowered by 1, and they become $\cup_{k\leq m-1} A_k + c_{m-1}$. Since the sets are ordered by their rank, we are always removing the leftmost set. In order to match the original set B, we move $\frac{(m-1)(m+2)}{4} + \frac{1}{2}$ to the right, since it is where B_m begins.

4.3 The *Grundlagen* and the ternary Cantor set

In the previous part, we discussed the possible behaviour when iterating set derivations. Cantor noticed that through different natures of iterated set derivations, a set could be separated into different parts based on their convergence. His first intuition was to split the set into two components. The first part is those points that will eventually disappear after finitely many set derivations, like the examples 4.4 and 4.5, and then the second part is the subset that does not vanish in finite steps. This core idea was later improved and formalized by I. Bendixson (1861-1935), and the result is now known as the Cantor-Bendixson theorem.

Before diving into the theorem, we need a lemma that characterizes perfect sets:

Lemma 4.10. *Let (M, d) be a metric space and A is a subset of M. Then A is perfect if and only if A is closed and has no isolated points.*

Proof. $\implies$) Suppose that $A = A'$. Let $x \in A' = A$, by the proposition 4.3, x is a nonisolated limit point.

$\impliedby$) Suppose that A is closed and has no isolated point. By proposition 4.3, all accumulation points are limit points, so $A' \subseteq \overline{A} = A$ where $\overline{A}$ is the topological closure of A. As A has no isolated point, all elements $x \in A$ admits a sequence $(x_n)_n \subseteq A$ such that $x_n \to x$, hence $A \subseteq A'$ and therefore, $A = A'$. □

Theorem 4.11. *(Cantor-Bendixson theorem) Given a closed subset $A \subseteq \mathbb{R}$ with the standard Euclidean topology, it can be partitioned into two components, the first one P is a perfect set and the second one N is countable: $A = P \bigsqcup N$.*

The hypothesis of A being a closed set is to avoid adding points in the first derivative like an open interval does: $(a, b)' = [a, b]$.

Proof. We know that the set of open intervals with rational endpoints forms a countable basis for the Euclidean topology in $\mathbb{R}$. Let $(I_1, I_2, I_3, ...)$ be an enumeration of the induced countable basis in A.

We define the following set M as a candidate for N:

$$M := \{x \in A \mid \exists\, U_x, \text{ an open and countable neighborhood of } x\}.$$

Given $x \in M$, by definition, there is an open, countable neighborhood U_x. Since we have an induced basis, we can represent U_x as a union of basic open sets. In particular, there will be a basic open set B_x such that $x \in B_x \subseteq U_x$. Notice that $B_x \subseteq M$ since any point in B_x admits B_x itself as an open countable neighborhood; thus, we have that $M = \cup_{x \in M} B_x$. This means that M is open by being the union of open sets. Since there are only countably many basic open subsets and for all given x, B_x is a basic open subset, M is at most a countable union of countable sets, which is countable.

Now we consider the complement space $Q = A \setminus M$. We aim to prove that it is perfect (i.e, $Q' = Q$). If $Q = \emptyset$, then it is trivially perfect, so we can suppose that $Q \neq \emptyset$. Since Q is the complement of an open set, by definition, it is closed. Given $x \in Q$, if x is isolated in Q, there exists an open subset V, a neighborhood of x such that $V \cap Q = \{x\}$. Consequently, $V \cap A \subseteq M \bigsqcup \{x\}$, which is a countable open neighborhood of x, contradicting that $x \notin M$. So M is a closed set with no isolated point, by lemma 4.10, M is perfect. □

In fact, we can also prove that if Q is nonempty, then any relatively open set of Q is not countable. Suppose that there is an open set V such that $V \cap Q$ is nonempty and countable, then $V = (V \cap M) \bigsqcup (V \cap Q)$ is a union of two at most countable sets, thus at most countable. Since V is open and can be considered as a neighborhood of its element, $V \subseteq M$. This leads to the contradiction that $V \cap Q \neq \emptyset$. Therefore, all nonempty relatively open subsets of Q are uncountable.

Within the theorem 4.11, Cantor believed that he had found the fundamental property of what he called the "continuum". By the 1880s, there were already many definitions of the real numbers. All of these constructions are based on arithmetical operations like sums and products. Cantor was searching for a more general and purely conceptual definition of the concept of the "continuum". By conceptual, Cantor meant to refer to what we nowadays call topological or set-theoretic. He asked himself:

When is a set $A \subseteq \mathbb{R}^n$ considered as a "continuum" body?

According to Dedekind [Ded05], a set is a "continuum" if and only if it is perfect. However, Cantor pointed out that this is not necessarily true, as there exist perfect sets that are not conceptually considered as a continuous body. In [Can83], Cantor gave a counterexample, which today is known as the ternary Cantor set.

Proposition 4.12. *The following set S is the Cantor set, and the ternary representation of $\mathcal{C}$ is unique.*

$$S = \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\} \right\}.$$

Proof. We will use a double inclusion to prove $\mathcal{C} = S$.

$\subseteq$) Let $x \in \mathcal{C}$, with ternary expansion $x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$.Consider the following inequalities:

$$0 \leq \sum_{n=N}^{\infty} \frac{a_n}{3^n} \leq \sum_{n=N}^{\infty} \frac{2}{3^n} = \frac{2}{3^N} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{2}{3^N} \cdot \frac{3}{2} = \frac{1}{3^{N-1}} \quad \forall N \geq 1.$$

If $x \in [0, 1/3] \cup [2/3, 1] \subset D_1$, then by the inequality, b_1 must be 0 in the first segment and 2 in the second one, because the tail of the series ($n \geq 2$) adds up to a maximum of 1/3. By induction, if $b_k \in \{0, 2\}$ then $b_{k+1} \in \{0, 2\}$ because the tail ($n \geq k + 1$) adds up to a maximum of $1/3^{k-1}$.

$\supseteq$) Take a point $x \in S$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. Since $a_1 \in \{0, 2\}$, we have $x \in D_1$. Now we suppose x is in one of the subintervals of stage n, by the inequality chain (the one before), $a_{n+1} \in \{0, 2\} \implies x \in D_{n+1}$.

Now for the uniqueness, given a given point $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in \mathcal{C}$. Let $\sum_{n=1}^{\infty} \frac{b_n}{3^n}$ be another representation, and let k be the minimal integer such that $a_k \neq b_k$ and we assume that $a_k = 2, b_k = 0$. Then:

$$0 = \sum_{n=k}^{\infty} \frac{a_n - b_n}{3^n} \geq 2/3^k - \sum_{n=k+1}^{\infty} \frac{2}{3^n} = 1/3^k.$$

This contradiction yields the uniqueness of the ternary representation of $\mathcal{C}$. □

Note that the ternary representation is generally NOT unique since:

$$1/3^k = \sum_{n=k+1}^{\infty} \frac{2}{3^n} \implies \sum_{n=1}^{k-1} \frac{a_n}{3^n} + \frac{1}{3^k} = \sum_{n=1}^{k-1} \frac{a_n}{3^n} + \sum_{n=k+1}^{\infty} \frac{2}{3^n}.$$

The ternary representation of $\mathcal{C}$ can be understood as every point in the Cantor set is equivalent to an infinite sequence of 0s and 2s: $(a_n)_{n \in \mathbb{N}}$. We may visualize it in this way: at stage 1, there are two subintervals (divided from the initial interval). If $a_1 = 0$, then you "move t" the subinterval that is on the left-hand side, and if it is 2, the one that is on the right-hand side. At stage 2, you are in an identical situation but being in a smaller subinterval decided by a_1, you repeat: if $a_2 = 0$, left subintervals, otherwise, to the right.

Proposition 4.13. *The Cantor set $\mathcal{C}$ is perfect and uncountable.*

Proof. Given $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in \mathcal{C}$, and $\varepsilon > 0$, there exists N such that $\sum_{n=N}^{\infty} \frac{a_n}{3^n} \leq \varepsilon$, thus we can choose the point $x_\varepsilon = \sum_{n=1}^{N-1} \frac{a_n}{3^n} + \sum_{n=N}^{\infty} \frac{0}{3^n}$, which is at most ε distance away from x. Now for every $n \geq 1$, we can define $x_{1/n}$ by choosing $\varepsilon = 1/n$ and the sequence $x_{1/n}$ converges to x. Hence $\mathcal{C}$ is perfect.

On the other hand, since the Cantor set is in bijection to the set of all sequences with entries 0 or 2, it has cardinality $2^{\mathbb{N}}$, which is not a countable cardinal. A more detailed explanation would be using Cantor's diagonal argument as follows. Consider the bijection:

$$T: \quad \begin{array}{ccc} \mathcal{C} & \longrightarrow & \{0,2\}^{\mathbb{N}} \\ \sum_{n=1}^{\infty} \frac{a_n}{3^n} & \longmapsto & (a_n)_{n\geq 1} \end{array}$$

Suppose we can enumerate $\{0,2\}^{\mathbb{N}} = (a_n^m)_{n,m} = ((a_n^1)_n, (a_n^2)_n, (a_n^3)_n, ...)$. Since $2-0 = 2and2-2=0$, the sequence $(2-a_n^n)_{n\geq 1} \in \{0,2\}^{\mathbb{N}}$ is different to every sequence in the enumeration. Therefore, $\mathcal{C}$ is uncountable. □

4.4 The "continuum" of Cantor

As we can see, a perfect set does not need to be continuous. Intuitively, a "continuum" cannot have holes in it, but a perfect set admits infinitely many pores, in the sense of having countably many open subsets between any two elements. Hence, there must exist more properties that define the "continuum". Cantor thought that the missing key was the idea of connectedness. If a subset is considered as a continuum, it must be connected.

Definition 4.14. A set $A \subseteq \mathbb{R}^n$ is said to be *Cantor connected* (or *C-connected*, for short) if for every $\varepsilon > 0$ and every pair of points $p, q \in A$, there exists a finite sequence

$$T = (t_0 = p,\, t_1,\, t_2,\, \dots,\, t_n = q),$$

where $\{t_j : j = 0, 1, \dots, n\} \subseteq A$, such that

$$d(t_k, t_{k+1}) < \varepsilon \quad \text{for all } k = 0, 1, \dots, n-1.$$

Here, d denotes the Euclidean distance in $\mathbb{R}^n$. Any such sequence is called an ε-chain between p and q.

The definition of C-connectedness is different than the usual connectedness definitions. For instance, a disconnected space can be C-connected:

Example 4.15. The set $A = (-1,0) \cup (0,1)$ is disconnected but C-connected. Given $\varepsilon > 0$, for any pair of points $x, y \in A$ and $x < y$, we can take n such that $(y-x)/n < \varepsilon$ and choose the sequence formed by $\{x + k(y-x)/n : k = 0, 1, 2, ..., n\} = \{t_k : k = 0, 1, 2, ..., n\}$. If $0 = x + k'(y-x)/n \in T$, then replace this point by the pair of points $0 \pm \varepsilon/2$.

On the other hand, path-connected spaces are also C-connected, meaning that the C-connectedness is a weaker condition compared to the path-connectedness.

Proposition 4.16. *If $A \subseteq \mathbb{R}^n$ is path-connected, then it is also C-connected.*

Proof. Let $\varepsilon > 0$ and $p, q \in A$ be a pair of different points of A. By being a path-connected set, there exists a continuous map $\gamma(t)$ from $[0, 1]$ into A such that $\gamma(0) = p$ and $\gamma(1) = q$. By the Heine-Cantor theorem (theorem A.3), γ is uniformly continuous, which means that there exists a $\delta > 0$ such that $d(x, y) < \delta \implies d(\gamma(x), \gamma(y)) < \varepsilon$. Now, by taking the sequence formed by $\{\gamma(k\delta) : k = 0, 1, 2, ..., \lfloor 1/\delta \rfloor\} \cup \{\gamma(1)\}$, we have that A is C-connected. □

Cantor believed that a continuous body is a set that is both perfect and C-connected. However, his main argument was that other people's theories were incorrect. In [Can83], Cantor explicitly criticized definitions proposed by Dedekind and Bolzano.

On the one hand, Bolzano claimed that a continuous body must be such that between any two elements, another element of the body can always be found, which is an early version of what we call density. On the other hand, Dedekind's definition was that the "continuum" is equivalent to being perfect. However, the Cantor set satisfies both definitions, but conceptually speaking, the object is not a continuous body. So he argued that both Dedekind and Bolzano's definition of the "continuum" were incomplete.

Even now, there is still no widely accepted definition of the "continuum". One of the most cited sources for the concept is [Mal69]:

> **Continuity.** "A material is continuous if it completely fills the space that it occupies, leaving no pores or empty spaces, and if furthermore its properties are describable by continuous functions."

We can see it is a conceptual or philosophical definition rather than a mathematical one. If we try to interpret this definition in purely mathematical terms, the part of being describable by continuous functions would mean purely topological. On the other hand, the idea of filling the space that it occupies, without any pores, would be asking the set to be open (filling), bounded (the occupied space), and simply connected (without any pores).

However, all these topological concepts were formalized in the early 20th century, as part of the broader effort to axiomatize mathematics, like the ZFC set theory. Thus, when Cantor originally introduced the idea of being perfect, he likely referred to an intuitive notion of an open set. That is, for any element x in the set, anything "surrounding" x is also inside the set. On the other hand, the part of C-connected is a primitive idea of topological connectedness, that the body is all linked together. However, Cantor's definition lacked the boundedness part, which was later added with the compactness condition.

It can be shown that in a compact metric space, C-connectedness is equivalent to usual connectedness.

Theorem 4.17. *Let (X, d) be a nonempty compact metric space. Then X is connected if and only if it is C-connected.*

Proof. $\Longrightarrow$) Let X be connected. Fix $a \in X$ and $\varepsilon > 0$. Define the set:

$$A_\varepsilon := \{x \in X \mid \exists \text{ an } \varepsilon\text{-chain from } a \text{ to } x\}.$$

1. $A_\varepsilon \neq \emptyset$: the trivial chain generated by $\{a\}$ connects a to itself.
2. A_ε is open: For any $x \in A_\varepsilon$, choose $\delta = \varepsilon/2$. If $d(x, y) < \delta < \varepsilon$, then the sequence of 2 elements, formed by $\{x, y\}$ is an ε-chain from x to y, therefore, A_ε is open.
3. A_ε is *closed*: Let $\{x_n\}_{n=1}^\infty \subset A_\varepsilon$ converge to $x \in X$. There exists N such that $d(x_N, x) < \varepsilon$. Hence, we can take an ε-chain from a to x_N and concatenate with x. The resulting sequence is an ε-chain from a to x, so $x \in A_\varepsilon$.

Therefore, A_ε is clopen and non-empty in a connected space, meaning that $A_\varepsilon = X$.

$\Longleftarrow$) Suppose X is disconnected but C-connected. Then $X = U \bigsqcup V$ where U and V are disjoint open sets that intersect X. Since $U = X \setminus V$ and V is open, U is closed; similarly, V is closed. Let $\delta := d(U, V)$. If $\delta = 0$, then there exists a pair of sequences $(u_n)_n \subset U$, $(v_n)_n \subset V$ such that:

$$\lim_{n \to \infty} d(u_n, v_n) = 0.$$

By the compactness of the space X, there exists a pair of convergent subsequences $(u_{n_j})_j \to \alpha$, $(v_{n_j})_j \to \beta$. Since U, V are both closed, $\alpha \in U$ and $\beta \in V$. On the other hand, by the continuity of the distance function,

$$\lim_{j \to \infty} d(u_{n_j}, v_{n_j}) = d(\alpha, \beta) = 0,$$

which yields $\alpha = \beta \in U \cap V$, contradicting that U, V are disjoint. Therefore, $\delta > 0$.

Choose $u \in U$, $v \in V$ and set $\varepsilon = \delta/2 > 0$. By C-connectedness, there exists an ε-chain $(u = t_0, t_1, \ldots, t_n = v)$. Let k be the smallest index such that $t_k \in U$ and $t_{k+1} \notin U$. Then $t_{k+1} \in V$, and we have:

$$d(t_k, t_{k+1}) < \varepsilon = \frac{\delta}{2} < \delta = d(U, V),$$

contradicting the definition of $d(U, V)$. Thus, X is connected. □

This theorem allows one to define the "continuum" without using any distance function.

Today, the most common mathematical definition of a continuous body is a compact, connected, metrizable space [Nad17]. Sometimes, less frequently but more abstractly, a compact connected Hausdorff space [Wil12]. Nevertheless, while mathematicians were trying to axiomatize mathematics during the early 20th century, quantum physics revealed a world governed by probabilistic behavior and discrete scales, namely the Planck length $\hbar \approx 1.616 \times 10^{-35}$. This fundamental granularity challenges the classical paradigm of smooth, deterministic continua. The mathematical continuum, for all its elegance and utility in modeling, may thus represent an overly idealized approximation of a physical reality.

A Some Basic Theorems and Definitions

In this appendix, we collect some theorems and definitions used throughout the main text.

Theorem A.1. *(Pigeonhole principle) Let A be a finite set of n elements, and A is divided into m disjoint subsets $A = \bigsqcup_{k\leq m} A_k$. If $m > n$, then there exists a member of the partition, say A_{k_1}, with at least 2 elements.*

Theorem A.2. *(Heine-Borel theorem) Consider the space $\mathbb{R}^n$ with the standard Euclidean topology and $T \subseteq \mathbb{R}^\ltimes$. Then T is compact if and only if it is closed and bounded.*

Theorem A.3. *(Heine-Cantor theorem) If $f : X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.*

Theorem A.4. *(Baire Category theorem) Let (M, d) be a complete metric space and $\{U_i\}_{i\geq 1}$ be a countable collection of dense open sets. Then $\cap_{i\geq 0} U_i$ is also dense.*

Definition A.5. (Measure) Let A be a set. A function $\mu : \mathcal{P}(A) \to [0, \infty]$ is a measure over A if and only if:

1. $\mu(\emptyset) = 0$,
2. If $\{E_i\}_{i\in\mathbb{N}}$ is a countable collection of pairwise disjoint subsets of A, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Theorem A.6. *(Monotone convergence theorem for sets) Let A be a set and μ a measure over A. If $\{A_n\}_{n\geq 1}$ is an increasing sequence of subsets of A. Then*

$$\mu(\cup_{n\geq 1} A_n) = \lim_{n\to\infty} \mu(A_n)$$

Theorem A.7. *(Moore-Osgood theorem) Let (X, d) be a metric space, E is a subset of X, and x is a limit point of E. Let $f_n : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$ be functions over X.*

If $f_n(x) \xrightarrow{n} f(x)$ uniformly on E and $f_n(x) \xrightarrow{x\to x_0} a_n \in \mathbb{R}$ for all n. Then the following limits exist, and they match:

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{n\to\infty} a_n = a.$$

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