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A Consistent Quantization Procedure with Deformation Quantization and Spinor Bundles

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Abstract

In 1946 it was proven that the canonical quantization procedure that identifies the Poisson bracket with the commutator in Hilbert space gives rise to contradictions in Quantum Mechanics. Our goal is to present a consistent and well defined quantization scheme that accounts for the non-commutativity of observables. We introduce Deformation Quantization as a consistent formalism to represent QM in phase space with the use of the so-called star products. A quantization map is provided by deriving a map that relates star products to the product of self-adjoint operators in Hilbert space. Furthermore, in order to prescribe a meaning to spinors, which correspond to phenomena with no classical analog, we introduce the so called spinor bundles. Spinorial matter is not quantum mechanical in the sense of phase space observables, but is rather dictated by the symmetries of the underlying manifold. These results suggest many conjectures on the fundamental aspects of QM and hint at the fact that the information of a Quantum Theory cannot be extracted from a classical setting without adding extra mathematical structure.

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1 Introduction

Theoretical physics in the twentieth century witnessed an important paradigm shift regarding our fundamental understanding of microscopic matter. This scientific revolution was spearheaded by the developments in the branch of Physics that came to be known as Quantum Mechanics (QM). There are two features that distinguish QM from its predecessor framework, Classical Mechanics: (i) the discrete spectrum of physical observables and (ii) the Heisenberg uncertainty principle. Both phenomena were explained in the 1920s by non-commutative self-adjoint linear operators acting on elements of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$. Dirac, Bohr and others noticed that many classical observables, which are taken to be C^∞ (smooth) functions in phase space could be represented in Hilbert space via a map $\mathscr{D} : C^\infty(T^*M) \to \text{Obs}(\mathcal{H})$, where we identify phase space with T^*M and denote the space of linear self-adjoint operators in $\mathcal{H}$ as $\text{Obs}(\mathcal{H})$. This map is defined by the assignments $q \mapsto \hat{q}$ and $p \mapsto \hat{p}$, so that any phase space observable $f(q,p)$ is mapped to $\hat{F} = f(\hat{q}, \hat{p})$. This map $\mathscr{D}$ is called **canonical quantization**. Dirac [1] also suggested that this map satisfied a Lie algebra homomorphism relation between the algebra of smooth functions equipped with the Poisson bracket $\{\cdot,\, \cdot\}$ and the algebra of linear operators in $\mathcal{H}$ with its associated commutator $[\cdot, \cdot]$:

$$\mathscr{D}(\{f,\, g\}) = \frac{1}{i\hbar}[\mathscr{D}(f), \mathscr{D}(g)] \quad \forall f, g \in C^\infty(T^*M)\,, \quad \text{or informally:} \quad \{\cdot,\, \cdot\} \to \frac{1}{i\hbar}[\cdot, \cdot]\,.$$

This quantization procedure has proved successful in many applications in QM. However, it was already known in the late 1920s and early 1930s that it gives rise to inconsistencies for observables involving high powers in position and momentum [2]. Furthermore, it is rather obscure how the Hilbert space disappears as $\hbar \to 0$ and gives rise to a totally different object in a smooth manifold.

In fact, in 1946 Groenewold presented his no-go theorem [3], by virtue of which it is proved that no map from classical to quantum observables, the former equipped with the Poisson bracket, and the latter with the commutator, could satisfy the Lie algebra homomorphism condition. In particular, canonical quantization is not a Lie algebra homomorphism, i.e.

$$\exists\, f, g \in C^\infty(T^*M) : \quad \mathscr{D}(\{f,\, g\}) \neq \frac{1}{i\hbar}[\mathscr{D}(f), \mathscr{D}(g)]\,.$$

Groenewold used as a counter example the fact that $\{q^3,\, p^3\}$, which classically equals $\{\{q^2,\, p^3\},\, \{q^3,\, p^2\}\}/12$, gives rise to an order $\hbar^2$ contradiction in $\mathcal{H}$. So the Lie algebra generated by smooth functions on phase space using the Poisson bracket is not isomorphic to linear operators on Hilbert space. A natural question arises.

> Is there any other algebra defined over smooth functions in phase space $C^\infty(T^*M)$ and a bijective map $\hat{\mathcal{O}} : C^\infty(T^*M) \to \text{Obs}(\mathcal{H})$ encoding a corresponding Lie algebra homomorphism?

Whether a well-defined quantization procedure exists or not is of crucial importance in modern fundamental physics, since all modern theoretical frameworks are stated in terms of a *classical* principle of stationary action which yields a system of equations of motion for some tensor or spinor field distributions. With canonical quantization, i.e. with a Lie algebra homomorphism between $\{\cdot,\, \cdot\}$ and $[\cdot,\, \cdot]$ we can ensure that the equations of motion of the quantum fields will coincide with those of the classical fields. Indeed, going from classical to quantum is a common procedure in Quantum Field Theory. However, since this map is not a morphism, we will need further mathematical structure in order to be able to reproduce the results of QM.

It turns out that in 1930 Hermann Weyl [4] had published a bijective quantization map, which, without knowing, encoded a Lie algebra homomorphism. Which was then this Lie algebra? Moyal [5] discovered it was a deformation of the commutative pointwise product of functions that later became known as a **star product**. This star product, together with the Weyl quantization map did satisfy the desired condition:

$$\hat{\mathcal{O}}(f \star g - g \star f) = \frac{1}{i\hbar}[\hat{\mathcal{O}}(f), \hat{\mathcal{O}}(g)]\,. \tag{1}$$

It was quickly acknowledged that the extra mathematical structure provided by this quantization procedure, which came to be known as **Deformation Quantization** (DQ) could be used to explain QM solely in phase space since the non-commutativity of observables is already taken care of by the star product. Nevertheless it was still unclear how one could relate the classical Poisson bracket structure with this new star product structure. It would not be until the late 1990s that Maxim Kontsevich[1] [7], proved that there is an isomorphism between certain equivalence classes of star products and Poisson brackets. Kontsevich proved that there is a standardized procedure that allows one to associate a specific star product structure with a given Poisson bracket on $\mathbb{R}^d$.

A big part of our work will be devoted to understand star products, both from a formal mathematical perspective and from a hands-on approach. Then, we will have constructed a consistent quantization procedure that is able to deal with points (i) and (ii) of the previous discussion without leaving classical phase space. However, we will have left a third point out of the discussion: spin. Indeed, QM as formulated in all elementary textbooks, explains that there are experiments in which purely quantum mechanical phenomena are manifested. They typically concern spin and have *no classical analogue*. Another question is then proposed:

> Can the non-commutative nature of spinor operators be captured without a Hilbert space description? How is it related to star products?

We will show how the metric tensor $g_{\mu\nu}$ and its symmetries over the manifold of study M determine the behavior of spinorial matter. We will explain spinors and their properties by constructing a geometrical framework around them. Spinorial operators will therefore not be subjected to any quantization procedure.

We will also discuss the philosophical implications of the results presented, for we will argue that the failure of canonical quantization to produce a consistent Quantum Theory lies in the fact that it does not preserve information, whereas DQ does preserve information and therefore constitutes an isomorphism at the algebra level.

[1]Kontsevich actually won the 1998 Fields Medal "for his contributions to algebraic geometry, topology, and mathematical physics, including the proof of Witten's conjecture of intersection numbers in moduli spaces of stable curves, construction of the universal Vassiliev invariant of knots, *and formal quantization of Poisson manifolds*" [6]

This dissertation is structured as follows. In section 2 we give a geometrical interpretation of Classical Mechanics in phase space, since it is the background from which the formalism of DQ is developed in section 3. We will investigate the key points of Kontsevich's work, and find a particular star product using his formalism that will serve useful when doing quantum mechanics in phase space. We will understand Weyl's quantization map $\mathcal{O}$, and how together with the star product, produces the desired Lie algebra homomorphism in equation (1). In section 4, we will borrow techniques from Yang-Mills theory such as principal fiber bundles, and we will construct the so-called **spinor bundles**. This construction will bring to light the fact that the non-commutativity of spinorial operators is fundamentally different from that of other quantum mechanical observables, since the latter inherits its properties from a star product structure, and the former from a Clifford algebra structure. Finally, in section 5 we will lay out the complete quantization scheme. In Appendix A we give a brief summary of the essential mathematical concepts and formulas utilized throughout the text.

2 Mathematical Structure of Classical Mechanics

In this section we will give a geometrical framework of Classical Mechanics that will be of use in our construction of the star product, as well as to contextualize some of the results that will be obtained throughout the text. Throughout this section, let M be an n-dimensional manifold (see A.1).

2.1 Symplectic Geometry

In Classical Mechanics, the possible states of a system of particles are determined by the positions and momenta of such particles. That is, a physical state corresponds to a pair $\mathbf{x} = (p, q)$ where each element corresponds to an n-dimensional vector (for simplicity the case $n = 1$ will be the most used in this section). The space of all possible configurations of such variables is known as **phase space** and is $2n$-dimensional. One can think of phase space as assigning to each point $q \in M$ a "direction" given by p. The most apt mathematical structure that we can construct with this information is the cotangent bundle T^*M (A.2) [8]. We also know that the directions at each point are not arbitrary, for they obey a set of Equations of Motion (EOM) given a smooth function $H(p, q)$:

$$\dot{q} = \frac{\partial H}{\partial p}\,, \quad \dot{p} = -\frac{\partial H}{\partial q}\,. \tag{2}$$

We recognize that the components of the one-form $dH = \partial_q H\, dq + \partial_p H\, dp$ are featured on the right hand side of the EOM. By a one-form we mean an object that given a vector field produces a scalar, so we may write it as $dH(\cdot)$. Also, they are known as $(0, 1)$ tensors, where as vector fields are referred to as $(1, 0)$ tensors. On the left-hand side, we find the components of the tangent vector to $\mathbf{x}$, which is indeed a vector. So we see that on one side of the equations we have dH which is a $(0, 1)$ tensor and on the other $\dot{\mathbf{x}}$ which is of type $(1, 0)$, both over T^*M. This is not mathematically correct unless there is some underlying object capable of transforming a vector into a one-form in such a way that the EOM in (2) are satisfied.

The sort of object that transforms a $(1, 0)$ tensor into a $(0, 1)$ tensor is a $(0, 2)$ tensor. By definition, a $(0, 2)$ tensor field $\omega(\cdot, \cdot)$ "takes" two vector fields to give a scalar field, i.e. an object in $C^\infty(T^*M)$. Now, if we feed a single vector field X, then $\omega(X, \cdot)$ is a $(0, 1)$ tensor, because by construction transforms a vector into a scalar. So we want to find a $(0, 2)$ tensor field satisfying $\omega(\dot{\mathbf{x}}, \cdot) = dH(\cdot)$ that reproduces the laws of physics. This translates to the following requirements [9]:

1. **Non-degeneracy**: if X is a vector field that is zero everywhere, then $\omega(X, Y) = 0$ for all vector fields Y over T^*M. This will ensure that we can always solve for $\dot{\mathbf{x}}$ given ω and H.

2. **Skew-symmetry**: in a system with no external forces the change in the Hamiltonian must be zero along its constant lines to enforce conservation of energy, so $dH(\dot{\mathbf{x}}) = 0 \iff \omega(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = 0$, which is to say that ω is alternating and therefore a two-form on T^*M.

3. **Closedeness**: we require that ω should be invariant under the time evolution of the system. This condition is enforced if the Lie derivative (A.3) of ω along the direction of $\dot{\mathbf{x}}$ is zero, i.e. if $L_{\dot{\mathbf{x}}}\omega = 0$. Using Cartan's magic formula (37) one finds

that necessarily $d\omega = 0$ if the EOM are to be invariant under time evolution. A form with zero exterior derivative is called closed.

A bilinear non-degenerate, closed two-form is called a **symplectic** form. A manifold together with a symplectic form is called a symplectic manifold. In particular, if we take $\omega = dq \wedge dp$, it is straightforward to check that the vector

$$\dot{\mathbf{x}} \equiv X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

satisfies $\omega(X_H, \cdot) = dH(\cdot)$.

2.2 Poisson Geometry

The construction of equations of motion given a symplectic form ω for a Hamiltonian H can be mirrored for any smooth function $f \in C^\infty(T^*M)$. That is, we may define X_f in a similar manner such that the symplectic form is preserved along its integral curves, and such that $\omega(X_f, \cdot) = df(\cdot)$. In general, if a vector field X preserves the symplectic form in the sense that $L_X\omega = 0$ it is called a **symplectic vector field**. For our canonical symplectic form $\omega = dq \wedge dp$ it will be of the form:

$$X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}\,.$$

We see that if we consider a smooth function g then $X_f(g)$ corresponds the classical Poisson bracket. Since $X_f(g) = df(X_g) = \omega(X_f, X_g)$, then $\{f,\, g\} := \omega(X_f, X_g)$ and we can always induce a Poisson bracket from a symplectic structure. One can verify that it satisfies:

1. **Leibniz identity**: $\{f,\, gh\} = g\{f,\, h\} + \{f,\, g\}h$. This means that the bracket is a derivation with respect to the point-wise product of functions.
2. **Jacobi identity**: $\{f,\, \{g,\, h\}\} + \{h,\, \{f,\, g\}\} + \{g,\, \{h,\, f\}\} = 0$. This means that the bracket is a derivation with respect to itself.

A **Poisson structure** on a manifold M consists of a (bilinear) Lie bracket $\{\cdot,\, \cdot\}$ on the algebra of smooth functions $C^\infty(M)$, which satisfies Leibniz and Jacobi. A Poisson structure on a manifold generalizes the symplectic structure [10], and will be the most useful concept to understand throughout the text. Poisson geometry is important because of its connection with Deformation Quantization, which lies in the fact that a Poisson manifold naturally gives rise to a precise Quantization procedure, as we will see.

Note that the Leibniz identity says that for any $H \in C^\infty(M)$ the operation $\{H,\, \cdot\}$ is a derivation of the algebra $C^\infty(M)$. Therefore, it can be used to define a vector field X_H on M via

$$\{H,\, f\} = L_{X_H}(f) \quad \forall f \in C^\infty(M)\,. \tag{3}$$

A consequence of the Leibniz rule is that Poisson brackets are local in the sense that they can be restricted to open subsets of the manifold. In a local chart (U, x) a Poisson bracket takes the form:

$$\{f,\, g\}|_U = \pi^{ij}\,\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{4}$$

where $\pi^{ij} := \{x^i,\, x^j\}|_U$ are called the **structure functions** of the Poisson bracket with respect to the chart. We will be using Einstein's sum convention unless otherwise stated.

This skew-symmetric functions determine the bracket locally. Indeed, the Jacobi identity yields a system of partial differential equations for the structure constants:

$$\pi^{il}\,\partial_l\,\pi^{jk} + \pi^{jl}\,\partial_l\,\pi^{ki} + \pi^{kl}\,\partial_l\,\pi^{ij} = 0 \qquad (1 \leq i < j < k \leq n)\,. \tag{5}$$

This system is an over-determined non-linear system of first-order PDEs. The space of local solutions of this system is poorly understood. Furthermore, the local properties of Poisson manifolds are encoded in the structure functions π^{ij}. This suggests that the Poisson bracket can be understood via an expression of the form:

$$\pi = \pi^{ij}\,\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\,.$$

Such an expression is an example of a skew-symmetric $(2,0)$ tensor field, which we will call a **bivector field**, which is not the same as a two form (a skew-symmetric $(0,2)$ tensor field). With a rewriting of the Jacobi identity specified in Appendix B in terms of the bivector, we will also call the pair (M,π) a Poisson structure throughout the rest of the work.

Finally, it will be useful for our later investigations to ask for a classification of different Poisson manifolds. To do this, we will consider **Poisson maps**. If (M_1,π_1) and (M_2,π_2) are two Poisson manifolds, then a smooth map $\phi : M_1 \to M_2$ is a **Poisson map** if $\phi_*\pi_1 = \pi_2$. That is, if $f,g \in C^\infty(M_2)$ then $(\phi_*\pi_1)(f,g) = \pi_1(f\circ\phi, g\circ\phi)$ and this should equal $\pi_2(f,g)$. In this work we will consider two Poisson structures on a given manifold to be equivalent if they can be related by a smooth map[2] $\exp(X)$ where X is a vector field over M. This maps form a Lie group (A.4) with the product given by the Baker-Campbell-Hausdorff (BCH) [11, p. 89] formula

$$e^X e^Y = e^Z \quad \text{where} \quad Z = X + Y + \frac{1}{2}[X,Y] + \cdots \tag{6}$$

and as stated above act on the bivector via the pushforward.

[2] actually it is a diffeomorphism

3 Deformation Quantization

In this section we present the formalism of Deformation Quantization in detail. We will begin by showing how from any given Poisson structure as presented in section 2, we can construct a star product satisfying certain desired properties. We will use Kontsevich's construction to derive a star product on a manifold with a symplectic structure, which is known as the Moyal product. We will then rederive the Moyal product by constructing the Weyl quantization map from phase space to Hilbert space. Finally, we will present the formalism of QM in phase space and outline the most interesting features.

3.1 Star Products

In mathematics we sometimes consider the so-called **formal power series**, defined by infinite sums of the form $a\,x^n$, considered independently from any notion of convergence. Note that this series may no longer represent a function of x, as would be the case with a power series, which within its radius of convergence does define a function. We can consider the set of formal power series with power $\hbar$ and coefficients being $C^\infty(M)$. We call this space $C^\infty(M)[\![\hbar]\!]$. In particular: $C^\infty(M) \subset C^\infty(M)[\![\hbar]\!]$. A **star product** on a manifold is an $\mathbb{R}[\![\hbar]\!]$ bilinear map $\star : C^\infty(M)[\![\hbar]\!] \times C^\infty(M)[\![\hbar]\!] \to C^\infty(M)[\![\hbar]\!]$ such that

1. $f \star g = fg + B_i(f,g)\,\hbar^i$
2. $(f \star g) \star h = f \star (g \star h)$ for all $f, g, h \in C^\infty(M)$
3. $1 \star f = f \star 1 = f$ for all $f \in C^\infty(M)$

The coefficients $B_i(f,g)$ are functions made up of derivatives of f and g, i.e. locally they take the form $b^{kl}\,\partial_k f\,\partial_l g$ for some multi-indices k and l. The third condition on the definition of the star product implies that the degree 0 term in the right-hand side of the first condition has to be the usual product and also ensures that the B_i are bidifferential operators in the sense that they have no term of order 0: $B_i(1,f) = B_i(f,1) = 0$ for all i. A further consequence of the previous requierements is the fact that the skew symmetric part of the first coefficient, defined by $B_1^-(f,g) = (B_1(f,g) - B_1(g,f))/2$ satisfies the conditions for the definition of a Poisson bracket from section 2. Indeed, given a star product $\star$ on $C^\infty(M)$, we can define a Poisson structure on it by

$$\{f,\, g\} = \frac{f \star g - g \star f}{\hbar} \quad \mathrm{mod} \quad \hbar\,. \tag{7}$$

The problem that we want to solve is the inverse: given a Poisson Manifold M can one define an associative but non-commutative product $\star$ on $C^\infty(M)$, which is a deformation of the point-wise product such that all the stated conditions hold? This problem will be thoroughly answered in the next section, but to end this section we would like to only look at star products that are not equivalent given some linear operator that relates them. Two star products $\star$ and $\star'$ on $C^\infty(M)$ are said to be equivalent if there exists a linear operator $A : C^\infty(M)[\![\hbar]\!] \to C^\infty(M)[\![\hbar]\!]$ of the form

$$Af = f + A_i(f)\,\hbar^i \quad \text{such that} \quad f \star' g = A^{-1}\,(Af \star Ag)\,, \tag{8}$$

where the A_i terms are differential operators. This should be read as in the Lie Algebra homomorphism condition $f(ab) = f(a)f(b)$.

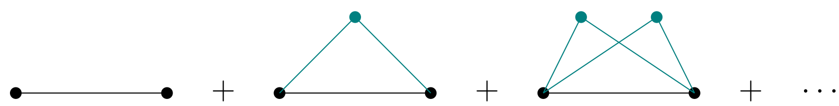
3.2 Kontsevich's Construction

The main difficulty in Deformation Quantization comes when one has a Poisson structure on $\mathbb{R}^d$ and one tries to find an associated star product. Kontsevich's main result was to find an indentification between the set of star products (or equivalently multidifferential operators) modulo the action of the differential operators A in equation (8) and the set of Poisson structures modulo the gauge group of diffeomorphisms mentioned in section 2. So we need somehow to relate bivectors with bidifferential operators so that given a bivector, we can readily find the associated bidifferential operator. This relationship between bivectors and bidifferential operators, however, cannot be achieved at the Lie algebra level and Kontsevich had to use the so called L_∞ algebras. The details of the proof are out of the scope of the text, and we leave them for the interested reader in Appendix C.

Kontsevich was able to construct a so-called L_∞ morphism U between multivectors modulo the gauge group of diffeomorphisms (see C) and multidifferential operators modulo the gauge group of equivalent star products. For a Poisson bivector π:

$$f \star g = fg + U(\pi) = fg + \sum_{j=1}^{\infty} \frac{(i\hbar)^j}{j!} U_j(\pi \wedge \cdots \wedge \pi) = fg + i\hbar U_1(\pi) + i^2 \frac{\hbar^2}{2!} U_2(\pi \wedge \pi) + \cdots .$$

What are these U_j terms that determine the multidifferential operator? Kontsevich showed that $U(\pi)$ can actually be calculated through a series of "Feynman rules" for certain graphs. Given a Poisson structure π, over $\mathbb{R}^d$, its corresponding star product $\star$ is given by a formal sum of graphs:

$f \star g =$  (9)

where the first term refers to the usual commutative product. To qualify as a Kontsevich diagram, we present the following rules and requirements.

Admissible graphs

Here we present the requirements that the graphs must satisfy:

1. $2\sum_j \bullet_j + \sum_k \bullet_k - 2 \geq 0$.
2. No loops for either type of edge.
3. All lines start from a $\bullet$ type vertex.

Kontsevich Rules

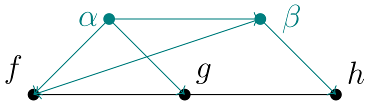
For each graph, apply the following rules to calculate the corresponding star product term:

1. Associate to each vertex $\bullet_i$ with k outgoing lines a multivector field $\xi_i \in \Gamma\left(\bigwedge^k \mathbb{R}^d\right)$.
2. In each vertex of the form $\bullet$ we place a $C^\infty(\mathbb{R}^d)$ function.
3. The j-th arrow (from left to right) outgoing from a k-vertex corresponds to a partial derivative with respect to the coordinate labeled by the j-th index of the multivector ξ_i associated with the vertex.
4. Multiply such elements in the ordering prescribed by the arrows.

5. Divide by the symmetry factor:

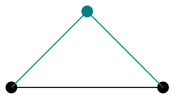
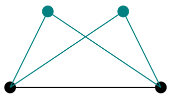
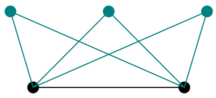
$$\frac{1}{(\sum_k \bullet_k)!}\left(\frac{i\hbar}{(\sum_i \bullet_i)!}\right)^{\sum_k \bullet_k}$$

As an example, consider a graph of three functions and a trivector α and bivector β associated to the $\bullet$ vertices. Then:

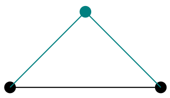


$$= \frac{1}{2}\left(\frac{i\hbar}{6}\right)^2 \alpha^{abc}(\partial_a\partial_d f)\,(\partial_b\, g)\,(\partial_e h)(\partial_c\beta^{de})\,.$$

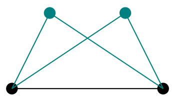
Now consider that our Poisson structure is that induced by the canonical symplectic structure as in section 2. Then the corresponding graphs can only have two lower inputs $\bullet$ corresponding to f and g and all the colored nodes $\bullet$ must only have two outgoing arrows corresponding to π. The only diagrams satisfying such requirements are of the form:

 ,  ,  , $\cdots$

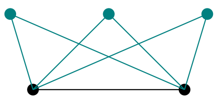
which are precisely those of equation (9). Note that in the symplectic case the π^{ij} are constants so all diagrams with derivatives on the bivector will not contribute. We can readily calculate its coefficients using Konsevich's rules:



$$= \frac{i\hbar}{2}\pi^{ij}\partial_i f\,\partial_j g = \frac{i\hbar}{2} f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right) g\,,$$



$$= \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 \pi^{ij}\pi^{kl}(\partial_i\partial_k f)(\partial_j\partial_l g) = \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)^2 g\,,$$



$$= \frac{1}{3!}\left(\frac{i\hbar}{2}\right)^3 \pi^{ij}\pi^{kl}\pi^{rs}(\partial_i\partial_k\partial_r f)(\partial_j\partial_l\partial_s g) = \frac{1}{3!}\left(\frac{i\hbar}{2}\right)^3 f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)^3 g\,,$$

and so on. We may then write our star product on our canonical symplectic structure as:

$$f\star g = fg + \frac{i\hbar}{2} f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)^2 g + \frac{1}{3!}\left(\frac{i\hbar}{2}\right)^3 f\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)^3 g + \cdots$$

This is called the **Moyal product** and it is usually written as:

$$f\star g = f\,\exp\left(\frac{i\hbar}{2}\pi^{ij}\overleftarrow{\partial}_i\,\overrightarrow{\partial}_j\right) g \tag{10}$$

which establishes a direct correspondence between the symplectic (and Poisson) structure given by π^{ij} and the non-commutative structure provided by the star product.

To get used to the Moyal product we work out a couple of properties to get familiar with the formalism.

Proposition 1. *The Moyal product of exponentials is non commutative in the Baker-Campbell-Haussdorff sense (see equation(6)):*

$$e^{\alpha_1 x+\beta_1 p} \star e^{\alpha_2 x+\beta_2 p} = e^{(\alpha_1+\alpha_2)x+(\beta_1+\beta_2)p}\, e^{\frac{i\hbar}{2}(\alpha_1\beta_2-\alpha_2\beta_1)}\,.$$

Proof. The first term in the right hand side of the BCH formula corresponds to the commutative product. The first term in the Kontsevich expansion is:

$$\frac{i\hbar}{2}e^{\alpha_1 x+\beta_1 p}\left(\pi^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)e^{\alpha_2 x+\beta_2 p} = \frac{i\hbar}{2}\{e^{\alpha_1 x+\beta_1 p},\, e^{\alpha_2 x+\beta_2 p}\} = \frac{i\hbar}{2}(\alpha_1\beta_2-\alpha_2\beta_1)$$

The second term is proportional to

$$\pi^{ij}\pi^{kl}(\partial_i\partial_k f)(\partial_j\partial_l g) = \pi^{ij}(\pi^{kl}\partial_k(\partial_i f)\partial_l(\partial_j g)) = \pi^{ij}\{\partial_i f,\, \partial_j g\}$$

which in our case is just:

$$\pi^{ij}\{\partial_i e^{\alpha_1 x+\beta_1 p},\, \partial_j e^{\alpha_2 x+\beta_2 p}\} = (\alpha_1\beta_2-\alpha_2\beta_1)\{e^{\alpha_1 x+\beta_1 p},\, e^{\alpha_2 x+\beta_2 p}\} = (\alpha_1\beta_2-\alpha_2\beta_1)^2$$

Continuing this procedure all successive powers will be touched. □

Proposition 2. *Lone Star Lemma:*

$$\int dx\,dp\, f\star g = \int dx\,dp\, fg = \int dx\,dp\, gf = \int dx\,dp\, g\star f\,.$$

Proof. This result is a consequence of the skew-symmetry of the Poisson bracket. Indeed the integral

$$\int dp\,dq\,\{f(p,q),\, g(p,q)\}$$

must be zero because by a trivial canonical transformation $p\mapsto q$, $q\mapsto p$ will switch the sign of the Poisson bracket but not of the volume element $dp\,dq$. Therefore the integral is zero. This is easily generalized to higher powers by separating in terms of smaller PB as in the first example. □

3.3 Recovering the Moyal Product Through Hilbert Space

We have already seen that given a Poisson structure on a manifold, there is a natural star product defined on it that naturally gives rise to a non-commutative structure on phase space. This was achieved in the 1990s, but the Moyal product was known since early after the World War II. It was recovered through a quantization map from Hilbert space used by Hermann Weyl around 1930. Then, during and after the war H. Groenewold in the Netherlands and J. Moyal in England took Weyl and Wigner's calculations seriously enough to consider them as an alternative picture for quantum mechanics. To begin, consider the Baker-Campbell-Hausdorff formula (6) applied to the $\hat{p}$ and $\hat{q}$ operators, which, of course, satisfy $[\hat{q},\hat{p}]=i\hbar$:

$$e^{\frac{i}{\hbar}(\hat{p}+\hat{q})} = e^{\frac{i}{\hbar}\hat{p}}\, e^{\frac{i}{\hbar}\hat{q}}\, e^{\frac{i}{\hbar}[\hat{p},\hat{q}]} = e^{\frac{i}{\hbar}\hat{p}}\, e^{\frac{i}{\hbar}\hat{q}}\, e^{-\frac{i}{2\hbar}}\,.$$

Now if we consider the transformation $\hat{p}\to x\hat{p}+y\hat{q}$, $\hat{q}\to x'\hat{p}+y'\hat{q}$, we may find that:

$$e^{\frac{i}{\hbar}((x+x')\hat{p}+(y+y')\hat{q})} = e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{\frac{i}{\hbar}(x'\hat{p}+y'\hat{q})}\, e^{-\frac{i}{2\hbar}(xy'-yx')} \qquad (11)$$

and, by applying BCH in the same fashion we obtain:

$$e^{-\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})} = e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{\frac{i}{\hbar}(x\eta-y\xi)}\,. \tag{12}$$

Using this equation and the definition of the delta function Weyl wrote:

$$\delta(x)\,\delta(y) = \frac{1}{(2\pi\hbar)^2}\int d\xi\, d\eta\, e^{-\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\,. \tag{13}$$

Now, any phase space function $f(q,p)$ can be written using delta functions:

$$f(q,p) = \frac{1}{(2\pi\hbar)^2}\int dx\, dy\, dp'\, dq'\, f(q',p')\, e^{-\frac{i}{\hbar}(x(p'-p)+y(q'-q))}$$

which is elementary and we are not adding any new information. Indeed, this equation just tells us that every normalizable function can be Fourier expanded:

$$f(q,p) = \int dx\, dy\, \mathscr{F}[f](x,y)\, e^{i(xp+yq)} \quad \Longleftrightarrow \quad \mathscr{F}[f](x,y) = \frac{1}{2\pi}\int dq\, dp\, f(q,p)\, e^{-i(xp+yq)}$$

Weyl now proposes a quantum analogy for this integral identity:

$$\hat{F} = \frac{1}{(2\pi\hbar)^2}\int dx\, dy\, d\xi\, d\eta\, e^{-\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\, \hat{F}\, e^{-\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\; e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}$$

That this is indeed an identity can be proven using (12) and (13). Continuing the analogy in phase-space means that every operator $\hat{F}$ can be expanded as

$$\hat{F} = \int dx\, dy\, \hat{f}(x,y)\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})} \tag{14}$$

where $f(x,y)$ corresponds to the middle term and can be reduced to

$$f(x,y) = \frac{1}{\hbar}\,\mathrm{tr}\left(\hat{F}\, e^{-\frac{i}{\hbar}(x\hat{p}-y\hat{q})}\right) \quad \text{where} \quad \mathrm{tr}(\hat{A}) = \int d\xi\, d\eta\, e^{-\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\, \hat{A}\, e^{\frac{i}{\hbar}(\xi\hat{q}+\eta\hat{p})}\,. \tag{15}$$

We are still missing our correspondence between $f(p,q)$ and $\hat{F}$. To do this, we will need to borrow some results a formalism long forgotten by theoretical physicists in the light of Bell's inequalities, namely that of **hidden variable theories**. At the time of Weyl and even Groenewold, hidden variables were still a very serious contender to substitute the Copenhagen interpretation. However in 1964 John Bell proved that *local* hidden variable theories cannot accurately describe QM [12]. Indeed, QM under the DQ framework can be interpreted as a hidden variable theory in the sense that quantum operators are associated to phase space observables by the operator $\hat{\mathcal{O}}$ of equation (1) and therefore are subject to the evolution of the deterministic hidden variables p and q. However, we will argue in the next section that Deformation Quantization yields a ***non***-local hidden variable theory, the non-locality coming from the global non-commutative structure provided from the star product.

Back to our exploration of the relationship between Hilbert space and phase space, we will need the so called known as Weyl correspondence principle of a hidden variable theory. If $a(q,p) \leftrightarrow \hat{A}$ and $b(q,p) \leftrightarrow \hat{B}$ then $\exists\, \rho(p,q)$ such that

$$\mathrm{tr}(\hat{A}\hat{B}) = \int dp\, dq\, \rho(p,q)\, a(p,q)\, b(p,q)\,.$$

This can be accomplished by choosing a kernel for the transformation:

$$a(p,q) = \mathrm{tr}(\hat{k}(p,q)\hat{A}) \quad \Longleftrightarrow \quad \hat{A} = \int dp\, dq\, \rho(q,p)\, \hat{k}(q,p)\, a(q,p) \tag{16}$$

satisfying certain properties (see sections 3 and 4 of [3]). Comparing (16) with (14) and (15) we can solve for both ρ and $\hat{k}$. The result is a direct correspondence between phase space functions and Hilbert space operators:

$$\hat{\mathcal{O}}(f) \equiv \hat{F} = \frac{1}{2\pi\hbar} \int dx\, dy\, \mathscr{F}[f](x,y)\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})} \tag{17}$$

$$f(q,p) = \frac{1}{2\pi\hbar} \int dx\, dy\, \mathrm{tr}\left(\hat{F}\, e^{-\frac{i}{\hbar}(x\hat{p}-y\hat{q})}\right) \tag{18}$$

which was already presented in section 1. Now we are in position to calculate the expression for the product of operators $\hat{\mathcal{O}}(f)\,\hat{\mathcal{O}}(g) = \hat{F}\hat{G}$:

$$\hat{F}\hat{G} = \frac{1}{(2\pi\hbar)^4} \int dx\, dy\, dx'\, dy'\, dp\, dq\, dp'\, dq' \left(e^{\frac{i}{\hbar}((x+x')\hat{p}+(y+y')\hat{q})}\, e^{\frac{i}{2\hbar}(xy'-yx')} \right.$$
$$\left. \cdot\, e^{-\frac{i}{\hbar}(xp+yq+x'p'+y'q')}\, f(p,q)\, g(p',q') \right)$$

where we used (11) and the explicit Fourier expansions of f and g. Using a suitable change of variables and evaluating the available delta functions:

$$\hat{F}\hat{G} = \frac{1}{(2\pi\hbar)^2} \int dx\, dy\, dp\, dq\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})}\, e^{-\frac{i}{\hbar}(xp+yq)}\, f\left(p+\frac{1}{4}y, q-\frac{1}{4}x\right)\, g\left(p-\frac{1}{4}y, q+\frac{1}{4}x\right) .$$

Now by Taylor expanding we see that:

$$f\left(p+\frac{1}{4}y, q-\frac{1}{4}x\right)\, g\left(p-\frac{1}{4}y, q+\frac{1}{4}x\right) = \left(e^{\frac{1}{4}(y\partial_p - x\partial_q)}\, f(p,q)\right)\left(e^{-\frac{1}{4}(y\partial_p - x\partial_q)}\, g(p,q)\right) .$$

Finally, one can integrate by parts and find

$$\hat{F}\hat{G} = \frac{1}{(2\pi\hbar)^2} \int dx\, dy\, e^{\frac{i}{\hbar}(x\hat{p}+y\hat{q})} \int dp\, dq\, e^{-\frac{i}{\hbar}(xp+yq)}\, f(q,p) \left(e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)}\right) g(q,p)$$

which reproduces the compatibility property of the Moyal product (10) with respect to the Weyl quantization map (18):

$$\hat{\mathcal{O}}(f)\hat{\mathcal{O}}(g) = \hat{\mathcal{O}}(f \star g) .$$

This in turn determines a well-defined Lie algebra homomorphism:

$$\hat{\mathcal{O}}(f \star g - g \star f) = \frac{1}{i\hbar}[\hat{\mathcal{O}}(f), \hat{\mathcal{O}}(g)]$$

which, as was mentioned in section 1 allows for a consistent Quantization procedure.

3.4 Quantum Mechanics in Phase Space

It was Wigner [5], together with Moyal and Groenewold, the ones who extended the analogy of wave functions to classical phase space. In this section we will explore how one would approach this subject, what changes, and what remains from the Copenhagen interpretation in Hilbert space. We will throughout this section introduce the Dirac notation in our formalism[3]. Our formula (17) for finding phase space variables in terms of quantum operators is then

$$f(q,p) = \frac{1}{\pi\hbar}\int d\xi \, \langle q+\xi| \, \hat{F} \, |q-\xi\rangle \, e^{-2i\xi\, p/\hbar} \,. \tag{19}$$

Let $|\Psi\rangle$ define a pure state in Hilbert space. We define the **Wigner Function** W as the operator associated to the density matrix $\hat{\rho} = |\Psi\rangle \langle\Psi|$:

$$W(x,p) := \frac{1}{\pi\hbar}\int d\xi \, \langle q+\xi| \, \hat{\rho} \, |q-\xi\rangle \, e^{-2i\xi\, p/\hbar} = \frac{1}{\pi\hbar}\int d\xi \, \psi(x+\xi)\psi^*(x-\xi) \, e^{-2i\xi\, p/\hbar}$$

where we used the expression of the wave function in terms of brackets $\psi(x) = \langle x| \, \Psi\rangle$. Wigner actually used this expression in the midst of finding quantum corrections for statistical mechanical processes. He sought an expression that captured QM in phase space. Supposing that our quantum states are normalized, the Cauchy-Schwartz inequality gives

$$|W(x,p)| \leq \frac{1}{\pi\hbar}\int d\xi \, |\psi(\xi)|^2 \quad \Longleftrightarrow \quad -\frac{1}{\pi\hbar} \leq W(x,p) \leq \frac{1}{\pi\hbar} \,.$$

We can see that Wigner functions can only give localized distributions (i.e. Dirac deltas) in the classical $\hbar \to 0$ limit. Table 1 shows how one can recover basic properties of QM in terms of Wigner functions, namely the normalization condition, orthonormality, eigenvalue equations, time evolution, etc.

It can be shown that in phase space one can define uncertainty relations for **real star-square** observables $G(x,p) = g(x,p)^* \star g(x,p)$ [5]. Also, the harmonic oscillator is thoroughly worked in this framework in [13]. As expected, a discrete set of eigenfunctions is obtained, but this time in terms of the Laguerre polynomials instead of Hermite.

To close this first part of our Quantization procedure, let us say a few words about the hidden variables that we used in the previous section. It was John Bell [12] who in 1964 proved that Quantum Mechanics cannot be described by a local hidden variable theory. By non-local, we mean that there exists a finite causal chain between events, and that the causes and effects propagate at most at the speed of light. Our construction throughout the text, on the other hand, is nothing more than a reformulation of QM, giving an origin to the non-commutativity of physical observables. This non-commutativity would be as fundamental as, for instance, the Riemann curvature tensor is in General Relativity. Both objects capture the non-local effects of their respective theoretical frameworks. Indeed, the non-commutativity of physical observables being endowed on our manifold globally, makes itself manifest even at cosmological distances, in exactly the same way as ordinary QM predicts.

[3] There are reasons to believe that over-relying on the Dirac notation can be mathematically obscure as exposed in this lecture by Frederic P. Schuller. Nevertheless we shall make our notation as simple as possible.

Table 1: Equivalence between Deformation Quantization and other formulations of QM.

Phase Space	**Density Matrix**	**Dirac Kets**
$\frac{1}{2\pi\hbar}\int W(q,p)\,dqdp = 1$	$\mathrm{Tr}(\|\psi\rangle\langle\psi\|) = 1$	$\langle\psi\|\,\psi\rangle = 1$
$W_E \star W_{E'} = \delta_{E,E'} W_E$	$\|\phi_i\rangle\langle\phi_i\|\,\phi_j\rangle\langle\phi_j\| = \delta_{ij}\|\phi_i\rangle\langle\phi_i\|$	$\langle\phi_i\|\,\phi_j\rangle = \delta_{ij}$
$H \star W_E = E\,W_E$	–	$H\|\phi_E\rangle = E\|\phi_E\rangle$
$H = \sum_E E\,W_E$	$H = \sum_j E_j \|\phi_j\rangle\langle\phi_j\|$	–
$i\hbar\frac{d}{dt}\mathrm{Exp}(Ht) = H \star \mathrm{Exp}(Ht)$	$i\hbar\frac{dU}{dt} = HU$	$i\hbar\frac{d\|\psi\rangle}{dt} = H\|\psi\rangle$

4 Spinors

We have seen how the Weyl map, toghether with Kontsevich's construction give a well defined mathematical framework for the quantization of a classical system. However, there is still one big QM character that we have not put in the spotlight: spin. In this section we shall show that the non-commutativity of, say, the Pauli matrices do not correspond to a Hilbert space operator structure, nor to a star product, but to a Clifford Algebra structure. At the level of Quantum Mechanics, spinors are geometrical objects that are defined on Riemannian manifolds.

4.1 Review of Spinorial Matter

Talking about spin is talking about the representations (see A.4) of the group $SO(3)$ on finite dimensional complex vector spaces. Let $R_{\phi,\mathbf{n}} \in SO(3)$ be a rotation of an angle ϕ around the $\mathbf{n}$ axis. Then $R^T_{\phi,\mathbf{n}} = R_{\phi,\mathbf{n}}$. What are the requirements that we wish to impose on the representation $U(R_{\phi,\mathbf{n}})$ that we seek? Firstly, it should be a linear and unitary representation so that the norm of $U(R_{\phi,\mathbf{n}})\,|\psi\rangle$ is equal to that of $|\psi\rangle$, in complete analogy with the natural $\mathbb{R}^3$ representation. Therefore, we require that $U^\dagger(R_{\phi,\mathbf{n}}) = U^{-1}(R_{\phi,\mathbf{n}})$. The remaining requirements we can obtain by investigating infinitesimal rotations:

$$(R_{\varepsilon,\mathbf{n}})_{ij} = \delta_{ij} + \omega_{ij} \quad \text{where } \omega_{ij} = -\omega_{ji} \text{ is of } \mathcal{O}(\varepsilon)\,.$$

Then this can be represented in $\mathbb{C}^2$ by

$$(U(R_{\varepsilon,\mathbf{n}}))_{a\,a'} = \delta_{a\,a'} + \frac{i}{2}\omega_{ij}(J^{ij})_{a\,a'} \quad \text{where } (J^{ij})^\dagger = J^{ij}.$$

We see that there are three independent coefficients ω_{ij} and therefore there must be three generators of rotations in both $\mathbb{R}^3$ and $\mathbb{C}^n$. Note that each J^{ij} is itself a matrix. Indeed, let $\mathbf{J} = (J^{23}, J^{31}, J^{12}) = (J^1, J^2, J^3)$, then it is a known result from elementary QM [11] that $[J^i, J^j] = i\hbar\epsilon_{ijk}J^k$. This in turn yields the so called "quantization of angular momentum", by virtue of which there exist a non-negative half integer j such that the operators J act on a $2j+1$ dimensional complex vector space. Each value of j, corresponding to the eigenvalue of $\mathbf{J}^2$, furnishes an irreducible and unitary representation of the rotation group in $2j+1$ dimensions. Each representation will then be constructed from the eigenstates of, say, J^3, which span from $-j$ to $+j$ as their eigenvalue in integer steps.

We see that the origin of the quantization of spin comes from the fact that we are trying to represent the rotation group in a finite-dimensional complex vector space. We can go back even further and recognize that $SO(3)$ can be understood as the metric-preserving transformations in euclidean space. Therefore, if we want to thoroughly describe spinors at the level of the original manifold M we will have to incorporate some kind of structure that uses the metric tensor g as a fundamental object in their understanding. Furthermore, when we perform a change of reference frame in our manifold, the value of the spin (or the value of its 3-direction component) will change. This must be accounted for in our picture if it is to yield a physical model. This is accomplished with the Clifford algebra structure, which we will implement at the manifold level using a spinor bundle.

4.2 Clifford Algebras

We start our journey defining the algebra that will account for the non-commutativity of observables. For a deep dive on Clifford algebras from the physicist viewpoint, see [14, 15]. Let V be a vector space and let $g : V \times V \to \mathbb{R}$ be a simetric bilinear form. Consider the tensor algebra

$$T(V) = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n} .$$

Then the **Clifford algebra** $\mathrm{Cl}(V, g)$ is defined as the quotient algebra

$$\mathrm{Cl}(V, g) = \frac{T(V)}{\langle u \otimes v + v \otimes u - 2g(u, v)\rangle} . \tag{20}$$

In particular, given an orthonormal basis $\{e_\alpha\}$ with respect to the metric g, we can easily check that $e_i \otimes e_i = g_{ii}$ and $e_i \otimes e_j = -e_j \otimes e_i$ at the level of the Clifford algebra. With this in mind we can define the so called **plurivectors** on V as elements of the form:

$$\alpha\, 1 + \sum_i a_i e_i + \sum_{i<j} \alpha_{ij} e_i\, e_j + \cdots + \sum_{i_1<\cdots<i_s} \alpha_{i_1\cdots i_s} e_{i_1} \cdots e_{i_s} + \cdots + \alpha_{12\cdots n} e_1 \cdots e_n$$

where we understand the products $e_1 \cdots e_k$ as tensor products modulo the identification in (20). Elements of the form $\alpha_i e_i$ are called vectors, elements of the form $\alpha_{ij} e_i e_j$ are called 2-vectors, and so on. Note that the Clifford product respects a $\mathbb{Z}_2$ grading: even products and odd products give even plurivectors, and even with odd (and vice-versa) give odd elements. For instance, a plurivector in 3 dimensions has the form:

$$a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{12} e_1 e_2 + a_{13} e_1 e_3 + a_{23} e_2 e_3 + a_{123} e_1 e_2 e_3$$

where we identify i with the pseudoscalar component $e_1 e_2 e_3$, since $(e_1 e_2 e_3)^2 = -1$. Now it is easy to check that $e_i e_j = \delta_{ij} + i\epsilon_{ijk} e_k$. This is a property satisfied by the Pauli matrices σ_i. In other words, the Pauli matrices can be understood to represent a basis of the Clifford Algebra of $\mathbb{R}^3$ equipped with the identity metric. Also:

$$(e_1 e_2)^2 = (e_2 e_3)^2 = (e_1 e_3)^2 = (e_1 e_2)(e_2 e_3)(e_1 e_3) = -1 . \tag{21}$$

This is precisely the equation that describes quaternion multiplication. So we can identify a scalar plus a 2-vector as a quaternion. Let's now look at three-dimensional rotations of vectors. For example, let $v = a_1 e_1 + a_2 e_2 + a_3 e_3$. When rotated by 90 degrees around the z axis the new components are $v' = -a_2 e_1 + a_1 e_2 + a_3 e_3$. By a straightforward computation we see that

$$\begin{aligned} v\, e_1 e_2 &= -a_2 e_1 + a_1 e_2 + e_1 e_2 e_3 , \\ e_2 e_1\, v &= -a_2 e_1 + a_1 e_2 - e_1 e_2 e_3 . \end{aligned}$$

If we apply both operations sequentially, we get $e_2 e_1 v e_1 e_2 = -a_1 e_1 - a_2 e_2 + a_3 e_3$. That is, we obtain a rotation by 180 degrees! To fix this, remember that $e_1 e_2$ corresponds to a 90 degree rotation in the $e_1 e_2$ plane, so it can be written as $\exp(e_1 e_2\, \pi/2)$. It can be shown that when using $\exp(e_1 e_2\, \pi/4)$, that is, a half rotation instead of the original one, we obtain the correct result. In general, to rotate a vector v an angle θ in the plane $\hat{I}$:

$$v' = e^{-\hat{I}\frac{\theta}{2}}\, v\, e^{\hat{I}\frac{\theta}{2}} .$$

This equation actually works in any dimension. Note that the rotor $e^{\hat{I}\frac{\theta}{2}}$ rotates half as fast as the vector, reminiscing of spinor rotations in Quantum Mechanics. Let us now explain what just happened in Group Theory language. In our formalism of the Clifford Algebras we wished to rotate vectors, which is accomplished via the action of the Lie Group $SO(3)$. We accomplished this by concatenating $SU(2)$ rotations and therefore constructing a group homomorphism between the two. As has been pointed out, the map is not injective because the same rotation (0 and 360 degrees) in $SO(3)$ yields different rotations in $SU(2)$. We say that $SU(2)$ is the **double cover** of $SO(3)$.

To be more mathematically precise, we define $\mathrm{Pin}(V)$ to be the group generated by all the unit vectors in V. Every element of $\mathrm{Pin}(V)$ is of the form $u_1 \cdots u_r$ with $g(u_i, u_i) = \pm 1$. There is a well defined action $\mathrm{Pin}(V) \times V \longrightarrow V$ given by

$$(p, x) \mapsto -pxp^{-1} = \frac{1}{g(p,p)}\, pxp = x - 2\,\frac{g(x,p)}{g(p,p)}\, p\,. \tag{22}$$

This is nothing more than a reflection on the hyperplane perpendicular to p. Since every element of the orthogonal group is a product of a finite number of reflections, this yields a group homomorphism $\mathrm{Pin}(V) \to O(V)$. Then the **spin group** of V is the subgroup $\mathrm{Spin}(V) \subset \mathrm{Pin}(V)$, which is defined as the pre-image of $SO(V)$ under the above group homomorphism. It follows that every element of $\mathrm{Spin}(V)$ can be written as an even number of unit vectors in V, because only an even number of reflections in equation (22) will preserve the determinant of the transformation. Therefore, every element of $\mathrm{Spin}(V)$ is of the form $u_1 \cdots u_{2p}$ with $g(u_i, u_i) = \pm 1$. Moreover, it can be proven [16, 17] that the spin group is a double cover of the special orthogonal group, because there is a two to one correspondence between elements of $\mathrm{Spin}(V)$ and elements of $SO(V)$.

In conclusion, we have found that given a Clifford algebra structure we can construct a corresponding spin group, which is inside the Clifford algebra. In the case of $\mathbb{R}^3$, we heuristically obtained that the spin group must be $SU(2)$. Now that the spin group is defined, we have to incorporate it to a 3-dimensional manifold M with its metric tensor such that it acts on $\mathbb{C}^2$ the most natural way and behaves "covariantly" under changes of reference frame.

4.3 Principal and Associated Bundles

As we have mentioned, spinors are geometrical objects that arise from the geometry of the manifold at hand. That is why we need to first be more aware of how to incorporate symmetries in the context of fiber bundles.

Both the tangent and the cotangent bundle are examples of fiber bundles (see A.2). A **fiber bundle** with **typical fiber** F, is a triple (E, π, M) where E is called the **total space** and E the **base space**. Here, E, F and M are smooth manifolds. The **projection** $\pi : E \longrightarrow M$ is a smooth map between manifolds. The projection π must obey the following property: there exist an open covering $\{U_i\}$ of M and diffeomorphisms $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times F$ and a map $\pi_1 : U_i \times F \longrightarrow U_i$ such that $\pi_1(m, f) = m \quad \forall m \in M, f \in F$. An intuitive drawing of this situation is depicted in Figure 1. Without loss of generality, one can identify F as a fiber over each point. Such ϕ_i maps are called **trivializations**. Now, if two open sets $U_i\,, U_j$ cover a same point $m \in M$, is there a concrete way in which we switch from a fiber point $f_i(m)$ associated to ϕ_i to a point

$f_j(m)$ associated to ϕ_j in the fiber? The answer is that there exist **transition maps** $\rho_{ij} : U_i \cap U_j \longrightarrow \text{Diff}(F)$ that assign to each point a certain transformation:

$$\phi_j \circ \phi_i^{-1}(m, f) = (m, (\rho_{ij}(m))f) \,. \tag{23}$$

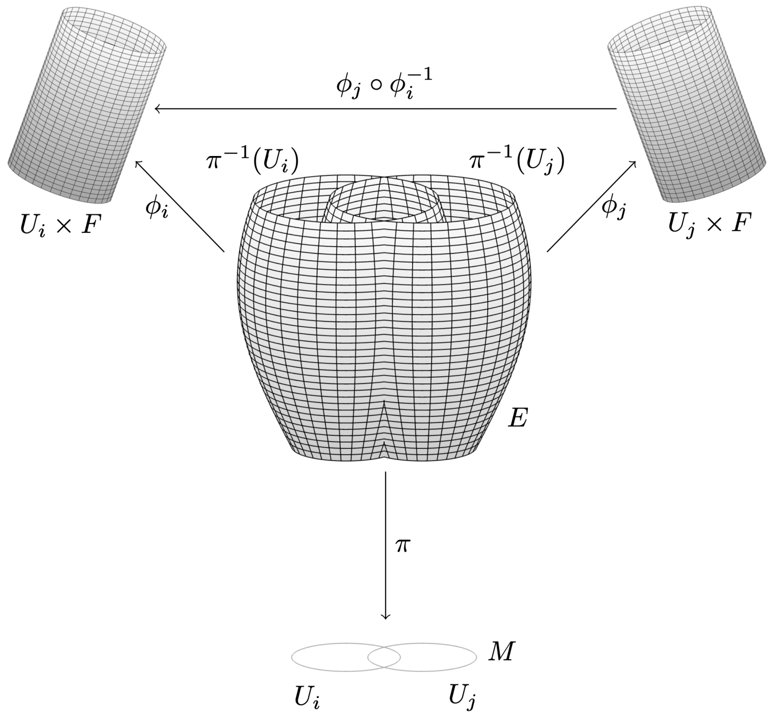


Figure 1: Intuitive image of fiber bundles.

A section of the fiber bundle $E \xrightarrow{\pi} M$ is a smooth map $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma = \text{id}_M$. We have already encountered sections since vector fields are sections of the tangent bundle. Indeed the tangent bundle is an example of a **vector bundle**, where the fiber F is just $\mathbb{R}^{\dim M}$ as a vector space.

A particular case of interest in theoretical physics arises when the typical fiber has the structure of a group. A **principal G-bundle** is a fiber bundle (P, π, M) whose fiber is the so-called **structure group** G. The trivializations $\phi : \pi^{-1}(U) \longrightarrow U \times G$ depicted in Figure 1 yield transition functions given by the left multiplication of A.4. More precisely, let $\rho_{ij} : U_i \cap U_j \longrightarrow G$, then ρ_{ij} acts on G by left multiplication and in the form of equation (23). Also, from the trivializations we can construct a **right action** $\triangleleft$ (A.4) such that:

$$p \triangleleft g = \phi_i^{-1}(\phi_i(p) \cdot g) \quad \text{where} \quad \phi_i(p) \cdot g = (p, h) \cdot g = (p, h \cdot g) \tag{24}$$

for some $h \in G$. The right action is well defined since it is independent of the trivialization that we choose. We see from equation (24) that this right action moves a point always in the same fiber. That is to say, the fiber over a point is its orbit under the right action. Since we defined the fiber to be G, then the orbit is homeomorphic to the group. This implies that the action is free (see A.4 for definitions of orbit and free action). The key idea of a principal bundle is that it encodes a locally trivial fibration of the total space

P over the base space M, where the fibers are homeomorphic to the group G. In other words, the structure of the total space is like a bunch of copies of the group G. The action of G on these fibers will eventually represent the symmetries of the bundle.

Our principal bundle of interest from which we will construct the spinor bundle is the **orthonormal frame bundle**, which we define at each point $p \in M$ to be homeomorphic to the group of proper rotations.We write the fiber over each point as L_pM. An intuitive picture is the following: imagine that we attach at each point p in M a reference frame, which is just a basis of T_pM. It is convenient to import such reference frame from $\mathbb{R}^n$:

$$L_pM := \{\varphi : \mathbb{R}^{\dim(M)} \to T_pM \,|\, \varphi \text{ orientation preserving diffeomorphism}^4\} . \tag{25}$$

Intuitively, we can think of φ as the matrix whose columns are the basis vectors of $\mathbb{R}^{\dim(M)}$. That way, one can also import a right $SO(\dim(M), \mathbb{R})-$action given by $\varphi \triangleleft R = \varphi \circ R$ for $R \in SO(\dim(M), \mathbb{R})$. We typically write $SO(3) \circlearrowleft L_pM$. The reference frames can be rotated but we will still be talking about the same point in our manifold. All the possible rotations that one can choose live on the fiber above the point p. The **frame bundle** is defined as:

$$LM = \bigsqcup_{p\in M} L_pM . \tag{26}$$

Just as with TM, one can equip LM with a smooth atlas inherited from M. We can also establish a projection map: $\pi : LM \longrightarrow M$ such that a basis $\{e_\alpha\}$ of T_xM is mapped to $x \in M$.

We would now like to *associate* a vector bundle to a principal bundle in a very specific fashion, such that the changes in frames on the principal bundle affect the "vectors" on the vector bundle. This is exactly what we want to do with a $\mathbb{C}^2$ vector bundle.

To do this, we will use the so called associated bundles (with respect to a principal bundle). Given a G-principal bundle $G \circlearrowleft P \xrightarrow{\pi} M$ and a smooth manifold F on which we have a left G-action $\triangleright : G \times F \to F$, we define the **associated bundle** $P(F) \xrightarrow{\pi_F} M$ as follows:

1. Let $\sim_G$ be the relation on $P \times F$: $(p, f) \sim (p \triangleleft g, g^{-1} \triangleright f)$ for some $g \in G$. Thus consider the quotient space $P(F) := (P \times F)/_{\sim_G}$.
2. Define $\pi_F : P(F) \longrightarrow M$ by $[(p, f)] \mapsto \pi(p)$.

The sections $\sigma : M \longrightarrow P(F)$ of an associated bundle are in one to one correspondence to F-valued functions $\phi_\sigma : P \longrightarrow F$ on the underlying principal bundle. This is of capital importance to us because we will define spinors as sections of a $F = \mathbb{C}^2$ associated bundle, which we are now sure that they will be $\mathbb{C}^2$- valued. In an abuse of notation, we represent associated bundles with the diagram

$$\begin{array}{ccc} F & \longrightarrow & LM\,(F) \\ & & \downarrow \\ & & M \end{array} \tag{27}$$

4.4 Spinor Bundles

Constructing spinor bundles from frame bundles and Clifford algebras is immediate. Let M be an 3−dimensional Riemannian manifold endowed with a positive definite metric, and consider the orthonormal frame bundle $LM \xrightarrow{\pi} M$. This is a principal bundle with structure group $SO(3)$, the rotation group, which has a double covering spin group $\mathrm{Spin}(3) = SU(2)$ contained in the Clifford algebra $\mathrm{Cl}(\mathbb{R}^3)$, as we obtained in subsection 4.2. A **spin structure** on M consists of a double covering

$$\begin{array}{ccc} SU(2) \circlearrowleft LM_{\mathrm{Spin}} & \xrightarrow{(2-1)} & SO(3) \circlearrowleft LM \\ & & \big\downarrow \pi \\ & & M \end{array} \tag{28}$$

such that $LM_{\mathrm{Spin}} \longrightarrow M$ is a principal bundle over M. Now let $F = \mathbb{C}^2$ be a vector space on which we represent $\mathrm{Cl}(\mathbb{R}^3)$ via the Pauli matrices. This restricts to a representation of the Spin Group $\rho : \mathrm{Spin}(3) = SU(2) \longrightarrow \mathrm{Aut}(\mathbb{C}^2)$ (automorphisms, i.e. invertible endomorphisms) such that this yields an associated vector bundle $LM(\mathbb{C}^2)$, in the sense of equation (27) with typical fiber $\mathbb{C}^2$, called the **spinor bundle**. An intuitive drawing of a spin structure over a manifold is depicted in Figure 2. The fibers of this bundle is what physicists call the internal space. Spinors are sections of the associated bundle, i.e. complex 2-tuples that transform under $SU(2)$ transformations, as we learn in elementary QM textbooks. Note that in our formalism the non commutativity of the Pauli matrices is of a fundamentally different nature from that of position and momentum.

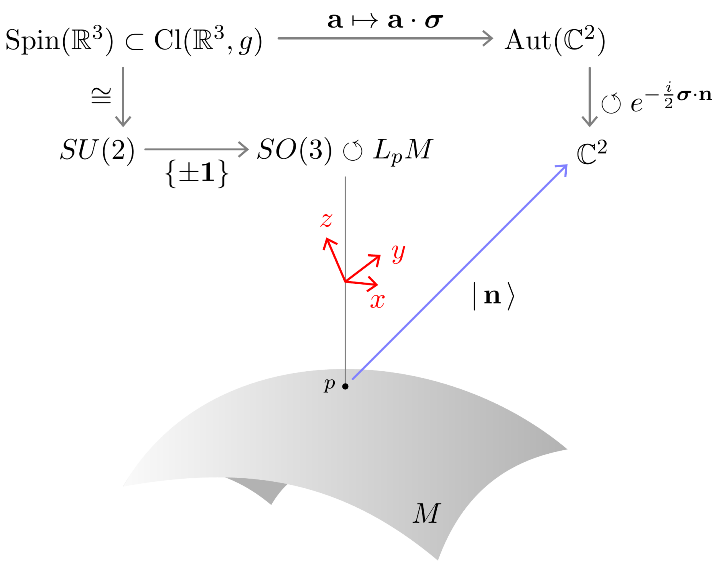


Figure 2: Intuitive image of a spin structure.

That spinorial behaviour is intrinsically related to the symmetries of the underlying spacetime is a well known fact in modern fundamental physics. It was Wigner [18] who developed a procedure to characterize elementary particles by how the Lorentz group acts on some complex Hilbert space. Indeed, Quantum Field theory uses this exact scheme to account for spinorial degrees of freedom but with the Lorentz group as the fiber of the principal bundle. Yang-Mills theories are also based on this formalism. Indeed, to each point we can associate a $U(1) \times SU(2) \times SU(3)$ principal bundle [16].[5]

[5]For more information on the subject, see the notes available here.

5 The Quantization Scheme

Over the course of the text, we have come up with a consistent quantization procedure that captures the non commutativity of QM so that this feature can be traced back to two different mathematical structures:

1. The non-commutativity of position, momentum, etc. can be explained by a non-commutative star product structure (10) that, toghether with the Weyl quantization maps (17) and (18) provide a Lie algebra homomorphism between the star product algebra and the algebra of linear self-adjoint operators in Hilbert space.
2. The non-commutativity of spinorial operators can be explained by equipping our manifold with a spinor bundle (27) inherited from a Clifford algebra structure and from the metric preserving transformations of the manifold as in (28).

The situation is best represented by the following diagram:

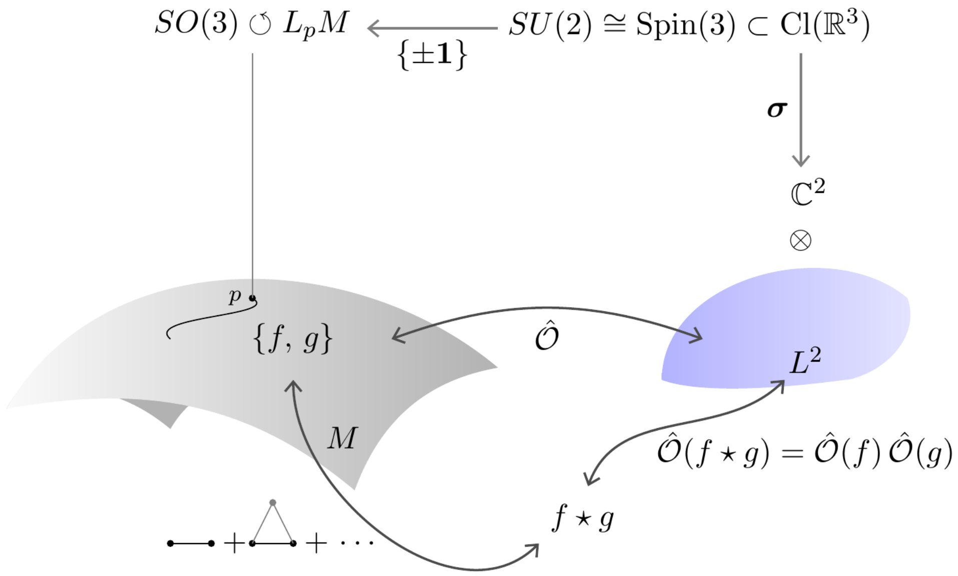


Figure 3: Intuitive image of the quantization procedure.

We see that our manifold is equipped with a Poisson bracket $\{\cdot, \cdot\}$ and an $SO(3)$ symmetry in the form of a principal bundle with structure group inherited from a Riemannian metric. The bivector is equivalent to a Poisson structure which by Kontsevich's construction will define a star product $\star$ on the cotangent bundle. It is this star product which gives rise to a Lie algebra homomorphism to a Hilbert space $\mathcal{H}$. On the other hand, the $SO(3)$ symmetry, toghether with a Clifford algebra structure gives rise to a spin structure mediated by the group $SU(2)$ and represented on $\mathbb{C}^2$.

We can also extract some epistemological conjectures out of our work. The first one is that a manifold with a symplectic/Poisson structure cannot by itself contain the information necessary to describe quantum mechanical phenomena. In other words:

> The information of a Quantum Theory cannot be extracted from Classical Mechanics unless some mathematical structure is added.

In order to make a consistent quantization procedure, we had to develop a non-commutative star product, which carries the information of QM to phase space. This could be worse because Kontsevich tells us that this extra structure is very closely related to the one we already have.

Consistent quantizations require that the information of the theory is conserved. In other words, a quantization map in which information is generated or lost will result in logical contradictions. A consistent quantization procedure requires the classical theory to have an equal amount of information on each space. This would be reasonable to expect of a consistent mathematical formalism because the defect or excess of information in a map will mean that it is not isomorphic.

A potential explanation given by Edward Witten in [19] would hinge of the fact that the symmetry groups of Classical Mechanics and Quantum Mechanics are different. In classical mechanics with a phase space T^*M, the group of symmetries is the group of canonical transformations of M. In quantum mechanics, the corresponding group is the group of unitary transformations of a Hilbert space $\mathcal{H}$. Since a classical system and its quantization have different symmetry groups, there cannot be an entirely natural passage from classical to Quantum Mechanics.

With regards with spinorial degrees of freedom, we have borrowed modern tools in mathematical physics to understand these quantum internal degrees of freedom. We have found that:

> The non-commutativity of spinorial operators traces back to a Clifford algebra structure, while the non-commutativity of phase space operators trace back to a star product. They are fundamentally different.

6 Conclusion

In this work we provided a complete quantization procedure that allows for a reformulation of classical mechanics without leaving the physical manifold of study, using novel mathematical structures like star products, Clifford algebras, and fiber bundles, each of which occupies its place in a consistent fashion. We have seen that consistent quantizations necessarily require that we add extra structure on the manifold, and that the structure for phase space observables like position and momentum is fundamentally different from that of spinorial observables like the Pauli matrices.

A promising topic for future research is whether there exists a consistent quantization procedure using Deformation Quantization in Field Theory [20], which would spark applications in the quantization of curved-space time geometries [21] and eventually of the gravitational field [22].

As a result of this work, we are left with a better understanding Quantum Theory and its relationship with Classical Mechanics, enriching our comprehension of their intricacies and paving the way for further investigation in the applications of our results to other areas in Theoretical Physics.

A Basic Concepts of Differential Geometry

We present basic concepts of Differential Geometry used throughout the text.

A.1 Manifolds and Tangent Space

Intuitively, a manifold M is a set that locally *looks like* $\mathbb{R}^d$ for some d, which is defined as the dimension of the manifold. This correspondence between the M and $\mathbb{R}^d$ is accomplished by choosing a coordinate system. More precisely, $\forall\, p \in U \subseteq M$ there exists a homeomorphism $x : U \to \mathbb{R}^d$. By doing this, the concepts like differentiability and smoothness, which are well known for euclidean spaces, can be *lifted up* to a general manifold. This homeomorphism x is called the **coordinate chart map**. The pair (U, x) is called a **chart**.

Let M be a smooth manifold. Let $\gamma : \mathbb{R} \longrightarrow M$ be a smooth curve through a point $p \in M$. Without loss of generality suppose $p = \gamma(0)$. Then the **tangent vector** at the point p along the curve γ is the linear map $X_{\gamma,p} : C^\infty(M) \to \mathbb{R}$ defined by $X_{\gamma,p} f := (f \circ \gamma)'(0)$. Intuitively, it represents the velocity of the curve at a point. The **tangent vector space** at a point $p \in M$ is defined as:

$$T_pM := \{X_{\gamma,p} \,|\, \gamma \text{ smooth through } p\} \tag{29}$$

together with a point wise addition and s-multiplication. A fundamental theorem in Differential Geometry states that the dimension of T_pM, treated as a vector space, is the same as that of M, treated as a manifold. A proof of this theorem can be found in [23, 24]. Since this is a vector space, it can be shown that every vector X_p in T_pM can be written in the form

$$X_p = X^a \left(\frac{\partial}{\partial x^a}\right)_p \qquad a = 1, \ldots, n \tag{30}$$

where the basis vectors depend on the choice of coordinates x that represent the manifold on a chart, hence the name **chart induced vector basis**.

A.2 Tangent and Cotangent Bundles

Let M be a smooth manifold. Then the **cotangent space** is defined as $T_p^*M := (T_pM)^*$. That is, the elements of the cotangent space are linear maps from T_pM to $\mathbb{R}$. Similarly, one can construct the set of all (r, s) tensors at p by creating copies of the tangent and cotangent bundles. The **cotangent bundle** is the set

$$T^*M := \bigsqcup_{p \in M} T_p^*M = \bigcup_{p \in M} \{p\} \times T_p^*M \tag{31}$$

equipped with the projection $\pi : T^*M \longrightarrow M$ such that $\omega_p \mapsto p$. We can define the tangent bundle analogously. It can be shown that the (co)tangent bundle has indeed the structure of a smooth manifold, so one can assign charts and coordinates to it. In particular $\dim(T^*M) = \dim(TM) = 2\dim(M)$. A **vector field** is a map $\sigma : M \to TM$. The set of all vector fields on TM is called

$$\Gamma(TM) := \{\sigma : M \longrightarrow TM \,|\, \pi \circ \sigma = \mathrm{id}_M\} \tag{32}$$

on the other hand, a **one form** is an element of $\Gamma(T^*M)$. Let $\phi : M \longrightarrow M$ be a smooth map between smooth manifolds. Then the **push-forward** ϕ_* of the map ϕ is $\phi_* : TM \to TN$ defined by $\phi_*(X)f := X(f \circ \phi)$. Intuitively, if X is tangent to a curve γ, then $\phi_*(X)$ is tangent to $\phi \circ \gamma$. Similarly, the **pull-back** ϕ^* of the map ϕ is the linear map $\phi^* : T^*N \longrightarrow T^*M$ defined by $\phi^*(\omega)X := \omega(\phi_*(X))$for some $X \in \Gamma(TM)$.

One can generalize this concept to construct the **tensor bundle** $\mathcal{T}_s{}^r(M)$ with the corresponding vector fields being the so called (r, s) tensor fields. Once this is constructed, we clearly can identify $\mathcal{T}_0{}^1(M)$ and $\mathcal{T}_1{}^0(M)$ as TM and T^*M, respectively.

A.3 Differential Forms and de Rham Cohomology

A (differential) k**-form** is a $(0, k)$ tensor field that is totally anti-symmetric. Here $0 \leq k \leq n$. We will write the set of all k-forms as $\Omega^k(M) \in \mathcal{T}_k^0$. Also $\Omega^k(M) = \Gamma(\bigwedge^k T^*M)$. Wehre the $\bigwedge^k$ sign precisely meand rank k antisymmetric. The **exterior** product $\wedge : \Omega^k(M) \times \Omega^m(M) \longrightarrow \Omega^{k+m}(M)$ is defined by:

$$(\omega \wedge \sigma)(X_1, \ldots, X_{k+m}) := \frac{1}{k!}\frac{1}{m!} \sum_{\pi \in \mathrm{Perm(k+m)}} \mathrm{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \ldots, X_{\pi(k+m)}) \tag{33}$$

where X_i are vector fields on on M. For example, for $k = m = 1$ we have $\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega$. If $\omega \in \Omega^k(M)$ and $\sigma \in \Omega^m(M)$ then $\omega \wedge \sigma = (-1)^{km}\sigma \wedge \omega$. The **exterior derivative operator** transforms k-forms into $(k+1)$-forms in the following manner:

$$(d\omega)(X_1, \ldots, X_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_k)) + \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j \ldots, X_k) \tag{34}$$

where the hat sign means the element is omitted. To give some meaning to this expression, consider a one-form ω. Then exterior differentiation amounts to: $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. An important property of the exterior derivative is its behaviour when differentiating a wedge product. If $\omega \in \Omega^k(M)$ and $\sigma \in \Omega^m(M)$ then $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^n \omega \wedge d\sigma$.

Let M be a smooth manifold with or without boundary, and let p be a nonnegative integer. Because $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is linear, its kernel and image are linear subspaces. We define:

$$\mathcal{Z}^p(M) = \ker\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\textbf{closed } p\textbf{-forms on } M\}$$
$$\mathcal{B}^p(M) = \mathrm{im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\textbf{exact } p\textbf{-forms on } M\}$$

The fact that every exact form is closed can be written as $\mathcal{Z}^p(M) \subseteq \mathcal{B}^p(M)$. The **de Rham cohomology group in degree** p **of** M is the quotient vector space:

$$H^p(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)} \tag{35}$$

The elements of $H^p(M)$ are equivalence classes of forms $[\omega]$, called **cohomology classes**. Two p-forms ω and ω' are said to be equivalent if they differ by an exact form. The more elements in $H^p(M)$, the more different forms that can't be related by exact ones, so there are many closed forms that are not exact. However, if all forms are exact, then they are all related to one another by an exact form, so the cohomology group has only one element, which we may call 0. An unexprected property of de Rham groups is that they are topological invariants. This is rather remarkable because the De Rham cohomology groups arise from the construction of the exterior derivative, which is an operator defined locally. We find, however, that its properties when acting on forms on a given manifold depend on the global features of the manifold itself.

A.4 Lie Groups

A **Lie Group**, $(G, \cdot)$ is a group with a group operation $\cdot$ satisfying associative, neutral element, and inverse properties, and where G has the structure of a smooth manifold. Furthermore, the multiplication and inverse maps are smooth. For every $g \in G$ we define the **left translation** with respect to g as a map $l_g : G \longrightarrow G$ such that $l_g(h) := g \cdot h$.

Let $(G, \cdot)$ be a Lie Group and let M be a smooth manifold. then a smooth map $\rhd$: $G \times M \longrightarrow M$ such that: (i) $e \rhd p = p \quad \forall p \in M$, where e is the neutral element in G; and (ii) $g_2 \rhd (g_1 \rhd p) = (g_1 \cdot g_2) \rhd p \quad \forall g_1, g_2 \in G$ is called a **left G-action** on the manifold M. One can define a **right $G-$action** $\lhd : M \times G \to M$ in an analogous manner. Also, given a left action $\rhd$ one can induce a right action acting on $p \in M$ as $p \lhd g = g^{-1} \rhd p$.

- The **orbit** of $p \in M$ is defined as $\mathcal{O}_p := \{q \in M \,|\, \exists\, p \in M,\, g \in G \,:\, q = g \rhd p\}$
- For $p, q \in M$, let $p \sim q$ if $\exists\, g \in G \,:\, q = g \rhd p$. So points on the same orbit are identified. Then we can define the **orbit space** as $M/_\sim$. Tipically one writes M/G.
- The **stabilizer** at a point $p \in M$ is defined as $S_p := \{g \in G \,|g \rhd p = p\} \subseteq G$.
- An action is called **free** if $\forall\, p \in M \,:\, S_p = \{e\}$. For a free G-action, each orbit $\mathcal{O}_p$ is a smooth manifold and can be homeomorphically identified with G.

Consider the one parameter subgroup $\{\varphi_t\}$ of local transformations on M generated by a vector field X. That is, take a point p in M and consider X at that point. Then there is a unique integral curve to X at p. Then the function φ_t is a local transformation that acts on p by $\varphi_t(p) = q$ where q is another point of the integral curve with associated parameter value t. It is easy to check that these operations form a one parameter subgroup. With this notion of **flow along an integral curve of a vector field**, we can define the Lie Derivative of a differential form as:

$$L_X\omega = \lim_{t\to 0} \frac{\varphi_t^*\omega - \omega}{t} \tag{36}$$

Here we will only present two useful properties of the Lie derivative. One is Cartan's magic formula:

$$L_X\eta = d(i_X\,\eta) + i_X(d\eta) \quad \forall\eta \in \Omega^k(M) \tag{37}$$

where the operator $i_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is defined by

$$(i_X\omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1})\,.$$

B Rewriting of the Jacobi Identity for π

Throughout this section, let $\mathcal{X}^i(M)$ denote the set of multivector fields over M.

In the same way that a vector field $X \in \mathcal{X}(M)$ can be identified with a derivation $d_X : C^\infty(M) \to C^\infty(M)$ on the algebra of smooth functions, a bivector field can be identified with a multiderivation operation:

$$L_\pi = \{\cdot,\, \cdot\}_\pi : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M) \quad \text{such that} \quad \{f,\, g\}_\pi = \pi(df, dg)$$

which is $\mathbb{R}$-bilinear, skewsymmetric, and satisfies the Leibniz identity. Actually, one can identify multivectors of degree k with k-multiderivations in the same fashion. Thus, this $\{\cdot,\, \cdot\}_\pi$ satisfies all the Poisson bracket conditions except the Jacobi identity. In order to enforce this last condition we need to generalize the notion of Lie bracket of vector fields to bivector fields, in a similar way in which the exterior derivative is a generalization of the gradient for differential forms. This generalization is provided by the **Schouten bracket**, which is the unique bilinear operation $[\![\cdot\,,\,\cdot]\!] : \mathcal{X}^{k+1} \times \mathcal{X}^{l+1} \longrightarrow \mathcal{X}^{k+l+1}$ such that it reduces to the Lie bracket for $k = l = 0$ and to the Lie derivative for $k = 0$, $l = -1$, is graded skew-symmetric and satisfies the graded Leibniz identity and the graded Jacobi identity. For bivector fields the Schouten bracket takes the following form:

$$\begin{aligned}[\![X_0 \wedge X_1, Y_0 \wedge Y_1]\!] = [X_0, Y_0] \wedge X_1 \wedge Y_1 - [X_0, Y_1] \wedge X_1 \wedge Y_0 - \\ - [X_1, Y_0] \wedge X_0 \wedge Y_1 + [X_1, Y_1] \wedge X_0 \wedge Y_0 \end{aligned} \quad (38)$$

which is a trivector field. In the same way that we can calculate the Lie derivative of a Lie bracket in terms of the commutator of Lie derivatives (**??**), the multiderivation associated to the Schouten bracket $[\![\pi, \pi]\!]$ turns out to be:

$$\begin{aligned} L_{[\![\pi,\pi]\!]}(f, g, h) &= 2(L_\pi(f, L_\pi(g, h)) + L_\pi(h, L_\pi(f, g)) + L_\pi(g, L_\pi(h, f))) = \\ &= 2(\{f,\, \{g,\, h\}_\pi\}_\pi + \{h,\, \{f,\, g\}_\pi\}_\pi + \{g,\, \{h,\, f\}_\pi\}_\pi) \end{aligned}$$

for all $f, g, h \in C^\infty(M)$. So if we want our multiderivation $L_\pi = \{\cdot,\, \cdot\}_\pi$ to be a Poisson manifold in its own right for all $C^\infty(M)$ smooth functions we would want to require that $[\![\pi, \pi]\!] = 0$. Actually, on any manifold M, the relation $\pi(df, dg) = \{f,\, g\}$ induces a one-to-one correspondence between Poisson brackets on M and bivector fields $\pi \in \mathcal{X}^2(M)$ satisfying $[\![\pi, \pi]\!] = 0$. A bivector field $\pi \in \mathcal{X}^2(M)$ such that $[\![\pi, \pi]\!] = 0$ is also called a Poisson structure on M.

C Details of Kontsevich Construction

In this section we try to explain how a bijection between multidifferential operators and multivectors was constructed by Kontsevich.

Since from basic Differential Geometry a vector field X can be thought of as a differential operator we might guess that we can construct a map $U_n^{(0)}$ by just applying the multivectors:

$$X_0 \wedge \cdots \wedge X_n(f_0, \ldots f_n) \mapsto \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\, X_{\sigma(0)}(f_0) \cdots X_{\sigma(n)}(f_n) \tag{39}$$

To understand why this guess is wrong we need to first generalize the concept of Lie algebra so that it can apply to our objects of study. For example, multivector fields, which we may write as

$$\mathcal{V}(M) = \bigoplus_{i=-1} \Gamma\left(\bigwedge^{i+1} TM\right) = \bigoplus_{i=-1} \mathcal{X}^{i+1}(M)$$

equipped with the Schouten bracket presented back in Appendix B, present a simple example of a Graded Lie Algebra (GLA). If we wish to add a derivation $d : \mathcal{V}^i \to \mathcal{V}^{i+1}$ such that $d^2 = 0$, then we have a Differential Graded Lie Algebra (DGLA). In our case, we will choose the $d = 0$ differential. Our bivector field $\pi \in \mathcal{V}^1$ then satisfies a Maurer (MC) equation:

$$d\pi + \frac{1}{2}[\![\pi, \pi]\!] = 0$$

This equation is important because it is invariant under DGLA morphisms, and also under gauge transformations of the diffeomorphism group that we mentioned back in section 2. When creating a map between this space and that of the multidifferential operators, we should be able to preserve the structure of this equation, and translate it into a similar expression in the target.

We can proceed analogously and construct the GLA $\mathcal{D}$ of multidifferential operators, which come from a restriction of the GLA of multilinear operators:

$$\mathcal{C} = \sum_{i=-1}^{\infty} \operatorname{Hom}_{\mathbb{K}}(C^\infty(M)^{\otimes(i+1)}, C^\infty(M))$$

with product structure induced by

$$\phi \circ \psi = \sum_{i=0}^{m} (-1)^{n \cdot i} \phi \circ_i \psi \qquad \text{for } \phi \in \mathcal{C}^{m+1} \text{ and } \psi \in \mathcal{C}^{n+1}$$

with induced Lie bracket $[\phi, \psi]_{\mathcal{C}} = \phi \circ \psi - (-1)^{mn} \psi \circ \phi$[6]. It can be turned into a DGLA by considering the multiplication operator $\mathbf{m} \in \mathcal{C}^1$ defined by $\mathbf{m}(f, g) = f \cdot g$ where we use the commutative point-wise product in $C^\infty(M)$. Note that the associativity of $\mathbf{m}$ can be expressed as $[\mathbf{m}, \mathbf{m}]_{\mathcal{C}} = 0$. Then we can define the **Hochschild differential** $d_{\mathbf{m}} : \mathcal{C}^i \to \mathcal{C}^{i+1}$ as $d_{\mathbf{m}}\psi = [\mathbf{m}, \psi]_{\mathcal{C}}$. Thanks to the Jacobi identity it is easy to check that $d_{\mathbf{m}} \circ d_{\mathbf{m}} = 0$. Now consider a bidifferential operator $B \in \mathcal{D}^1$. Then we can consider $\mathbf{m} + B$

[6] $\circ_i$ is to be understood as having the *ith* entry of ϕ to be $\psi(\cdot)$.

as a deformation of the original product. Note that since this product is to be associative $[\mathbf{m}+B,\mathbf{m}+B]_{\mathcal{C}}=0$, and since $[\mathbf{m},B]_{\mathcal{C}}=[B,\mathbf{m}]_{\mathcal{C}}=d_{\mathbf{m}}B$ then we have another MC equation

$$d_{\mathbf{m}}B+\frac{1}{2}[B,B]_{\mathcal{C}}=0$$

This should be invariant under $DGLA$ morphism and under gauge transformations like (8). Finally, we are ready to discuss what is wrong with (39), namely the fact that it does not preserve the DGLA structure. For the $n=1$ case we have that in general

$$U_1^{(0)}[\![(X_0\wedge X_1),(Y_0\wedge Y_1)]\!]\neq[U_1^{(0)}(X_0\wedge X_1),U_1^{(0)}(Y_0\wedge Y_1)]_{\mathcal{C}}$$

which can be checked explicitly. This is a problem because it may happen that the MC equation may fail to be satisfied and therefore we may break the symmetries that we constructed for star products. However, the extra term we get in the right hand side is exact in the cohomology of $\mathcal{D}$ (subsection A.3). This is crucial because there must be some way of controlling this defect by treating the problem at the level of cohomologies. We know that multivector fields are isomorphic to their own cohomology because $d=0$. If we can somehow relate them to multidifferential operators, then we can work with them at the level of cohomology and the problem would be solved. The details of this final part are out of the scope of this work, but the key point is that Kontsevich knew that there was am isomorphism at the cohomology level between multidifferential operators and multivectors. This result was already partially known, but by working at the DGLA level nothing guarantees that there exist an quasi-inverse. By a quasi-inverse we mean another map which at the cohomology level is precisely the inverse of the above mentioned isomorphism.

However, a quasi-inverse can be constructed by equipping our DGLAs with the so called L_∞ structure. L_∞ algebras are a generalization of DGLAs that have the ideal property that their quasi-isomorphisms always have a quasi-inverse [25].

Finally, it can be shown that in the first equation in Appendix C the U_1 term actually corresponded to (39), so it can be interpreted as a first order approximation of the true correspondence between algebras.

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