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HIGH ENERGY PHYSICS, ASTROPHYSICS AND COSMOLOGY

MASTER'S THESIS

Exceptional Generalized Geometry, Topological p-branes and Wess-Zumino-Witten Terms

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Abstract

We study the interplay between the AKSZ construction of σ-models, the Hamiltonian formalism in the language of symplectic dg-geometry, the encoding of dynamics and symmetries inside algebroid structures and the exceptional generalized geometric description of supergravity theories. By utilizing the power of these higher geometric structure and constructions, we study the M5-brane Wess-Zumino term in the Green-Schwarz Lagrangian of M5-brane coupled to the associated background flux in supergravity theories.

We introduce the AKSZ construction and clarify its nature as a construction formalism for higher Chern-Simons theories, together with the Hamiltonian formalism nature of its geometric background. After embedding the E_6 generalized tangent bundle into a symplectic dg-manifold, via AKSZ construction we obtain a 7-dimensional higher Chern-Simons theory, then from the boundary of the theory we obtain the Wess-Zumino term of the Green-Schwarz action functional for the Abelian M5-brane.

We discuss and propose speculations on a three-fold coincidence between the D-branes in a transitive algebroid structure, well definedness of p-dimensional AKSZ σ-model boundary terms which can at the same time be seen as WZW terms of a p-dimensional Green Schwarz action functional, and the forcing of AKSZ σ-model boundary terms to lie in a Lagrangian submanifold. This is based on various observations made about the Lagrangian submanifolds of the symplectic dg-manifolds in the corresponding Hamiltonian formalism and Dirac structures of higher Courant algebroids found in the literature.

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Contents

Chapter 1

Introduction

1.1 Motivations and Strategy

String theory is a theoretical framework in which the point-like particles of particle physics are replaced by one-dimensional objects called strings. While string theory may or may not be the dreamt theory of everything, the understanding of string theory and its dualities has resulted in an deepening understanding of quantum field theories. The studies of AdS/CFT and AGT correspondence [1], Entanglement entropy, *etc.* cannot be isolated from the framework of string theory, and when studying gauge theories, conformal theories, supersymmetries, the problems can more or less be mapped to some corresponding string or brane models. Hence a proper understanding of string theory is still relevant and important.

After taking into account all the resonable consistency conditions, there are five possible superstring theories: type I, type IIA, type IIB, and heterotic $SO(32)$ and $E_8 \times E_8$. In 1990s, various dualities between these superstring theories implied that all these are different formulations of a theory which is more fundamental. It is conjectured to be the quantum theory of 11-dimensional supergravity (SUGRA). As a conjectured theory that unifies various string theories, M-theory still lacks a proper formulation. The studies imply that *higher structures* play a crucial role in the description of these theories of extended objects. Formulations of SUGRA theories with string duality manifest results in their description in terms of (exceptional) generalized geometry. In the second quantization of the string, string field theory, the Hilbert space can carry higher homotopy algebras such as an L_∞ and A_∞ algebra. In Batalin-Vilkovisky formalism, the L_∞ algebra structure appears in every classical field theory [2]. Higher-degree form fields appearing in supergravity theories are connections on higher principal bundles. Topological p-brane σ-models can be constructed through AKSZ construction of field theories, which are based on the geometry of symplectic dg-manifolds.

Nahm showed that 11 dimensions is the largest number of dimensions consistent with a single graviton with no higher-spin (greater than 2) particles, hence the maximal 11D SUGRA, along with the fact that the largest space-time dimension consistent with superconformal symmetry is 6. The quantization of 11D SUGRA leads to a theory of 2-branes and 5-branes, and is precisely the aformentioned M-theory. Hence the 2- and 5-branes are called M-branes. The quantum field theory on the worldvolume of M5-branes is a 6-dimensional (2,0) superconformal field theory, argued to be crucial to the understanding of deep physics and mathematics, for instance Khovanov homology, geometric Langlands duality and Montonen-Olive duality [3]. From this $(2,0)$-theory one can obtain, for example, AGT correspondence and 3d-3d duality through topological twist or compactifiction. A proper formulation of the theory is more than desireable.

However, the theory is far from being well understood. The Lagrangian description of several coincident M5-branes, being the classical limit of the $(2,0)$-theory, is still unknown, and was argued to be non-existent. The argument for the non-existence is roughly as follows, M5-branes are where M2-brane ends, while the boundary of an M2-brane is a self-dual string, a 1-dimensional object. and sweeps a surface σ. The self-dual string is charged under certain group G, hence parallel transported along the surface σ. There is no reparametrization invariant notion of surface ordering, so the parallel transport of self-dual strings forces G to be Abelian, so that the 2-form connection B's integral over the surface $\int_\sigma B$ is well defined.

The obstruction that leads to this argument now seems to be circumventable, after introducing 2-morphisms and 2-categories [4]. Consider a parallel transport of a 1-dimensional object subdivided into two pieces, with g_i denoting the group action. The original requirement that $(g_1'g_2')(g_1g_2) = (g_1'g_1)(g_2'g_2)$ can be expressed in a rather different way, that is

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2)$$

this interchange law no longer forces the group action g_i to be Abelian. Also successful M2-brane models have been constructed even if there are no continuous parameters in these models which suggests that there should be no Lagrangian description. Construction of a Lagrangian of coincident (hence non-Abelian) M5-branes is still a question worthy to ask.

Now the problem is what the proper tools and frameworks for the study of M5-brane are. Here the notion of *generalized geometry* [5], *Batalin Vilkovisky formalism* and in particular *AKSZ construction* [6] come into mind. Inspired by various dualities in the string theory, in particular *T-duality*, the attempt to geometrize string theories and make the duality manifest have led to the formulation in terms of generalized geometry. Generalized geometry takes the metric and the 2-form gauge field, the 2-form B-field appearing in type II strings, in an equal footing and provides a natural framework for the study of string theory and supergravity. The consideration of the 'larger' duality, The *U-duality* [7] then led to the formulation of *Exceptional Generalized Geometry* [8], capturing the features of the backgrounds of string and SUGRA and organizing the family of $p-$form fields and the branes to which the form fields couple into a unified geometric structure, encoding the strange and complicated symmetries arising from string theory. The symmetries of type II string, without introducing flux, corresponds to the structure studied by plain generalized geometry is encoded in the geometric structure known as *Courant algebroids*. For the case of flux being taken into account and for the case of 11D SUGRA, there are also corresponding algebroid structures that encode the symmetry.

The formalism of Batalin-Vilkovisky, which can be regarded as a systematic refinement of the BRST formalism, is a powerful tool for the construction of quantizable actions with complicated symmetries by introducing ghosts, anti-fields, anti-ghosts, *etc*, and in this framework AKSZ constructions provide a systemic and near mechanical way for the production of σ-models which are in some sense 'topological': From a higher Chern-Weil theoretical perspective, the action functional comming from the AKSZ procedures are higher analogous of Chern-Simons theory [9].

The Green-Schwarz action functional for branes includes a kinetic term and a Wess-Zumino-Witten term [10]. The latter encodes the coupling of the brane to the background gauge field, and is conformal in the sense of 'conformal block', which means through an analogue of CS-WZW correspondence, which seems to be highly related to the celebrated AdS/CFT correspondence between SUGRA and conformal field theory, there should be an higher analogous of Chern-Simons theory that on the boundary corresponds to the Wess-Zumino-Witten term. It is possible that the AKSZ construction can lead to this higher Chern-Simons theory.

Then where do one obtain the data that contains the information of complicated (higher) gauge symmetries of SUGRA theories,for the construction of this higher Chern-Simons action? The data needed for the construction are for worldvolume a *tangent Lie* n*-algebroid* and for target a *symplectic Lie* n*-algebroid*, and it can be shown that the category of *symplectic dg-manifolds* is a full subcategory inside the category of symplectic Lie $n-$algebroids. Symplectic dg-manifolds can be obtained by degree shift of ordinary tangent (and cotangent) bundles. It has been shown that symplectic Lie 2-algebroids are isomorphic to Courant algebroids which underly the geometric structure of generalized geometry. Hence given a theory and its corresponding exceptional geometric structure, one expects to find a corresponding symplectic dg-manifold, and from this datum construct an AKSZ σ-model action that captures the topological sector of the theory. Given the description of 11D SUGRA in terms of exceptional generalized geometry, the generalized Wess-Zumino term, which can be seen as the 'topological sector' of the full Lagrangian of the theory, could be obtained by considering the dg-manifold description of the associated generalized tangent bundle.

There are more phenomena providing indications for this approach. For example the Polyakov action can be obtained from the boundary of a 3-dimensional AKSZ σ-model. Wess-Zumino-Witten model, describing the propagation of the string on a group coupled to gravity and $B-$field can also be obtained by Kaluza-Klein reduction from Chern-Simons theory. It was identified by Henneaux and Mezincescu that Green-Schwarz action functionals are some kind of higher-dimensional WZW models [10].

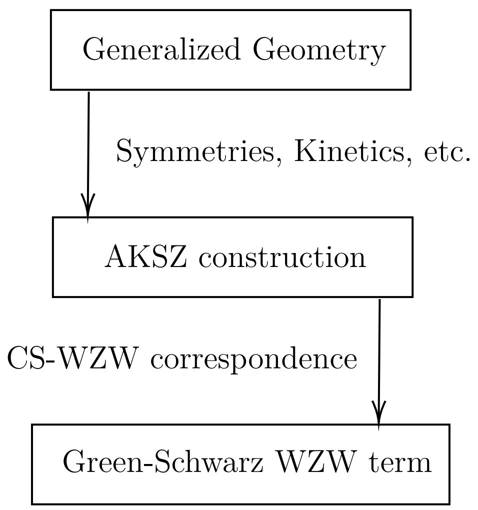

Figure 1.1: Strategy we follow

The thesis studies the interplay between AKSZ construction of σ-model and the Hamiltonian formalism in the language of symplectic dg-geometry, the encoding of dynamics and symmetries inside algebroid structures and exceptional generalized geometric description of supergravity theories. By utilizing the power of these higher geometric structure and constructions, we study the M5-brane Wess-Zumino term in the Green-Schwarz Lagrangian of M5-brane coupled to the associated background flux in supergravity theories.

The strategy is concisely expressed in Figure 1.1.

There are also some other mysterious phenomena that could be hints for arguments supplying the strategy of our derivation of Wess-Zumino terms. There seems to be correspondences between the choice of boundary conditions and the current algebra, and thence the anomally cancellation mechanism for the Wess-Zumino-Witten term derived [11–15]. Also the underlying geometric language of AKSZ construction is closely related to the variational problems of extended objects. While these are unsettled speculations, they are still worth mentioning.

The thesis will not try to tackle with the problem of generalizing the higher gauge theory to non-Abelian one, and it is in fact an open problem many physicists and mathematicians are trying to answer. Since what is of interest is a geometric understanding of, for the time being Abelian, M5-brane action, the focus will be on 11D SUGRA, where M5-branes live.

1.2 Outline

In Chapter 2, some necessary mathematical backgrounds are introduced in a very brief manner.

In Chapter 3, after briefly reviewing elements of differential graded (dg) geometry, the AKSZ construction of σ-models is introduced in the language of symplectic Lie n-algebroid. Symplectic Lie n-algebroids are equivalently symplectic dg-manifold of grade n in the dg-geometric picture. The fact that AKSZ σ-models can be regarded as a certain higher generalization of Chern-Simons theory will be shown in the language of higher Chern-Weil theory.

In Chapter 4, we study the properties of AKSZ σ-models, and after giving some concrete examples, we further discuss the underlying geometric language of the construction from the perspective of variational problem. AKSZ construction, with certain target manifolds chosen, can be used to obtain the σ-model action of 'topological' p-brane theories, we clarify the word 'topological'. Also we discuss the significance of Lagrangian submanifolds of the target symplectic dg-manifold, its relation to D-branes and anomaly cancellation.

In Chapter 5, we do some speculations to argue for our strategy. The higher Courant algebroid structure associated to $TM \oplus \wedge^p T^*M$ arises when considering the variational problem of the propagation of a p-dimensional object, since when $p = 1$ the corresponding action is Nambu-Goto action, with general p the corresponding actions should be related to Green-Schwarz action functionals. Taking into account the fact that AKSZ σ-models are in fact some certain generlization of Chern-Simons theory to higher dimensions, and the correspondence between 3d Chern-Simons bulk field theory and the 2d Wess-Zumino-Witten model on a suitable Lie group (more precisely, the duality between the space of quantum states of 3D Chern-Simons theory on a surface Σ and the conformal blocks of the WZW model on Σ), which is related to AdS/CFT correspondence, might shed some light on the Lagrangian formulation of the worldvolume theory of non-Abelian M5-brane, 6D $(2,0)$-theory, which was believed to be non-existent. Compactifying the 11D SUGRA on a 4-sphere to get a 7-dimensional theory (AdS_7), a 7-dimensional Chern-Simons term arises, then there can be a correspondence between this term and the conformal field theory in 6 dimension (CFT_6).

In Chapter 6, aspects of various dualities in string theory and their description in the language of generalized, exceptional generalized geometry are reviewed. Here the focus is on 11D SUGRA, which is the low energy limit theory of the conjectured M-theory. Then, the descriptions of the 11D SUGRA in terms of the exceptional generalized geometry with the generalized tangent bundle $TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$ in $\dim M = 6$ is presented.

In Chapter 7, we derive the M2-brane Wess-Zumino term as an exercise.

In Chapter 8, finally the M5-brane Wess-Zumino term (in Abelian case, *i.e.* no coincident branes) is derived, from the exceptional geometric structure and the AKSZ construction, on the boundary of a 1-dimensional higher σ-model obtained through AKSZ construction.

In Chapter 9, first we make conclusions. Next, based on various observations made about the Lagrangian submanifolds of the symplectic dg-manifolds in the corresponding Hamiltonian formalism and Dirac structures of higher Courant algebroids found in the literature, we discuss and propose speculations on a three-fold coincidence between the D-branes in a transitive algebroid structure, well definedness of p-dimensional AKSZ σ-model boundary terms which can at the same time be seen as WZW terms of a p-dimensional Green Schwarz action functional, and the forcing of AKSZ σ-model boundary terms to lie in a Lagrangian submanifold. Finally, we give outlooks for future research.

1.3 Conventions and Nomenclature

- $\iota_X \theta$ will denote the contraction (interior product) of vector field X with a differential form θ.
- Lie derivatives along X will be denoted by $\mathcal{L}_X$.
- $\Gamma(X)$ denotes the space of (global or local) sections of X.
- d (without subscript) will be the de Rham differential in most circumstances.
- All the manifolds are smooth, the categories manifolds form will also be the categories of smooth $X-$manifolds with X being qualifiers, so the word 'smooth' will be occasionally omitted.
- What are usually called QP-manifolds, NQ-manifolds, Σ_n-manifolds, *etc.* are called symplectic dg-manifolds, with the associated Q-structure, or the odd vector field, homological vector field, *etc.* being called cohomological vector field.
- Internal degrees of graded manifolds are called 'grade', instead of 'degree'. However the standard notation $\deg(\cdot)$ or $|\cdot|$ will still be used for the grade of an element, since degrees in the sense of 'k-th exterior product' will not appear.

We will assume familiarity with some mathematical concepts such as manifold, sheaf, fiber bundle, representation theory of Lie groups and category theory.

Chapter 2

Mathematical Preliminaries

2.1 Some Symplectic and Poisson Preliminaries

Symplectic dg-manifolds which will be introduced below and used throughout are higher generalizations of symplectic manifolds. We will also encounter structures analogous to Poisson structure.

Definition 2.1.1. *A symplectic manifold (M, ω) is a smooth manifold equipped with a symplectic structure ω, i.e. a closed non-degenerate 2-form.*

An important theorem of Darboux ensures that we can find a chart for (M, ω) with coordinates being $\{x^a\}$ such that $\omega = \omega_{ab} dx^a dx^b$ with ω_{ab} constant. The chart is called the *Darboux chart* and the coordinates *Darboux coordinates*. The theorem basically tells us that symplectic manifolds are topologically and globally interesting, but locally dull.

Now let $H : M \to \mathbb{R}$ be a smooth function, dH is then a 1-form. By nondegeneracy there must be a unique vector field X_H on $\Gamma(TM)$ with

$$\iota_{X_H} \omega = dH \tag{2.1.1}$$

Definition 2.1.2. *The smooth function $H : M \to \mathbb{R}$ is called a Hamiltonian function associated to the Hamiltonian vector field X_H, if $\iota_{X_H} \omega = dH$.*

One says that the smooth function H generates the Hamiltonian vector field X_H on M.

Equivalently for every 1-form θ there is a unique vector field Ω_θ such that $\iota_{\Omega_\theta} \omega = \theta$ (or $\Omega_{dH} = \omega^{-1}(dH)$) because of the non-degeneracy of the symplectic form, since then it will be able to induce an isomorphism. It is just the induction of a functor $C \to Set$ or a presheaf (contravariant functor $C^{op} \to Set$) from a Hom-functor $C \times C^{op} \to Set$, in our case $\omega(X_f, \cdot) = df(\cdot)$ and $\omega(\cdot, X_g) = \mathcal{L}_{X_g}$.

Every symplectic manifold comes with a *Poisson bracket*,

Definition 2.1.3. *The Poisson bracket is a bilinear operation on smooth functions, $\{\cdot, \cdot\}_P : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, defined by*

$$\{f, g\}_P = \omega(X_f, X_g) \tag{2.1.2}$$

The Poisson bracket is clearly antisymmetric, and furthermore

$$\{f, g\}_P = \mathcal{L}_{X_g} f \tag{2.1.3}$$

So the Poisson bracket is also a derivation, satisfying the well-known Leibniz rule. The Poisson bracket itself as an algebraic bilinear operation should be familiar, we omit the details.

What immediately follows the definition of Hamiltonian vector fields is that

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H}d\omega = d^2H + 0 = 0 \tag{2.1.4}$$

Hence every one-parameter family of diffeomorphisms $\rho_t : M \times \mathbb{R} \to M$ generated by X_H ($\frac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H$) preserves ω, so is a family of symplectomorphisms. Also the Hamiltonian function H itself is preserved under Hamiltonian vector fields, that is

$$\mathcal{L}_{X_H}H = \iota_{X_H}\iota_{X_H}\omega = 0 \tag{2.1.5}$$

This is often reinterpreted in terms of the Poisson bracket relation

$$\{H, H\}_P = 0 \tag{2.1.6}$$

The notion of *Lagrangian submanifold* is important in the whole field of symplectic geometry, and also will play an important role in our language

Definition 2.1.4. *A submanifold L of a $2n$-dimensional symplectic manifold (M, ω) is Lagrangian, if every tangent space T_pL of L is a Lagrangian subspace of T_pM, i.e. $\omega_p|_{T_pY} \equiv 0$, it is an isotropic submanifold, and if it is maximally isotropic,* $\dim T_pL = n$.

2.2 Graded Algebraic Structures

For completeness we present some basic definitions. See [16] for more.

Definition 2.2.1. *A $\mathbb{Z}$-graded vector space is a direct sum $V = \oplus_i V_i$ of vector spaces, a nonzero element $v \in V_i$ is said to be homogeneous of grade i.*

We only consider graded vector spaces such that $\dim V_i < \infty$ for all $i \in \mathbb{Z}$. $V[k]$ denotes a graded vector space with degree increased by k, so $(V[k])_i = V_{i-k}$ for all $i \in \mathbb{Z}$. For details on grade shifted vector spaces, see [2].

Definition 2.2.2. *A $\mathbb{Z}$-graded commutative ring $\mathcal{R}$ is a ring with a decomposition $\mathcal{R} = \oplus_i \mathcal{R}_i$. For any homogeneous elements $a, b \in \mathcal{R}$, under multiplication they obey the Koszul sign rule: $ab = (-1)^{|a||b|}ba$, with $|a|$ denoting the grade of the homogeneous element a.*

Definition 2.2.3. *A $\mathbb{Z}$-graded algebra is a $\mathbb{Z}$-graded ring with the structure of an $\mathbb{R}$-module.*

Definition 2.2.4. *A $\mathbb{Z}$-graded differential graded algebra is a $\mathbb{Z}$-graded associative algebra A equipped with a nilpotent derivation of grade 1 or -1, denoted by $d : A \to A$.*

- *Nilpotency:* $d \circ d = d^2 = 0$.
- *Derivation: satisfying the graded Leibniz rule* $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} \cdot (db)$

Now we introduce the notion of *Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

Definition 2.2.5. *The Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ *of a finite-dimensional Lie algebra* $\mathfrak{g}$ *is the semifree differential graded commutative algebra, with the underlying graded algebra being the Grassmann algebra* $\wedge^\bullet \mathfrak{g}^*$*, and the differential* d*, which is of grade 1, such that*

$$d|_{\mathfrak{g}^*} = [\cdot,\cdot]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

The nilpotency of the Chevalley-Eilenberg differential, $d \circ d = 0$, translates to the fact that the Lie bracket satisfies the Jacobi identity. When the module $\mathfrak{g}^*$ is now equipped with a grading, $\mathfrak{g}$ is equipped with the structure of an L_∞ algebra, whose horizontal categorification is a L_∞-algebroid, which we will discuss later. See [17] for the relations between differential graded algebras and higher (L_∞, A_∞, *etc*) algebras.

2.3 Jet bundle approach to Classical Field Theory

When discussing the Hamiltonian mechanics of p-dimensional objects and their relation to σ-models with the worldvolume set to tangent Lie algebroids (equivalently shifted tangent bundle) and target space set to symplectic dg-manifolds, we will briefly use some concepts of jet bundles. For a complete treatment, see [18].

Given a fibre bundle $\pi : E \to M$ with the base manifold M being the spacetime, the *k-jet space* $J^k E$ comprises the equivalence classes j^k_x, $x \in M$, of sections s of E identified by the first $k+1$-terms of their Taylor series at points $x \in M$. In other words

Definition 2.3.1. *The k-jet of sections of π is the bundle, of which the fibre over $x \in M$ is the space of equivalence classes of germs of sections of π. The equivalence is given by the coincidence of their first k partial derivatives.*

We are concerned only with first order Lagrangian and Hamiltonian systems, for most of the field theories nowadays are described by first order Lagrangian densities. Denoting the jet bundle by $j_1 E \to M$, a *first order Lagrangian density* on the configuration space $j_1 E$ is represented by a n-form, where $\dim(M) = n$,

$$L = \mathcal{L} dx^1 \wedge \ldots \wedge dx^n$$

where x are M-coordinates. With a Lagrangian density L given, the associated *Legendre morphism* is the morphism

$$\hat{L} : j_1 E \to \Pi = \wedge^n T^* M \otimes_E TM \otimes_E V^* E$$

where V^*E is the *vertical cotangent bundle*, dual to the *vertical tangent bundle*.

Definition 2.3.2. *The vertical tangent bundle $VE \to E$ of E a fiber bundle is defined through*

$$VE = \mathrm{Ker}\ T\pi$$

where $T\pi : TE \to TM$ is induced by π

The $j_1 E$ coordinates (x^μ, e^i, e^i_μ), where (x^μ, e^i) are fibred coordinates of E and y^i_μ are derivative coordinates (velocities), induces coordinates on the Legendre bundle Π, (x^μ, e^i, p^μ_i), such that

$$p_i^\mu \circ \hat{L} = \pi_i^\mu$$

The multisymplectic form carried by Π can be expressed by $\Omega = dp_i^\mu \wedge de^i \wedge \omega \otimes \partial_\mu$. On the Legendre bundle $\Pi \to M$, a connection γ is said to be Hamiltonian associated to a Hamiltonian form H if $dH = \iota_\gamma \Omega$. This is the multimomentum Hamiltonian formalism. The jet bundle approach can be used when the Lagrangian density is not regular, *i.e.* partially defined or degenerate, so there is no Hamiltonian in the traditional sense, which happens frequently in gauge theory. However, the fact that we need to know about the jet bundle approach is the following one.

Theorem 2.3.3. *There is a 1:1 correspondence between sections of the first jet $j_1 E \to E$ and connections on the fibred bundle $E \to M$*

Chapter 3

AKSZ construction, Chern-Weil theory and Symplectic dg-geometry

3.1 Differential Graded Geometry

The original AKSZ construction is based on the language of symplectic dg-manifolds, so we briefly review these objects. We follow [9] for the main logic, but occasionally refer to others, for example [19–21].

Definition 3.1.1. *A (smooth) graded manifold M over a base M_0 is a sheaf of $\mathbb{Z}-$graded commutative algebras $C^\bullet(M)$, over a smooth manifold M_0, locally isomorphic to an algebra of the form $C^\infty(U) \otimes S^\bullet(V)$ ($U \subset M_0$ an open set). Here V is a graded vector space, and $S^\bullet$ the free graded-commutative algebra on V.*

We only consider $\mathbb{N}$-graded manifolds here.

Remark. An equivalent, but more abstract definition of the $\mathbb{N}$-graded manifolds based on Isbell duality [22] is as follows [9]. Let $V \to M_0$ be an $\mathbb{N}-$graded smooth vector bundle, V^* its dual bundle, and $\Gamma(V^*)$ the graded $C^\infty(M)$-module of smooth sections of V^*. From these data one can obtain Grassmann algebras of the form $\wedge^\bullet_{C^\infty(M_0)}\Gamma(V^*)$. Then $\mathbb{N}-$graded manifolds form a category, which is the full subcategory of the opposite category of $\mathbb{N}-$graded commutative algebras, on those isomorphic to $\wedge^\bullet_{C^\infty(M_0)}\Gamma(V^*)$.

The generators of the algebra (of functions) $C^\bullet(U) \simeq C^\infty(U) \otimes S^\bullet(V)$ are viewed as local coordinates on M. According to the grades, we have a decomposition $C^\bullet(M) = \oplus_k C^k(M)$, with each $C^k(M)$ being a sheaf of locally free $C^\infty(M_0)$-modules. For nonnegative graded case, $C^\infty(M_0) = C^\infty(M)$.

Now we equip the graded manifolds with a nilpotent vector field, hence the name of 'differential'.

Definition 3.1.2. *A differential graded (dg-) manifold is a graded manifold (M, Q), equipped with a grade-1 (or internal degree 1) vector field Q satisfying $[Q, Q] \equiv Q^2 = 0$, which is often called a cohomological vector field or a $Q-$structure (hence the name of $Q-$manifold) in the literature. The bracket $[\cdot, \cdot]$ here is a graded Lie bracket. Q, being a vector field, can be seen as a grade-1 derivation $Q : C^\infty(M) \to C^\infty(M)$.*

Remark. The category of ($\mathbb{N}$-graded) dg-manifolds DgMfd hence is the full subcategory of the opposite of differential graded-commutative algebras, on objects whose underlying graded algebra comes from the category of graded manifolds GrMfd.

Given a dg-manifold, one can define the de Rham complex over it for the study of the differential structure, through the *de Rham complex functor*

Definition 3.1.3. *The de Rham complex functor $\Omega^\bullet(\cdot) : \mathrm{DgMfd} \to \mathrm{cdgAlg}_{\mathbb{R}}^{op}$, is a functor that sends a dg-manifold M with its algebra of functions (isomorphic to $\wedge^\bullet_{C^\infty(M_0)}\Gamma(V^*)$) to the Grassmann algebra over $C^\infty(M_0)$ on the graded $C^\infty(M_0)$-module $\Gamma(T^*M) \oplus \Gamma(V^*) \oplus \Gamma(V^*[-1])$.*

- *$V^*[-1]$ is V^* with its grades increased by 1.*
- *The differential $\mathbf{d}$ accompanying the functor is defined on generators, s.t. $\mathbf{d}|_{C^\infty(M_0)} = d_{\mathrm{dR}}$ (the ordinary de Rham differential $C^\infty(M_0) \to \Gamma(T^*M)$).*
- *$\mathbf{d}|_{\Gamma(V^*)}$ the grade-shift isomorphism $\Gamma(V^*) \to \Gamma(V^*[-1])$.*
- *$\mathbf{d}$ vanishes on all remaining generators.*

Remark. The Cartan calculus of differential geometry generalizes directly to graded manifolds. For M agraded manifold and a vector field $v \in \Gamma(TM)$. For example,

- The contraction on the de Rham complex, $\iota_v : \Omega^\bullet(M) \to \Omega^\bullet(M)$.
- The Lie derivative $\mathcal{L}_v \equiv [\iota_v, \mathbf{d}] : \Omega^\bullet(M) \to \Omega^\bullet(M)$.

Definition 3.1.4. *Over any coordinate patch $U \to M$, and the corresponding basis of generators $\{x^a\}$, the Euler vector field $\epsilon \in \Gamma(TM)$ is defined, given by*

$$\epsilon|_U = \sum_a \deg(x^a) x^a \frac{\partial}{\partial x^a}$$

The Euler vector field assigns to every homogeneous element $\alpha \in \Omega^\bullet(M)$ a unique natural number $n \in \mathbb{N}$, the *grade* of α, by

$$\mathcal{L}_\epsilon \alpha = n\alpha$$

We introduce the notion of *symplectic dg manifolds.*

Definition 3.1.5. *The tuple (M, Q, ω) is a symplectic dg manifold obtained by equipping a dg-manifold (M, Q) with a symplectic form $\omega \in \Omega^2(M)$ with its grade being p, i.e. $\mathcal{L}_\epsilon \omega = p\omega$, and require it to be Q invariant, $\mathcal{L}_Q \omega = 0$.*

Locally, with coordinates $\{x^a\}$, $\omega|_U = \frac{1}{2}\mathbf{d}x^a \omega_{ab} \mathbf{d}x^b$. The Darboux theorem is still applicable [1]. Also the function algebra of the grade n symplectic dg-manifold is naturally equipped with a Poisson bracket, just like in the symplectic geometric case. It should be noted however that the Poisson bracket

$$\{\cdot,\cdot\}_P : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$$

lowers the grade by n, so it is $-n$-graded.

The Hamiltonian vector field and the Hamiltonian function can also be defined in a completely identical manner, *i.e.*

[1] See Appendix B.5 of [19].

Definition 3.1.6. *$H \in C^\infty(M)$, is the Hamiltonian function for v, the Hamiltonian vector field, if*

$$\iota_v\omega = \mathbf{d}H$$

In local coordinates $\{x^a\}$ this becomes $\omega_{ab}v^b = \frac{\partial H}{\partial x^a}$.

Associated to the graded Poisson bracket and the cohomological vector field, we define the *cohomological function*

Definition 3.1.7. *The cohomological function of a symplectic dg-manifold $(M, \omega, v)$of grade n is a generator $\Theta \in C^\infty(M)$ of v with respect to the canonically equipped graded Poisson bracket $\{\cdot,\cdot\}_P$, such that*

$$v = \{\Theta, \cdot\}_P$$

It is clear that cohomological function is precisely the Hamiltonian function associated to the cohomological vector field. The grade of the cohomological function should be $n+1$ for grade n symplectic dg-manifold. The nilpotency of the cohomological vector field $v^2 = 0$ translates directly to

$$\{\Theta, \Theta\}_P = 0$$

Given a cohomological function, a new bilinear bracket called *derived bracket* can be defined [19, 23],

Definition 3.1.8. *A derived bracket for functions on the symplectic dg-manifold with Θ as its cohomological function, is a bilinear bracket, associated to the graded Poisson bracket $\{\cdot,\cdot\}_P$, such that*

$$[f, g]_d \equiv -\{\{\Theta, f\}_P, g\}_P$$

nilpotency of v, so that $\{\Theta, \Theta\}_P = 0$, leads to the (modified) Jacobi identity for the derived bracket

$$\{f, \{g, h\}\}_d = \{\{f, g\}, h\}_d + (-1)^{(|\deg(f)+n|)(\deg(g)+n)}\{g, \{f, h\}_d\}_d$$

where n is the grade for the Poisson bracket.

3.2 Symplectic L_∞-algebroids

Symplectic dg-manifolds are higher geometric objects we need for the formulation of the AKSZ construction, but a Lie theoretical perspective should be helpful for the identification of $\sigma-$models obtained with higher Chern-Simons action functionals, hence the introduction of L_∞ algebroids. We define that

Definition 3.2.1. *The category of $L_\infty-$algebroids, L_∞*Algd *is equivalent to the category of dg-manifolds* DgMfd*. The corresponding dg-algebra of $\mathfrak{a} \in L_\infty$*Algd *is denoted by* $\mathrm{CE}(\mathfrak{a})$*, the Chevalley-Eilenberg algebra of $\mathfrak{a}$. The differential is now the Chevalley-Eilenberg differential $d_{\mathrm{CE}(\mathfrak{a})}$.*

Definition 3.2.2. *For M a smooth manifold, the tangent Lie algebroid $\mathfrak{a} = \mathfrak{T}M$ is defined by* $\mathrm{CE}(\mathfrak{T}X)(\Omega^\bullet(X), d_{\mathrm{dR}})$

Remark. A tangent Lie algebroid is equivalently a shifted tangent bundle $T[1]M$.

If the graded algebra generators are of at most grade n, the algebroid is a *Lie n-algebroid*. To study the characteristic class and invariant polynomials of these geometric objects, we need Lie algebra cohomology.

Definition 3.2.3. *The Weil algebra* $(\mathrm{Weil}(\mathfrak{a}), d_{\mathrm{W}(\mathfrak{a})})$ *of a L_∞-algebroid is given by*

- $\mathrm{Weil}(\mathfrak{a}) = \mathrm{CE}(\mathfrak{T}\mathfrak{a})$
- $d_{\mathrm{W}}(\mathfrak{a}) = \mathbf{d} + \mathcal{L}_v$, *where $\mathcal{L}_v$ is the Lie derivative along the vector field corresponding to the Chevalley-Eilenberg differential: locally, $v|_U = v^i \frac{\partial}{\partial x^i}$ with $v^i = d_{\mathrm{CE}(\mathfrak{a})} x^i$.*

In other words it is the Chevalley-Eilenberg algebra of the tangent L_∞-algebroid $\mathfrak{T}\mathfrak{a}$. Note that there is a canonical dg-algebra homomorphism $i^* : \mathrm{W}(\mathfrak{a}) \to \mathrm{CE}(\mathfrak{a})$. On the level of complexes this is a projection of complexes.

We introduce the notion of *L_∞-cocycle*, *invariant polynomial* and *transgression element* or *Chern-Simons element*

Definition 3.2.4. *An L_∞-cocycle on $\mathfrak{a}$ is an element $\mu \in \mathrm{CE}(\mathfrak{a})$ that is closed.*

Definition 3.2.5. *An invariant polynomial $\langle\cdot\rangle$ on $\mathfrak{a}$ is a closed element in the subalgebra generated by the shifted elements in the Weil algebra $\mathrm{W}(\mathfrak{a})$.*

Definition 3.2.6. *If there exists an element $\mathrm{cs} \in \mathrm{W}(\mathfrak{a})$ such that $i^*\mathrm{cs} = \mu$ and $d_{\mathrm{W}(\mathfrak{a})}\mathrm{cs} = \langle\cdot\rangle$ with μ a cocycle, μ is said to be in transgression with $\langle\cdot\rangle$. The element cs is said to be a Chern-Simons element or a transgression element witnessing the transgression.*

Now we can define *symplectic Lie n-algebroids* which are Lie theoretic analogues of symplectic dg-manifolds.

Definition 3.2.7. *A symplectic Lie n-algebroid $(\mathfrak{S}, \omega)$ is defined to be a Lie n-algebroid $\mathfrak{S}$ equipped with a quadratic non-degenerate invariant polynomial $\omega \in \mathrm{W}(\mathfrak{S})$ of degree $n+2$.*

Some important facts about symplectic Lie n-algebroids are as follows

- Every symplectic dg-manifold of grade n can be seen as a symplectic Lie n-algebroid. More precisely, there is a full and faithful embedding of symplectic dg-manifolds of grade n into symplectic Lie n-algebroids.
- A symplectic Lie n-algebroid $(\mathfrak{S}, \omega)$ given by the embedding of a symplectic dg-manifold carries a canonical L_∞-algebroid cocycle π, and is the Hamiltonian of $d_{\mathrm{CE}(\mathfrak{S})}$
- $\frac{1}{n}\pi$ is in transgression with ω, with the Chern-Simons element being $\mathrm{cs} = \frac{1}{n}(\iota_\epsilon \omega + \pi)$

The proof of the statements are given in Appendix. Now, locally with coordinates $\{x^a\}$,

$$\pi|_U = \frac{1}{n+1}\omega_{ab}\mathrm{deg}(x^a) x^a \wedge v^b$$

Recall that $d_{\mathrm{CE}} x^b = v^b$. Upon identifying $\mathbf{d} = d_{\mathrm{W}} - d_{\mathrm{CE}}$, the Chern-Simons element can be expressed in this coordinate as

$$\mathrm{cs}|_U = \frac{1}{n}\left(\mathrm{deg}(x^a)\omega_{ab} x^a \wedge d_W x^b - n\pi\right)$$

Finally we let L_∞-algebroid make contact with (higher) gauge theory.

Definition 3.2.8. *For* $\mathfrak{a} \in L_\infty\mathrm{Algd}$ *and* Σ *a smooth manifold, a morphism*

$$A : \mathrm{W}(\mathfrak{a}) \to \Omega^\bullet(\Sigma) \tag{3.2.1}$$

is defined to be a degree 1 $\mathfrak{a}$*-valued differential form on* Σ.

This is the Ehresmann connection if in ordinary gauge theory. One is then led to define its *curvature* F_A

Definition 3.2.9. *The curvature* F_A *associated to a degree 1* $\mathfrak{a}$ *valued differential form on* Σ *is the induced morphism of graded vector spaces given by*

$$\Omega^\bullet(\Sigma) \xleftarrow{A} \mathrm{W}(\mathfrak{a}) \leftarrow \wedge^1 V[1] : F_A \tag{3.2.2}$$

The degree 1 $\mathfrak{a}$-valued differential form A is *flat* if it factors throught i^*, or equivalently $F_A = 0$. Locally, in a coordinate $\{x^a\}$ of $\mathfrak{a}$, assigned to the generator x^a, the differential form is $A^a = A(x^a) \in \Omega^{\deg(x^a)}(\Sigma)$, hence the components of the curvature $F_A^a = A(\mathbf{d}x^a) = A(d_{\mathrm{W}}x^a - d_{\mathrm{CE}}x^a) \in \Omega^{\deg(x^a)+1}(\Sigma)$. A is a dg-algebra homomorphism, so $A(d_{\mathrm{W}}x^a) = d_{\mathrm{dR}}A^a$ and $A(d_{\mathrm{CE}}x^a) = d_{\mathrm{dR}}A^a - F_A^a$.

Associated to A, a Chern-Simons element cs have its image in $\Omega^\bullet(\Sigma)$, $A(\mathrm{cs})$. Equivalently cs can be seen as a map from the space of degree 1 $\mathfrak{a}$-valued differential forms on Σ to its de Rham complex $\Omega^\bullet(\Sigma)$, denoted by $\mathrm{cs}(A)$, the *Chern-Simons differential form* associated to A. Similarly for the invariant polynomial $\langle\cdot\rangle$ transgressed by cs on $\mathfrak{a}$, the evalutation of which is denoted by $\langle F_A\rangle$, is called the *curvature characteristic form* of A with respect to $\langle\cdot\rangle$.

3.3 AKSZ σ-Models, and AKSZ actions as Chern-Simons functionals

Originally, the AKSZ formalism (vaguely introduced in [6] and explicit construction introduced in [24]) is introduced in the language of symplectic dg-manifold. Taking the target space of the σ-model to be a symplectic dg-manifold (M, ω), and the worldvolume to be the shifted tangent bundle of a compact smooth manifold Σ, then a mapping space $\mathcal{M} = \mathrm{Maps}(\mathfrak{T}\Sigma, M)$ of graded manifolds can be formed. The sum of cohomological vector fields $v_\Sigma + v_M$ equips the mapping space with the structure of a dg-manifold. Then given an n-form α on $\Omega^n(M)$, it can be lifted by transgression to $\tilde{\alpha} = \Omega^n(\mathcal{M})$

$$\tilde{\alpha} = \int_{\mathfrak{T}\Sigma} \mathrm{ev}^*\alpha = \mathrm{p}_!\mathrm{ev}^*\alpha \tag{3.3.1}$$

by pull-push through the canonical correspondence

$$\mathcal{M} \xleftarrow{p} \mathcal{M} \times \mathfrak{T}\Sigma \xrightarrow{\mathrm{ev}} M$$

Hence the symplectic structure ω can also be lifted to a symplectic sturcutre on the mapping space $\mathcal{M}$. So associated to the canonical cohomological vector field $v = v_\Sigma + v_M$ there is a Hamiltonian $\mathbf{S} \in C^\infty(\mathcal{M})$, with

$$\mathbf{dS} = \iota_v \int_{\mathfrak{T}\Sigma} \mathrm{ev}^*\omega$$

Now take the grade-0 component $S_{\text{AKSZ}} = \mathbf{S}|_{\mathcal{M}_0}$, this is a functional on the space of graded manifold morphisms $\mathfrak{T}\Sigma \to M$. *In the case that M admits a global Darboux coordinate* $\{x^a\}$ with Σ being $n+1$-dimensional, an explicit formula can be given. Corresponding to the symplectic structure ω, there must be a Hamiltonian π of v_M, the action functional is then

$$S_{\text{AKSZ}} : \phi \mapsto \int_\Sigma \left(\frac{1}{2}\omega_{ab}\phi^a \wedge d_{\text{dR}}\phi^b - \phi^*\pi \right)$$

The maps ϕ's here are interpreted as $\phi \in \Omega^\bullet(\Sigma, V)$, $V-$valued differential forms on Σ, where $V \to M_0$ is the vector bundle corresponding to the graded manifold M, since there is a natural bijection $\mathcal{M} \simeq \Omega^\bullet(\Sigma, V)$. $\phi_0 \in \Omega^0(\Sigma, V)$ are smooth functions $\Sigma \to X_0$, and for $n > 1$, $\phi_n \in \Omega^n(\Sigma, \phi_0^* V_{n-1})$. The classical master equation $\{\mathbf{S}, \mathbf{S}\} = 0$ is automatically satisfied[2]. The degree-0 cohomology of $\mathcal{M}$ with respect to the cohomological vector field v is the space of functions corresponding to fields satisfying the Euler-Lagrange equations of S_{AKSZ}.

Now we expose the Chern-Weil theorectic nature of the construction, given the fact that any symplectic dg-manifold of grade n can be seen as a symplectic Lie n-algebroid: An L_∞-algebroid $\mathfrak{a}$, equipped with (π, cs, ω) triple (hence arising from a symplectic Lie n-algebroid $(\mathfrak{S}, \omega)$), with the element cs transgressing the invariant polynomial ω to a cocycle π, defines an AKSZ σ-model action.

- As target space for the σ-model take a tangent L_∞-algebroid $\mathfrak{T}\mathfrak{a}$.
- The worldvolume is taken as shifted tangent space $\mathfrak{T}\Sigma$.
- The space of fields is then the space of maps $\text{Maps}(\mathfrak{T}\Sigma, \mathfrak{T}\mathfrak{a})$. This is dually the space of morphisms of dgcas from $\text{W}(\mathfrak{a})$ to $\Omega^\bullet(\Sigma)$, the space of degree 1 $\mathfrak{a}$-valued differential forms on Σ.
- Finally, the σ-model action functional $A \mapsto \int_\Sigma \text{cs}(A)$.

The connection to the previously shown AKSZ construction is stated as the following theorem:

Theorem 3.3.1. *For $(\mathfrak{S}, \omega)$ a symplectic Lie $n-$algebroid associated to a symplectic dg-manifold of positive grade n, with global Darboux chart, the action functional induced by the Chern-Simons element cs= $\frac{1}{n}(\iota_\epsilon\omega + \pi)$ where $\pi = \frac{1}{n+1}\iota_\epsilon\iota_v\omega \in \text{W}(\mathfrak{S})$, is the AKSZ action, i.e.,* $\int_\Sigma \text{cs} = \int_\Sigma L_{\text{AKSZ}}$

See 9.0.4 for a proof. It is a direct consequence of the above theorem that

Corollary 3.3.1.1. *If the $(n+1)$-dimensional smooth manifold Σ is the boundary of an $(n+2)$-dimensional compact oriented manifold, $\partial N = \Sigma$, then $\int_\Sigma L_{\text{AKSZ}} = \int_N \omega(F_A)$ where $\omega(F_A)$ is the symplectic form evaluated on the curvature of $A : \text{W}(\mathfrak{S} \to \Omega^\bullet(\Sigma)$, since it can be seen as an invariant polynomial.*

[2]See [25] for details on the BV formalism

Chapter 4

Properties of AKSZ σ-models and higher Courant algebroids

4.1 AKSZ σ-models with Boundary

Since CS-WZW correspondence [26] is a crucial ingredient of the logic of this thesis, for later use the AKSZ construction with a open target should be briefly reviewed. For an $n+1$-manifold with boundary Σ and a symplectic dg-manifold $E = T^*[n]T[1]M$, one can construct a topological σ-model on $\mathrm{Map}(T[1]\Sigma, T^*[n]T[1]M)$ through the AKSZ procedure. It is important that the boundary conditions on ∂M should be set consistent with the dg-structure, *i.e.*, the cohomological vector field v and the symplectic structure ω. We follow [21] in this section.

Let $i_\partial : \partial M \to M$ be the inclusion map, and $\mu_{\partial M}$ the boundary measure induced from a (possibly Berezin) measure μ by i_∂. $(i_\partial \times \mathrm{id})^*$ is the restriction of the bulk graded differential forms on the mapping space to the boundary.μ_* is induced from $p_!$ in (3.3.1), with μ as its measure. Take $\omega = -d\vartheta$ and Θ the cohomological function. With the identification of Θ with the target space part of cocycle π, immediately we have $S = S_0 + S_1 = \iota_v \mu_* \mathrm{ev}^* \vartheta + \mu_* \mathrm{ev}^* \Theta$, and

Theorem 4.1.1. *If manifold M is open, i.e. $\partial M \neq 0$. Consider*

$$S = S_0 + S_1 = \iota_v \mu_* \mathrm{ev}^* \vartheta + \mu_* \mathrm{ev}^* \Theta \tag{4.1.1}$$

The classical master equation $\{S, S\} = 0$ requires

$$\iota_{v\mu_{\partial M *}} (i_\partial \times \mathrm{id})^* \mathrm{ev}^* \vartheta + \mu_{\partial X *} (i_\partial \times \mathrm{id})^* \mathrm{ev}^* \Theta = 0$$

The two terms must vanish independently if the consistency with the variational principle of a field theory is considered. Denote local coordinates by

- $p_{a(n-i)}$ of grade $n-i$, for $\lfloor n/2 \rfloor < i \leq n$
- $q^{a(i)}$ of grade i, for $0 \leq i \leq \lfloor n/2 \rfloor$

Then the symplectic form in a Darboux coordinates with (p, q) can be written as,

$$\omega = \sum_{i=0}^{\lfloor n/2 \rfloor} \int_M d^{n+1}\sigma d^{n+1}\theta (-1)^{ni} dq^{a(i)} \wedge dp_{a(i)}$$

with (σ, θ) the local coordinate on the worldvolume, of grade $(0,1)$ respectively.

The kinetic term in a AKSZ sigma model reads

$$S_0 = \int_M d^{n+1}\sigma d^{n+1}\theta \sum_{0\leq i\leq \lfloor n/2 \rfloor} (-1)^{n+1-i} p_{a(i)} dq^{a(i)}$$

Upon taking variation and setting it to zero, the boundary integration of the variation of the total action should vanish for consistency

$$\delta S|_{\partial M} \int_{\partial M} d^n\sigma d^n\theta \sum_{0\leq i\leq \lfloor n/2 \rfloor} (-1)^{n+1-i} p_{a(i)} \delta q^{a(i)} = 0$$

which imposes the boundary conditions $p_{a(i)} = 0$ or $\delta q^{a(i)=0}$ on ∂M, so the image of the boundary must lie in a Lagrangian submanifold $L \in E$. Under this condition the first term in $\{S, S\}$ (appearing in (4.1.1)) vanishes, and the theorem could be reduced to a simpler form

Theorem 4.1.2. *Let L be a Lagrangian submanifold of E (then it follows that $\vartheta|_L = 0$), then $\{S, S\} = 0$ if $\Theta|_L = 0$.*

4.2 Examples

The AKSZ procedure introduced above is heavily abstract, so albeit clear in the concept and in the geometric meaning, it is not very helpful for practical situations. Some examples should be presented.

4.2.1 Courant sigma model and ordinary 3d Chern-Simons theory

In the case of $n = 2$, the target is a symplectic Lie 2-algebroid, which is equivalently a *Courant algebroid*. Courant algebroids can be defined in the language of ordinary differential geometry.

Definition 4.2.1. *A (general) Courant algebroid is a vector bundle $E \to M$, equipped with*

- *a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E*
- *a bilinear operation $\circ$ on $\Gamma(E)$*
- *an anchor map $\rho : E \to TM$*

satisfying axioms, for $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$

1. *Jacobi identity for the operation, $e \circ (e_1 \circ e_2) = (e \circ e_1) \circ e_2 + e_1 \circ (e \circ e_2)$*
2. *Leibniz rule for the operation, $e_1 \circ (fe_2) = f(e_1 \circ e_2) + (\rho(e_1) \cdot f)e_2$*
3. *$\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$*
4. *$e_1 \circ e_2 = \frac{1}{2}\mathcal{D}\langle e_1, e_2\rangle$, where $\mathcal{D}$ is defined as $\langle \mathcal{D}f, e\rangle = \rho(e)f$ (explicitly $\mathcal{D} = \rho^* d : C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{\rho^*} E^* \simeq E$)*

5. $\rho(e)\langle e_1, e_2\rangle = \langle e \circ e_1, e_2\rangle + \langle e_1, e \circ e_2\rangle$

Important examples of Courant algebroids are *exact Courant algebroid*, with $E = TM \oplus T^*M$. Let $X, Y \in TM$ and $\alpha, \beta \in T^*M$. Let $X + \alpha$ denote a formal sum, which in some literature is expressed through (X, α), a notation we will not adopt.

- The bilinear operation $\circ$ is now called a *Dorfman bracket*, $(X + \alpha) \circ (Y + \beta) = [X + \alpha, Y + \beta]_D = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha$ which is in general not antisymmetric.
- The symmetric bilinear form is now $\langle X + \alpha, Y + \beta\rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha)$
- The anchor map is now the natural projecion to the tangent bundle $\rho(X + \alpha) = X$

The qualifier 'exact' comes from the short exact sequence

$$0 \to T^*M \xrightarrow{\rho^{\mathrm{T}}} E \xrightarrow{\rho} TM \to 0$$

Later it will be shown that this structure underlies the $O(d, d)$-generalized geometry.

The corresponding AKSZ σ-model for a general Courant algebroid is called the Courant sigma model. To accomodate the Courant algebroid to a symplectic dg-manifold, one does the identification $\mathcal{M} = T^*[2]T[1]M$, equipped with its canonical grade 2 symplectic structure ω. Now choose a Darboux coordinate $\{q^i, \xi^a, p_i\}$, with their degree being respectively $\{0, 1, 2\}$, such that

$$\omega = \mathbf{d}p_i \wedge \mathbf{d}q^i + \frac{1}{2} g_{ab} \mathbf{d}\xi^a \wedge \mathbf{d}\xi^b$$

g_{ab} are some constant symmetric matrices. Corresponding to the Chevalley-Eilenberg differential and hence the cohomological vector field, the most general form for the grade $2 + 1 = 3$ cohomological function Θ should be

$$\Theta = P^i_a(q)\xi^a p_i + \frac{1}{3!} T_{abc}(q)\xi^a \xi^b \xi^c$$

The equation $v^2 = 0$ translates to $\{\Theta, \Theta\}_P = 0$, which imposes the following constraints on the grade 0 functions

$$\begin{aligned}
0 =& g^{ab} P^i_a P^j_b = 0 \\
0 =& P^j_c \partial_{q^j} P^i_b - P^j_b \partial_{q^j} P^i_c + g^{ez} P^i_e T_{zbc} \\
0 =& P^i_d \partial_{q^i} T_{abc} - P^i_c \partial_{q^i} T_{dab} + P^i_b \partial_{q^i} T_{cda} - P^i_a \partial_{q^i} T_{bcd} \\
&+ g^{ef}(T_{eab} T_{cdf} + T_{eac} T_{dbf} + T_{ead} T_{bcf})
\end{aligned}$$

Hence the cohomological vector field is

$$v = P^i_a \xi^a \partial_{q^i} + g^{ab}(P^i_b p_i - \frac{1}{2} T_{bcd} \xi^c \xi^d)\partial_{\xi^a} + \left(-\partial_{q^i} P^j_a \xi^a p_j + \frac{1}{6}\partial_{q^i} T_{abc} \xi^a \xi^b \xi^c\right)\partial_{p_i}$$

Translating to the Weil differential, we get

$$\begin{aligned}
d_{\mathrm{W}} q^i &= P^i_a \xi^a + \mathbf{d}q^i \\
d_{\mathrm{W}} \xi^a &= g^{ab}(P^i_b p_i - \frac{1}{2} T_{bcd} \xi^c \xi^d) + \mathbf{d}\xi^a \\
d_{\mathrm{W}} p_i &= -\partial_{q^i} P^j_a \xi^a p_j + \frac{1}{6}\partial_{q^i} T_{abc} \xi^a \xi^b \xi^c + \mathbf{d}p_i
\end{aligned}$$

The canonical cocycle π is then

$$\pi = \frac{1}{n+1}\omega_{ab}\deg(x^a)x^a \wedge v^b = P^i_a\xi^a p_i - \frac{1}{6}T_{abc}\xi^a\xi^b\xi^c$$

Finally the Chern-Simons element is

$$\begin{aligned}\mathrm{cs} &= \frac{1}{2}\left(\sum_{a,b}\deg(x^a)\omega_{ab}x^a \wedge d_{\mathrm{W}}x^b - 2\pi\right) \\ &= p_i d_{\mathrm{W}}q^i + \frac{1}{2}g_{ab}d_{\mathrm{W}}\xi^b - P^i_a\xi^a p_i + \frac{1}{6}T_{abc}\xi^a\xi^b\xi^c\end{aligned}$$

Also the Dorfman derivative and anchor maps translates to the derived bracket.

Now if there are three symplectic Lie 2-algebroid valued degree-1 differential form (X, A, F), and for simplicity we assume that the target manifold is closed, then

$$S_{\mathrm{AKSZ}} = \int_\Sigma F_i \wedge d_{dR}X^i + \frac{1}{2}A^a d_{dR}A^b - P^i_a A^a F_i + \frac{1}{6}T_{abc}A^a A^b A^c$$

If $M = \mathfrak{g}$ a Lie algebra, so there are no q^i's and p^i's, and hence no X and F. T and g becomes constant. Courant sigma model becomes Chern-Simons theory

$$S = \int_{\mathfrak{g}} \frac{1}{2}g_{ab}A^a d_{dR}A^b + \frac{1}{6}T_{abc}A^a A^b A^c$$

4.2.2 String sigma model in the case of exact Courant algebroids

Following [27], we show that Polyakov action emerges out of the boundary of σ-model defined on an exact Courant algebroid. If the algebroid is exact, and in local coordinates $(x^i, \xi^i = dx^i, \pi_i, p_i)$, we have $\omega = dp_i dx^i + d\pi_i d\xi^i$ and $\Theta = p_i\xi^i - \frac{1}{6}\eta_{ijk}(k)\xi^i\xi^j\xi^k$. After integration by parts the Courant sigma model action becomes

$$S = \int_M \left[p_i(dx^i - \xi^i) + \pi_i d\xi^i + \frac{1}{6}\eta_{ijk}(x)\xi^i\xi^j\xi^k\right] + \frac{1}{2}\int_{\partial M}\pi_i\xi^i$$

The equation of motions of p_i and π_i shows that in the bulk integral they are Lagrange multipliers imposing $\xi^i = dx^i$ and $d\xi^i = 0$, hence there is a map $f : X \to M$ such that the bulk integral is $\int_X f^*\eta$, with components x^i.

Now impose a boundary condition given by a Lagrangian submanifold, on which the boundary term of the variation of the action functional δS vanishes, or is exact. With a linear transformation $h : E \to E$ such that

$$h^2 = 1, \langle hV, hW\rangle = \langle V, W\rangle, \mathrm{Tr}h = 0, \langle V, hV\rangle > 0 \ (\forall 0 \neq V \in \Gamma(E))$$

introduced, the boundary condition

$$\begin{aligned}(\pi_i)_+ &= h_{ij}(\xi^i)_+ \\ (\pi_i)_- &= -h_{ji}(\xi^i)_-\end{aligned}$$

will do the work, and the boundary part becomes

$$S_{\partial M} = \int h_{ij}\partial_+ f^i\partial_- f^j$$

which is precisely the Polyakov action. With the bulk part present, the σ-model is twisted by the closed 3-form η,

$$S_{\partial M} = \int_{\partial M} h_{ij}\partial_+ f^i \partial_- f^j + \int_M f^*\eta$$

which on the boundary becomes B if $B = d\eta$, hence the nonlinear σ-model.

4.3 Exact Courant Algebroids and Variational Problems

From the emergence of Polaykov action out of the boundary of σ-model defined on an exact Courant algebroid one is inclined to suspect that exact Courant algebroids are strongly related to string theory. Now it will be shown that they indeed appear naturally in 2-dimensional variational problems [28].

On a manifold M, the trajectory of a propagating 1-dimensional object is a surface. To get the stationary surface the 'volume' of this surface, a 2-form ω, is integrated. Then one follows the standard procedure.

The Noether's theorem can be translated to the statement that if v is a vector field preserving ω, $\mathcal{L}_v\omega = 0$, then (from Cartan's magic formula, $\mathcal{L}_v\omega = d(\iota_v\omega) + \iota_v d\omega$, the second term vanishes since the integrated ω is a top form) on extremal surfaces $\iota_v\omega$ is closed. Also note that for θ a 1-form, $\mathcal{L}_v\omega + d\theta = 0$ implies the conservation of $\iota_v\omega + \theta$ as well. Hence (v, θ), $v \in \Gamma(TM)$ and $\theta \in \Gamma(T^*M)$ are conserved, thence the formal sum $v + \theta \in \Gamma(E = TM \oplus T^*M)$ is conserved.

Take ω and induce a map $\omega : TM \to T^*M$ for $x \in M$, then the graph of ω induces a subbundle $D \subset E$. It is clear that $v+\theta$ preserves D if and only if $\mathcal{L}_v\omega + d\theta = 0$, with the corresponding 'generalized Lie derivative' being the Dorfman derivative, *i.e.*,

$$\mathcal{L}^D_{v_1+\theta_1}(v_2 + \theta_2) = \mathcal{L}_{v_1}(v_2 + \theta_2) - \iota_{v_2} d\theta_1 \tag{4.3.1}$$

Together with the natural projection $\rho : E \to TM$ and the inner product (nondegenerate symmetric bilinear mentioned before) $\langle v_1 + \theta_1, v_2 + \theta_2\rangle = \frac{1}{2}(\iota_{v_1}\theta_2 + \iota_{v_2}\theta_1)$, the algebroid strucure emerges. Also $\omega \mapsto \omega + \beta$ with $\beta \in \Omega^1(M)$ closed induces an automorphism (called *B-transformation*) $v + \theta \mapsto v + \theta + \iota_v\beta$ of E, meaning that what is of importance is the exact sequence $0 \to T^*M \to E \to TM \to 0$ and an isotropic splitting of E. For later use we also introduce the *Courant bracket*, which is the antisymmetrization of the Dorfman bracket

$$[v_1 + \theta_1, v_2 + \theta_2]_D = [v_1, v_2] + \mathcal{L}_{v_1}\theta_2 - \mathcal{L}_{v_2}\theta_1 - \frac{1}{2}d(\iota_{v_1}\theta_2 - \iota_{v_2}\theta_1) \tag{4.3.2}$$

We will find the WZW terms of M-branes on the boundary of some certain algebroids, to get a feeling about what is going on, we study D-branes. It is well-known that the open strings with Dirichlet boundary conditions ends on D-branes, now it will be demonstrated that D-branes are *Dirac structures*, which we introduce now, in the exact Courant algebroid. The constrain for open strings can be expressed as

$$\iota_v\omega = 0$$

for v tangent to the boundary of the strings. The graph of the splitting ω_x is by definition involutive, i.e., closed under Dorfman derivative. The graph of $\iota_v\omega = 0$ for every x defines an isotropic subbundle of the bundle E, which is also automatically maximally isotropic. This defines a Dirac structure

Definition 4.3.1. *A maximally isotropic involutive subbundle L of an exact Courant algebroid E is called a Dirac structure. Where*

- *Isotropic:* $\langle \cdot, \cdot \rangle = 0$ *for all sections of L*
- *Involutive:* $[\cdot, \cdot]_D \in \Gamma(L)$ *for all sections of L*
- *Maximal:* $\dim(L) = \frac{1}{2}\dim(E)$

hence the identification of D-branes with Dirac structures in the exact Courant algebroid case [28, 29]. We also introduce the Dirac structures of general Courant algebroids.

Definition 4.3.2. *A maximally isotropic involutive subbundle L of an exact Courant algebroid E is called a Dirac structure whose sectons are involutive with respect to the Dorfman derivative.*

Note that maximal no longer means $\dim(L) = \frac{1}{2}\dim(E)$. After the identification of a Courant algebroid with a symplectic dg-manifold of grade 2, the Dirac structure translates to the Lagrangian submanifold of the latter. See [30], Section 4.

4.4 Higher Courant Algebroid, Variational Problems

Since (exact) Courant algebroid structure emerges naturally in 2 dimensional variational problems, it is natural to study the case of $p+1$-dimensional objects propagating. The argument for the variational problem is completely analogous. Having considered the case of exact Courant algebroid $E_1 = TM \oplus T^*M$ one only needs to consider its higher generalization, *viz.*, those isomorphic to $E_p = TM \oplus \wedge^p T^*M$, with $p = 1$ being the special case of the exact Courant algebroid. Sections of the vector bundle E_p correspond to a formal sum of vectors and p-forms, that is, $X + \eta \in \Gamma(E_p)$, $X \in \Gamma(TM)$ and $\eta \in \Gamma(\wedge^p T^*M)$.

The non-degenerate symmetric bilinear form is now the fibre pairing $\langle \cdot, \cdot \rangle : E_p \times E_p \to \wedge^{p-1} T^*M$ given by a symmetrization of contractions between vectors and $p-$forms $\langle X + \eta, Y + \xi \rangle = \frac{1}{2}(\iota_X \xi + \iota_Y \eta)$.

The higher Dorfman bracket, or the generalized Lie derivative when thinking geometrically, is now defined as

$$[X + \eta, Y + \xi]_D = [X, Y] + \mathcal{L}_X \xi - \iota_Y d\eta$$

so essentially there is no change. Now we show that all these things can be re-expressed in the language of σ-models whose worldvolumes Σ's are tangent Lie algebroids and the target spaces E's are symplectic dg-manifolds [30].

Traditionally a Lagrangian assigns a density on Σ to any map $\Sigma \to M$, hence it lifts $\Sigma \to M$ to $\Omega^\bullet \Sigma \simeq T[1]\Sigma \to T[1]M \times \mathbb{R}[p]$, *i.e.*, at a specific $x \in \Sigma$ a $\wedge^n T_x^*\Sigma$-valued function on the space of linear maps $T_x\Sigma \to TM$. Now since a Lagrangian is defined naturally only up to a closed n-form $\in \Omega^\bullet(M)$, the lift should more appropriately be from $\Sigma \to M$ to $\Omega^\bullet \Sigma \simeq T[1]\Sigma \to E$, where $E \to T[1]M$ is a principal $\mathbb{R}[n]-$bundle. In other words a Lagrangian at x is an element of $\Gamma(\wedge^p T^*\Sigma, U_x)$-bundle

$\mathrm{Maps}(T_x[1]\Sigma, E) \to \mathrm{Maps}(T_x[1]\Sigma, T[1]M)$. Finally, upon re-defining Lagrangians as Legendrian submanifolds in the space of 1-jets of sections of the bundle $j^1\mathrm{Maps}(T_x[1]\Sigma, E)$ [1], the natural isomorphism leads to $j^1\mathrm{Maps}(T_x[1]\Sigma, E) \simeq \mathrm{Maps}(T_x[1]\Sigma, j^1E)$. Setting $X = T^*[n]T[1]M$ and $\tilde{X} = X \times \mathbb{R}[n]$ results in the identification $\tilde{X} = j^1E$, meaning that $T[1]\Sigma \to X = T^*[n]T[1]M$ defines an Hamiltonian mechanics of $n-$dimensional objects.

It was shown by Zambon in [32] that $TM \oplus \wedge^p T^*M$ is naturally embedded into $T^*[p]T[1]M$ by degree shifting. In [33], a 1-1 correspondence between symplectic dg-manifolds and higher Courant algebroids $TM \oplus \wedge^p T^*M$ is found, which will not be demonstrated here to avoid descending endlessly into details. However, it should be pointed out that the embedding is such that the Hamiltonian or Lagrangian mechanics (which are virtually identified in the above discussion) or more appropriately the dynamics and (continuous) symmetries of σ-models with $T^*[p]T[1]M$ as target spaces are encoded in the axioms of these higher Courant algebroids.

Now notice that the definition of Hamiltonian mechanics with $T[1]\Sigma \to X = T^*[n]T[1]M$ is familiar: X, being a dg-manifold with the naturally equipped symplectic form of grade $n+1$ originating from the associated variational problem, is in fact a symplectic dg-manifold. From the discussion of AKSZ σ-model construction from the perspective of (higher) Chern-Weil theory, the σ-models constructed in this procedure are characteristic classes associated to symplectic dg-manifolds, hence the name 'AKSZ topological theories' in the literature: These σ-models are not really 'topological' when the geometry and topology is of the target spacetime manifold, the body M, but are indeed 'topological' when considering the dg-manifold $E \to M$ associated to M, though the meaning of the word 'topological' itself in the physics literature is rather vague.

4.5 Lagrangian submanifolds and Current algebra

The Lagrangian submanifolds of the targets have more interesting properties, which we discuss now.

The setting for the variational problem of p-dimensional extended objects leads to the Hamiltonian formalism for σ-models with worldvolume being a tangent Lie algebroid and the target space being a symplectic dg manifold. In [15] a unified formulation of the current algebra theory in this setting is proposed. There, in the Corollary 4.3 and below, it is proven that in a Lagrangian submanifold of the target, the anomalous terms in the current algebra vanish before twists[2].

We give a very rough explanation, or rather speculation, in physics perspective. The action functional of AKSZ $\sigma-$models are (transgressions of) characteristic classes canonically associated to symplectic dg-manifolds, they encodes the topology of the manifolds, corresponding to some dynamics and symmetries of the physical models. Anomalies in quantum field theories, are the consequences of being able to detect the global geometry and in particular topology after quantization. To avoid anomalies, one should restrict to those submanifolds where symmetries are always preserved.

In the particular example of exact Courant algebroid encoding the dynamics, the analogue of Lagrangian submanifolds, Dirac structures, are those preserved by Dorfman derivative under the flow of a conserved current. When the theory is reduced to a boundary, which also is a submanifold, to preserve the symmetry the Dirac structure itself should

[1] See [31], example 1.11.

[2] See there for the notion of twists in this context.

be preserved. Anomally cancellation is indeed the preservation of symmetry, which are fully encoded by the exact Courant algebroid structure, hence the choice of Dirac structures is the choice of submanifolds where there are no anomalies in the sectors that are captured.

We can define higher analogues of Dirac structures [32–34]. There are different notions of higher Dirac structures, we might adopt the notion defined in [33] to fit into the dg-manifold picture. In analogy to the discussion of Dirac structures as D-branes, the Lagrangian submanifolds of the targets, corresponding to the higher Dirac structures, again might be the space where D-branes live, possibly with the background gauge field changed. The characteristic classes encode some information about the topology of the manifolds, hence the coincidence [12] between the requirement that the boundary submanifold of AKSZ type σ-models to be Lagrangian and the choice of isotropic involutive subbundle[3] for the anomalous terms of current algebras of topological p-branes to vanish. The relations between these different notions of 'generalized' or 'higher' Dirac structures are not clear yet.

It can be speculated that, as in the case of the requirement that should be satisfied for D-branes to live in Lagrangian submanifolds, if the algebroid structure embedded into the symplectic dg-manifold possesses a surjective anchor $\rho : E \to TM$, the algebroid possesses a structure analogous to the Dirac structure, and the restriction to the Lagrangian submanifold is equivalent to the restriction to the subbundles that are preserved under the Dorfman derivative. It is also worth mentioning that this can be related to [13].

Summing up these observations, we propose a speculation that there is a three-fold coincidence associated to the Lagrangian submanifolds of the target, which might be reinterpreted as a 'generalized Dirac structure' from the perspective of algebroid or generalized geometry[4]

- After embedding a transitive (equipped with a surjective anchor $\rho : E \to TM$) algebroid structure inside a symplectic dg-manifold, the Lagrangian submanifolds of the latter are where D-branes live in the variational problem associated. If the algebroid is a higher Courant algebroid isomorphic to $TM \oplus \wedge^n T^*M$, this corresponds to the higher Dirac structures.

- For consistency, the boundary of a AKSZ σ-model must lie in a Lagrangian submanifold.

- The current algebra is anomaly-free when restricted to a Lagrangian submanifold.

where only the second statement is a theorem.

[3] See [12], around equation 3.14, for the definition in this particular case.

[4] These generalized Dirac structure might be defined in the approach of [33].

Chapter 5

Discussion: Bulk-Boundary duality and p-brane actions

Let us review what we learned,

- Courant algebroids and their higher analogous algebroids encodes the dynamics of p-dimensional objects in their axioms.
- The framework of higher-dimensional Hamiltonian (or Lagrangian, since they are essentially identified) mechanics in the language of symplectic dg-manifolds accomodates these geometrical structures into the target space.
- The AKSZ formalism provides a way to construct $\sigma-$models associated to the characteristic classes of the target symplectic dg-manifolds in the special and simple case when the underlying higher bundle structures are trivial[1].
- On the boundary, AKSZ formalism forces the boundary submanifolds to be lying in a Lagrangian submanifold.
- The Dirac structures, or the involutive Lagrangian subbundles, in higher Courant algebroids are D-branes.
- More generally, if the embedded algebroid structure in the symplectic dg-manifold is equipped with a surjective anchor $\rho : E \to TM$, the Lagrangian submanifolds of the dg-manifold is where D-brane lives.

From these phenomena we come up with a strategy for the construction of a Dp-brane action functional, which is a $p+1$-dimensional object, or at least its topological sector:

- One constructs a $p+2$-dimensional $\sigma-$model through the AKSZ procedure.
- The body of the target symplectic dg-manifold should bea manifold with boundary.
- On the boundary one obtains the topological sector of a Dp-brane action.
- The dynamics and symmetries should be encoded in some kind of algebroid that is expected to be embedded into the target symplectic dg-manifold.

The strategy looks reckless, since the words 'topological sector', 'dynamics' are all vague. We discuss some more phenomena to argue for the validness of the strategy, and shed some light on what we are actually going to do.

[1]See [9], Section 5.

5.1 Green-Schwarz action functional for p-branes

The Green-Schwarz action functional is an action functional for a σ-model describing the propagation of a super p-brane in the spacetime. It exhibits manifest spacetime supersymmetry. The basic idea is the construction of thr action functional as a propagating extended object, *i.e.*

$$S_{\text{kin}} = \int_{\Sigma} \text{vol}_{\phi^* g}$$

where g is the metric of the target space, and Σ is the worldvolume. Its variational problem then leads us directly to the Hamiltonian formalism with symplectic dg-manifold as the underlying language that we discussed in Section 4.4. However, with this action functional, which has only a kinetic term, the dynamics derived are not equivalent to that of the RNS formalism. Green and Schwarz observed that with the addition of extra term, called Wess-Zumino term S_{WZW}, the dynamics of the full action

$$S_{GS} = S_{\text{kin}} + S_{\text{WZW}}$$

becomes, at least classically, equivalent to that of the RNS superstring. In the view of our Hamiltonian formalism, it is similar to the case where the underlying algebroid is twisted by a flux.

Moreover, the Green-Schwarz action functional can be seen as a Wess-Zumino-Witten (WZW) model [10], with the identifications

- The target space being locally super Minkowski spacetime.
- The WZW term is 'topological', in the sense that it is a local potential for some super Lie algebra cocycle.

What is crucial is that

1. The WZW terms are local potentials, which means they should be seen as boundary modes if the target space is with boundary.
2. The WZW terms, as local potentials for some super Lie algebra cocycle encoding the gauge field degrees of freedom. It should be well defined under changes of boundary. See [35] for details.
3. Green-Schwarz action functionals are some kind of higher WZW model [36], just like the higher Chern-Simons theories constructed through AKSZ procedure are some kind of Chern-Simons theory, which will lead us to speculate some kind of holographic relation between a d-dimensional Green-Schwarz action functional and a $(d+1)$-dimensional higher Chern-Simons theory.

5.2 AdS/CFT and CS-WZW correspondence

Firstly, there is a well known correspondence between 3d Chern-Simons theory as the bulk field theory and the 2d Wess-Zumino-Witten model as the boundary field theory on a suitable Lie group. The space of quantum states of Chern-Simons theory on a surface Σ are the space of conformal blocks of the WZW model [26].

Also, 3-dimensional gravity theory, whose Einstein-Hilbert action with cosmological constant Λ is

$$S_{\text{EH}} = \int_M \sqrt{g}(R - 2\Lambda)$$

looks very similar to a Chern-Simons theory in its Cartan-Palatin (or first-order) formulation [37], namely

$$S = \int_M \text{Tr}\left(e \wedge F + \frac{\Lambda}{3} e \wedge e \wedge e\right)$$

Where A is an $SU(2)$ gauge connection, whose curvature is F and $e \in \Omega^1(M)$ is an $SU(2)-$valued 1-form. After constructing a metric with

$$g_{ab} = -\frac{1}{2}\text{Tr}(e_a e_b)$$

then e^i can be interpreted as the vielbein and A the corresponding spin connection. A similar reformulation in terms of Chern-Simons gauge theory exists, in which the action is now

$$S = \int_M \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

The Lie algebra where the gauge connection A take values depends on the value of the cosmological constant (the groups are $SL(2, \mathbb{C}), ISO(3), SU(2) \times SU(2)$ for $\Lambda < 0, = 0, > 0$, repectively), and an be viewed as the isometry group of the underlying geometric structure. It is a well-known conjecture that SUGRA on a 3-dimensional anti de-Sitter space is in holographic dual to a conformal field theory on its boundary, $\text{AdS}_3/\text{CFT}_2$.

The AdS/CFT picture extends to higher dimensions, compactifying the 11D SUGRA on a 4-sphere to get a 7-dimensional theory (AdS_7), there a 7-dimensional Chern-Simons term arises, then there can be a correspondence between this term and the conformal field theory in 6 dimensions (CFT_6). In [38], it was argued that the conformal blocks of the 6d conformal field theory in AdS/CFT are given by the Chern-Simons sector of the dual gravity in the AdS side. Then the worldvolume theory of M5-branes is a superconformal theory in 6 dimensions, its conformal blocks should be related to some kind of 7-dimensional Chern-Simons theory.

There seems to be some kind of duality between n-dimensional bulk *topological* field theory and its boundary theory which might be a conformal field theory, or the conformal blocks of the CFT, or a *topological conformal field theory* (TCFT). Note that it is known that the chiral part of any 2d conformal field theory defines a 3d topological quantum field theory [39].

Finally, there are concrete examples to support the speculation

- We have seen that a Courant σ-model on the boundary becomes the Polyakov action, which *is* a conformal field theory.

- And upon a twist by a closed 3-form the Polyakov action becomes a nonlinear σ-model, which will become a WZW model on a group manifold.

- Courant σ-model with various additional structure reduce to topological A on its boundaries, which are topologically twisted superconformal field theories. See [40].

Now the question becomes: how should one encode the dynamics of a M5-brane, and in what kind of algebroid? The dynamics of a plain propagating string is encoded by a Courant algebroid, but what about an M-theory brane? The answer is given by the notion of *exceptional generalized geometry*.

Chapter 6

Exceptional Generalized Geometry and 11D SUGRA

6.1 String Dualities preliminary

While there are many different perspectives whence one can view the subject of generalized geometry, we will start from the geometralization of various string dualities [7]. And indeed T-duality, S-duality and U-duality are correlated to the gauge structure and the dynamics of superstring (and SUGRA) theories, since as it will be shown, upon geometrizing the various string dualities the algebroid structrues that controls the dynamics and symmetries of the theories naturally emerges. This is remarkable since various different string theories are related through a web of dualities, which means that the algebroid structures and hence the dynamics and symmetries of the theories are correlated. Unifying these dualities in the Kaluza-Klein apporach, one is led to 11D SUGRA which is also a 'maximal' supergravity theory when the assumption that there the allowed amount of time directions is one, suggesting that the corresponding string (or membrane) theory of 11D SUGRA is the unifying theory of strings. The dualities, will be the guiding principle in the search of proper geometries that encode the dynamics and symmetries of various string theories. We follow [41] in this section.

6.1.1 T-duality

When a string is quantized with one spatial direction on S^1 of radius R, along the circle the string has quantised momentum states $p = n/R, n \in \mathbb{Z}$ and winding modes characterized by $w \in \mathbb{Z}$. The mass squared of a string state with momentum n/R and winding w, restricted to the circle, is

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{l_s^2}\right)^2 + ...$$

where l_s is the string length. There is a transformation that leaves the spectrum of the string unchanged, given by

$$R \mapsto R' = \frac{l_s^2}{R}$$
$$(n, w) \mapsto (n', w') = (w, n)$$

which interchanges the momentum and winding modes and inverting the radius of the S^1. This is called a *T-duality transformation*. Now when the compact space is a d-dimensional torus, there are d winding numbers w^i and d momenta associated to numbers n_i. The tuple in the T-duality transformation becomes now $2d$-tuple, $(n_i, w^i) \equiv p_M$ with $M = 1, ..., 2d$. Suppose that the torus has a constant metric g_{ij} and a 2-form field B_{ij}, the mass squared is now

$$M^2 = \frac{1}{l_s^2} p_M H^{MN} p_N + ...$$

with the level-matching condition

$$N_R - N_L = \frac{1}{2}\eta^{MN} p_M p_N \tag{6.1.1}$$

H^{MN} is a $2d \times 2d$ matrix

$$H^{MN} = \begin{pmatrix} l_s^2 g^{ij} & -g^{ik}B_{kj} \\ B_{ik}g^{kj} & \frac{1}{l_s^2}\left(g_{ij} - B_{ik}g^{kl}B_{lj}\right) \end{pmatrix} \tag{6.1.2}$$

and

$$\eta^{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

There are $O(d, d; \mathbb{Z})$ transformations, the *generalized T-duality transformations*, that leaves η invariant

$$\eta^{MN} \mapsto \eta'^{MN} = U^M{}_K U^N{}_L \eta^{KL} = \eta^{MN}$$

leaving these formula invariant, *i.e.*,

$$p_M \mapsto p'_M = (U^{-1})^K{}_M p_K$$
$$H^{MN} \mapsto H'^{MN} = U^M{}_K U^N{}_L H^{KL}$$

The inverse of η^{MN}, η_{MN}, is called the $O(d, d)$ *metric*, and can be identified with the nondegenerate symmetric bilinear form of an exact Courant algebroid $\langle \cdot, \cdot \rangle$ when the formal sum $v + \theta$ is split into (v, θ). The H matrix encoding the metric and B-field, obeys

$$H^{MK} H^{NL} \eta_{KL} = \eta^{MN}$$

The matrix H^{MN} itself defines an element of $O(d, d; \mathbb{R})$, its inverse H_{MN} is called the *generalized metric*.

B and g states are common to bosonic and type II strings, and they exhibit T-dualities. Compactification of these theories on a d-torus thence leads to the $O(d, d; \mathbb{Z})$ duality symmetry introduced above. When $\det(U)$ is minus one, the transformation exchanges the type IIA theory on one torus with the type IIB theory on a dual torus, while with unit determinant the duality symmetry acts within IIA or IIB.

In the low energy limit, the effective actions are type II supergravities for superstrings, when KK-reduced on a T^d, the reduced theory exhibits a global $O(d, d; \mathbb{R})$ symmetry.

6.1.2 S-duality

For type IIB superstring there is a non-perturbative duality that relates the type IIB at strong coupling to the weak coupling, the S-duality transformation under which the theory is invariant is

$$g_s \mapsto \frac{1}{g_s}$$

In the type IIB SUGRA, the S-duality appears as a classical global $SL(2;\mathbb{R})$ symmetry (while in the superstring it is an action of $SL(2;\mathbb{Z})$ on the BPS states). In type IIB SUGRA, we have dilaton, RR-sector p-forms, of even rank, C_0, C_2, C_4, the metric and NSNS 2-form. The dilaton Φ and RR 0-form can be combined into a complex scalar that transforms under $SL(2)$ via a modular transformation

$$\tau = C_0 + ie^{-\Phi}$$
$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

The transformation can be expressed as a unit determinant matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$$

or in terms of a matrix

$$M = \frac{1}{\text{Im } \tau} \begin{pmatrix} |\tau|^2 & \text{Re } \tau \\ \text{Re}\tau & 1 \end{pmatrix} = e^{\Phi} \begin{pmatrix} C_0^2 + e^{-2\Phi} & C_0 \\ C_0 & 1 \end{pmatrix}$$

which transforms as $M \mapsto UMU^T$. With the choice of

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the transformation becomes the original transformation that inverts the coupling constant. The doublet on which these transformations act is (C_2, B_2), formed by the two 2-forms,

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \mapsto U^{-T} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}$$

while the 4-form C_4 and the Einstein frame metric (g^{E} such that $g_{\mu\nu} = e^{\Phi/2} g^{\text{E}}_{\mu\nu}$, which is the metric that appears naturally in the string σ-model with no dilaton factors) is invariant.

6.1.3 U-duality

S- and T-duality transformation do not commute, and their combinations generate a larger group of dualities of type II theories, the U-duality. U-duality is a non-perturbative duality of the type II string on a torus. The strong coupling limit of the type IIA string is the conjectured M-theory, which is 11-dimensional since as the type IIA string coupling goes to infinity an eleventh dimension decompactifies. The compactification of the 11D SUGRA on T^2 can be mapped to the reduction of the type IIB SUGRA on a circle, and its Kaluza-Klein reduction on S^1 leads to the type IIA SUGRA. The U-duality is then also a duality of M-theory.

The 11-dimensional radius R_{11} and Planck length l_p are related to the 10-dimensional string coupling constant g_s and string length l_s by

$$R_{11} = l_s g_s \tag{6.1.3}$$
$$l_p = g_s^{1/3} l_s \tag{6.1.4}$$

corresponding to a reduction ansatz for the 11-dimensional metric of the form

$$\frac{1}{ds_{11}^2} = e^{4\Phi/3}(dx^{11} + A)^2 + e^{-2\Phi/3}\frac{ds_{10}^2}{l_s^2}$$

Generating 11-dimensional U-duality transformations is to combine the two transformations

$$T_i : R_i \mapsto \frac{l_s^2}{R_i}, \quad g_s \mapsto g_s\frac{l_s}{R_i}$$
$$S : g_s \mapsto \frac{1}{g_s}, \quad l_s^2 \mapsto g_s l_s^2$$

where i is the i-th direction, and even number of T-duality transformations should be considered as the relation (6.1.3) holds only for IIA string variables. The uplifting of this to a symmetrical U-duality transformation is natural, for example on 3 directions, on the 11-dimensional Planck length, with (I, J, K) distinct and be any of the indices $(i, j, 11)$,

$$U_{IJK} : R_I \mapsto \frac{l_p^3}{R_J R_K}, \quad l_p^3 \mapsto \frac{l_p^6}{R_I R_J R_K}$$

On the T^3 with the definition of $V = R_I R_J R_K$, the U-duality transformation can be seen as a volume inversion, generalizing the radius inversion of the basic T-duality. The dual volume and the dual Planck length denoted V', l_p', we have $V'/l_p^3 = l_p^3/V$. The full U-duality group is generated by these transformations. The U-duality group just discussed is a special case where the compact dimension is 3. When the dimension of the compact dimension changes, the U-duality group changes accordingly.

6.2 Generalized Geometry

Now we consider the generalized geometry introduced by Hitchin and Gualtieri [5], and follow mainly [42]. Consider an exact Courant algebroid over d-dimensional manifold M, $E \to M$ now called *generalized tangent bundle*, isomorphic to $TM \oplus T^*M$ at any point on M, sections of which we call *generalized vectors*. The introduction of the generalized tangent bundle geometrize the gauge transformation of a two form. On a coordinate patch U_α, we have

$$V_{(\alpha)} = X_{(\alpha)} + \mu_{(\alpha)} \in \Gamma(E, U_\alpha).$$

At the intersection $U_\alpha \cap U_\beta$, the gluing condition reads

$$V_{(\alpha)} = X_{(\alpha)} + \mu_{(\alpha)} = g_{(\alpha\beta)}X_{(\beta)} + g^{-T}_{(\alpha\beta)}\mu_\beta - \iota_{g(\alpha\beta)X_\beta} d\Lambda_{(\alpha\beta)}$$

where $g_{(\alpha\beta)}$ is a transition function of $GL(d, \mathbb{R})$, and Λ is a 1-form gauge parameter. It should satisfy the folowing co-cycle condition on the overlap $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = -ih^{-1}_{(\alpha\beta\gamma)}dg_{(\alpha\beta\gamma)}$$

with $h \in U(1)$, satisfying another gluing condition for transition functions on quadruple overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$h_{(\beta\gamma\delta)}h^{-1}_{(\alpha\gamma\delta)}h_{(\alpha\beta\delta)}h^{-1}_{(\alpha\beta\gamma)} = 1$$

so there is a 2-form B transforming as

$$B_{(\alpha\beta)} \equiv B_{(\alpha)} - B_{(\beta)} = d\Lambda_{(\alpha\beta)}$$

generalizing the $U(1)$-connection of principal $U(1)$-bundle to a connection on a *gerbe*, and generalizing the connection over a fibre bundle to a higher rank form, which cannot appear in ordinary gauge theory. Now we are in fact dealing with *higher gauge theory*. The $O(d,d)$ metric introduced earlier in the context of analyzing T-duality is the nondegenerate symmetric bilinear form appeared when introducing exact Courant algebroids, *i.e.*,

$$\langle \cdot, \cdot \rangle = \eta(\cdot, \cdot)$$

such that

$$\eta(V,W) = \eta(X+\mu, Y+\lambda) = \frac{1}{2}\left(\iota_X \lambda + \iota_Y \mu\right)$$

in matrix form this is just

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad W = \begin{pmatrix} w \\ \lambda \end{pmatrix} \quad X = \begin{pmatrix} X \\ \mu \end{pmatrix}$$

upon diagonalization the $O(d,d)$ metric becomes

$$\eta' = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

The analogue of generalized T-duality transformation can be introduced by defining a *generalized metric* on the generalized tangent bundle, in addition it should be a Riemannian metric which is positive definite. This is done by explicitly splitting the generlized tangent bundle E into two sub-bundles $E \simeq C_+ \oplus C_-$ where C_+ is the space where the $O(d,d)$-metric, or more appropriately the inner product η is positive definite, and C_- is negative definite. The splitting induces generalized metric, that is, a

$$G(\cdot,\cdot) = \eta(\cdot,\cdot)|_{C_+} - \eta(\cdot,\cdot)|_{C_-}$$

This is in pair with the level matching condition (6.1.1)

The generalized metric is hence an $O(d) \times O(d)$ structure over the base manifold M. Note that if $V \in TM$ and $W \in T^*M$, we have $\eta(V,W) = 0$, so $TM \cap C_\pm = \emptyset = T^*M \cap C_\pm$. Now define $\alpha' : TM \to T^*M$ which can be seen as $\alpha \in T^*M \otimes T^*M$, such that C_+ is the graph of α', that is

$$C_+ = \{X + \iota_X \alpha | X \in \Gamma(TM)\}$$

Then α' which is essentially α provides an isomorphism $TM \simeq C_+$. We can write α as a sum of its symmetric and antisymmetric parts, since $T^*M \otimes T^*M \simeq \mathrm{Sym}^2 T^*M \oplus \wedge^2 T^*M$, $\alpha = g + B$. Hence an element $V_+ \in C_+$ can be written as $V_+ = X + \iota_X(B+g)$. As for C_-,

$$C_- = \{X + \iota_X(B-g) | X \in \Gamma(TM)\}$$

Under B-shifts (recall the discussion of B-transformation when discussing the exact Courant algebroid), $e^B \cdot (X+\mu) = X + \mu + \iota_x B$, the $O(d,d)$ metric is invariant, so g can be identified with an ordinary Riemannian metric on M, that is

$$\begin{aligned}
\eta(V_+, W_+) &= \eta(X + \iota_X B + \iota_X g, Y + \iota_Y B + \iota_Y w) \\
&= \eta(X + \iota_X g, Y + \iota_Y g) \\
&= \frac{1}{2}(\iota_X \iota_Y g + \iota_Y \iota_X g) \\
&= g(X,Y)
\end{aligned}$$

The generalized T-duality transformation H appears in generalized geometry, it is in fact the generalized metric G introduced above. This can be seen by writing down the explicit form for G. Defining that

$$\mathcal{G}(V) = G(V, \cdot) = V_+ - V_-$$

We have that $\mathcal{G}^2 = \mathbb{1}$, and the eigenspaces are $C_\pm$, with respect to eigenvalues ± 1 of $\mathcal{G}$, inducing the splitting of $TM \oplus T^*M$ into $C_{g\pm} = \{X \pm g(X, \cdot)|X \in TM\}$. Then for $V_{g\pm} \in C_{g\pm}$ and $V_{g\pm} = X \pm g(X, \cdot)$,

$$2\mathcal{G}(V) = V_{g+} - V_{g-} = 2g(X, \cdot)$$

thence

$$2\mathcal{G}^2(V) = V_{g+} + V_{g-} = 2X$$

So the simplest form for G is

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

now B field can be reintroduced through a B-transformation on $V_{g\pm}$:

$$\begin{aligned} e^B V_{g\pm} &= (X \pm g(X, \cdot) + \iota_X B) = V_\pm \\ &= \pm e^B G_g V_{g\pm} \\ &= \pm e^B G_g e^{-B} e^B V_{g\pm} \\ &= \pm e^B G_g e^{-B} V_\pm \end{aligned}$$

which can be true if and only if

$$G = e^{-B} G_g e^B = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}$$

The form is identical (with some re-arrangement of blocks) to (6.1.2).

Dorfman derivative (4.3.1) and Courant bracket (4.3.2) can be introduced without any modification. Under B-shifts parametrized by a closed 2-form β, the Courant bracket is not invariant

$$[e^\beta \cdot V, e^\beta, \cdot W]_C = e^\beta [V, W]_C + \iota_X \iota_Y d\beta$$

which suggests the introduction of a *twist* by a 3-from H, that is

$$[V, W]_H = [V, W]_C + \iota_X \iota_Y H$$

6.3 Exceptional Generalized Geometry

Having review the basic idea of generalizing the geometry in order to account for the string duality, which amounts to treating the NSNS B-field and the metric in equal footing, we now turn to the quest of generalizing the geometry to 11D SUGRA [8, 43]. The bosonic sector of 11D SUGRA action in form language is

$$S_M = \frac{1}{2\kappa}\left[\gamma \int d^{11}x\sqrt{|g|}\mathcal{R} - \frac{\alpha}{2}\int G_4 \wedge \star G_4 - \frac{\beta}{6}\int C_3 \wedge G_4 \wedge G_4\right]$$

where $G = dC$. The field strengths are G_4 and $\star G_4$[1], the former coming from the potential, or (higher) gauge connection C, the latter being a 7-form coming from a 6-form potential. The 3-form connection is related to the M2-brane charge, while the 6-form one is related to the M5-brane charge. Also for compact manifold with dimension greater or equal to 6, there should be the Kaluza-Klein monopole charge, but we will not discuss it here.

For our purpose, the case when the dimension of the compact manifold $d = 4$ and $d = 6$ will be discussed, so as to study M2- and M5-branes which are 3 and 6-dimensional correspondingly. For $d = 4$, we suspect that the generalized tangent bundle should be

$$E = TM \oplus \wedge^2 T^* M$$

since there cannot be M5-brane in 4 dimensions, but there can be M2-branes, and for $d = 6$

$$E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M$$

Relevant to our discussion is the U-duality group in these cases. For $d = 4$, it is $E_4 = SL(5, \mathbb{R})$, and for $d = 6$, it is $E_{6(6)}$. We only need to verify that the generalized tangent bundle chosen for $E_{6(6)}$ reproduces the U-duality, because for $SL(5, \mathbb{R})$ we simply need to discard the 5-form section.

6.4 $E_{6(6)}$ Exceptional generalized geometry

The bundle we suspect is $E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M$, with the section given by the formal sum

$$V = v + \theta + \sigma$$

where v is a vector, θ is a 2-form and σ is a 5-form. The fibres are 27-dimensional, transforming in the $6 + 15 + 6$ representaion of $SL(6, \mathbb{R})$, with a natural action of E_6 acting in the **27** representation. The adjoint of E_6 decomposes, under $SL(6, \mathbb{R})$, as

$$\mathbf{78} = \mathbf{35} + \mathbf{1} + \mathbf{20} + \mathbf{20} + \mathbf{1} + \mathbf{1}$$

The **35** corresponds to the natural action of $SL(5, \mathbb{R})$ on tangent vectors and forms on a 5-fold, the first **1** is a scaling transformation, the two **20** are the action of a 3-form and a trivector. The last two **1** are singlets, but regarding the fact that there are 6-forms and 6-vectors in 7 dimensions, they should be seen as the action of a 6-form and a 6-vector.

$E_{6(6)}$ can be reduced to its maximal compact subgroup $H_6 = Sp(4)/\mathbb{Z}_2$, the coset E_6/H_6 is then 42-dimensional, can be parametrized by a symmetric matrix G, a 3-form C and a 6-form $\tilde{C}$. Descending to the base manifold, the generalized metric is parametrized their counterparts, reproducing C_3 gauge connection corresponding to the M2-brane charge and the dual connection corresponding to the M5-brane charge. Hence, the generalized tangent bundle chosen are correct.

The Dorfman derivative can be defined, in analogy to the Lie derivative of ordinary geometry, that is

$$\mathcal{L}_V V' = V \cdot \partial V' - (\partial \times_{\text{ad}} V) \cdot V'$$

where $\times_{\text{ad}}$ is the projection onto the adjoint bundle $\text{ad} = E\mathbb{R} \oplus (TM \otimes T^* M) \oplus \wedge^3 T^* M \oplus \wedge^3 TM$

$$\times_{\text{ad}} : E^* \times E \to \text{ad}$$

[1] $\star$ is the Hodge star operator.

and ∂ should be seen as the embedding of the action of the partial derivative operator via the map T^*M to E^*. The operator $\cdot$ denotes the adjoint action of $E_{6(6)}$.

Hence the Dorfman derivative can be expressed as

$$\mathcal{L}_V^D V' = \mathcal{L}_v v' + (\mathcal{L}_v \theta' - \iota_{v'} d\theta) + (\mathcal{L}_v \sigma' - \iota_{v'} d\sigma - \theta' \wedge d\theta)$$

for $SL(5, \mathbb{R})$ this reduces to

$$\mathcal{L}_V^D V' = \mathcal{L}_v v' + (\mathcal{L}_v \theta' - \iota_{v'} d\theta) - \theta' \wedge d\theta$$

The 3-form C and its 6-form $\tilde{C}$ should again be interpreted as gerbe connections, patched on an overlap $U_\alpha \cap U_\beta$, such that

$$\begin{aligned} C_{(\alpha)} &= C_{(\beta)} + d\Lambda_{(\alpha\beta)} \\ \tilde{C}_{(\alpha)} &= \tilde{C}_{(\beta)} + d\tilde{\Lambda}_{(\alpha\beta)} - \frac{1}{2} d\Lambda_{(\alpha\beta)} \wedge C_{(\beta)} \end{aligned}$$

with Λ a 2-form, and $\tilde{\Lambda}$ a 5-form. Reproducing the field strengths

$$\begin{aligned} G &= dC \\ \tilde{G} &= d\tilde{C} - \frac{1}{2} C \wedge G \end{aligned}$$

Chapter 7

M2-brane action from AKSZ

As an excercise we derived the full single M2-brane action via AKSZ procedure. The action was derived in [44], and was studied in detail in [45].

The generalized tangent bundle is $E = TM \oplus \wedge^2 T^*M$, of which the structure is a higher Courant algebroid. The structure can be embedded fully inside the symplectic dg-manifold of grade 3, $T^*[3]T[1]M$ [32]. The Darboux coordinates we need are $({}_0x^a, {}_1\psi^a, {}_3\xi_a)$, the pre-subscripts representing the grade associated to the coordinates.

The cohomological function Θ should be of grade 4, in this coordinate system,

$$\Theta = A^a_b(x)\xi_a\psi^b + \frac{1}{4!}F_{abcd}(x)\psi^a\psi^b\psi^c\psi^d$$

The classical master equation $\{\Theta, \Theta\}_P = 0$ yields

$$A^k{}_{[a}\partial_{x_k}F_{bcde]} = 0$$

since the embedding is proven to be correct, we will not do the verification that the derived Poisson bracket $\{\{\Theta, \cdot\}, \cdot\}_P$ reproduces the 2-Courant bracket, *etc.*, and proceed straightly to the construction of AKSZ σ-model. With a closed 4-form G twisting the 2-Courant bracket, the 3-brane σ-model becomes

$$S = \int_M \left(\xi_a(dx^a + \psi^a) + \frac{1}{4!}G_{abcd}(x)\psi^a\psi^b\psi^c\psi^d\right)$$

The equation of motion for the 3-form ξ gives

$$-dx^a = \psi^a$$

with $G = dC$ (locally this is always possible), leading to the Wess-Zumino term of the M2-brane action for a closed M2-brane coupled to a 3-form C-field on the boundary

$$S_{M2,WZ} = \int_{\partial M} \frac{1}{3!}C_{abc}dx^a \wedge dx^b \wedge dx^c$$

Notice that if we add a boundary term

$$S_\partial = \int_{\partial M} \frac{1}{2}g_{ab}(x)\psi^a \wedge \star\psi^b$$

we get the full M2-brane action

$$S_{M2} = \int_{\partial M} \left[\frac{1}{2}g_{ab}dx^a \wedge \star dx^b + \frac{1}{3!}C_{abc}dx^a \wedge dx^b \wedge dx^c\right]$$

Note that without the twist, which is a flux in the physics perspective, the σ-model that results in the M2-brane action is just the action arising from the variational problem of a 3-dimensional object. We can imagine turning the G-flux on its boundary, which is forced to live in the Lagrangian submanifold of the corresponding symplectic dg-manifold, the volume now is 'twisted' by the higher gauge field residing in its boundary.

Chapter 8

M5-brane Wess-Zumino term from AKSZ

8.1 Symplectic dg-geometry of E_6 Exceptional Geometry

In this section we follow [46, 47]. The corresponding generalised geometry is $TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M$ (E_6 exceptional geometry) whose sections are given by $v + \tau + \sigma$, for $v \in \Gamma(TM)$,$\tau \in \Gamma(\wedge^2 T^*M)$ and $\sigma \in \Gamma(\wedge^5 T^*M)$.

The Dorfman derivative, without twist, is $\mathcal{L}^D_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \tau' - \iota_{v'} d\tau) + (\mathcal{L}_v \sigma' - \iota_{v'} d\sigma - \tau' \wedge d\tau)$. As in the standard choice, the 5-form σ and v can be accomodated by a symplectic dg manifold $T^*[6]T[1]M$ of grade 6. The 2-form τ should also be encoded, which can be done by

$$E = T^*[6]T[1]M \times \mathbb{R}[3]$$

The coordinates are

$$(x^a, {}_1\psi^a, {}_3\xi, {}_5\gamma_a, {}_6p_a)$$

with the pre-subscript denoting the grade of the coordinates. The symplectic sturcture ${}_6\omega$ in a Darboux coordinate is then

$${}_6\omega = dp_a dx^a - d\gamma_a d\psi^a - \frac{1}{2} d\xi d\xi$$

The section of the generalized tangent bundle E, *viz.*, a generalized vector field, should be a grade 5 function, with the most general form

$$V = v^a(x)\gamma_a + \frac{1}{2}\tau_{ab}(x)\psi^a\psi^b - \frac{1}{5}\sigma_{abcde}\psi^a\psi^b\psi^c\psi^d\psi^e$$

Since the grade 0 functions will be constrained with the Poisson bracket $\{\cdot,\cdot\}_P$, the cohomological function Θ,without twist, need not involve arbitrary functions. It should also be of grade 7, the only possible one is

$$\Theta = p_a\psi^a$$

Now we check that the Dorfmann derivative can be recovered from this symplectic dg-manifold structure, *i.e.*, $\{\{\Theta, V\}_P, V'\}_P = -\mathcal{L}^D_V V'$

$$
\begin{aligned}
-\{\{\Theta, V\}_P, V'\}_P =&(v^a\partial_a v'^b - v'^a\partial_a v^b)\gamma_b + \\
&\frac{1}{2}(v^a\partial_a \omega'_{bc} + 2\partial_b\partial^z\omega'_{zc} - 3v'^z\partial_{[z}\omega_{bc]})\xi\psi^b\psi^c - \\
&\left[v^a\partial_a\sigma'_{bcdef} + 5\partial_b v^z\sigma'_{zcdef} - 6v'^z\partial_{[z}\sigma_{bcdef]} - \frac{5!}{4}\partial_b\omega_{cd}\omega'_{ef}\right]\frac{1}{5!}\psi^b\psi^c\psi^d\psi^e\psi^f
\end{aligned}
$$

Next we twist the Dorfmann derivative by considering the most general cohomological function Θ such that $\{\Theta, \Theta\}_P = 0$. Also note that for the anchor map $E \to TM$ to be surjective, the derived bracket $\{\{\Theta, V\}_P, f\}_P$ defined by Θ with $f \in C^\infty(M)$ should have the correct grade 0. The most general Θ, of grade 7, is

$$\Theta = P^a{}_b(x)p_a\psi^b + T^a{}_{bc}(x)\gamma_a\psi^b\psi^c + F_4(x)\xi\psi^4 - F_7(x)\psi^7$$

where

$$F_i(x)\psi^i \equiv \frac{1}{i!}(F_i)_{a_1a_2\ldots a_i}\psi^{a_1}\psi^{a_2}\ldots\psi^{a_7}$$

Since $P^a{}_b$ is invertible, with a symplectomorphism $g = a^{-1}$, it can be replaced with δ^a_b, with $\psi^a = g^a{}_b(x)\psi'^b$ and $\gamma_a = g_a{}^b\gamma'_b$. When imposing the constraint $\{\Theta, \Theta\}_P = 0$, T should vanish, hence we find

$$\Theta = p_a\psi^a + F_4(x)\xi\psi^4 - F_7(x)\psi^7$$

then the Poisson bracket imposes the constraints

$$
\begin{aligned}
dF_4 &= 0 \\
dF_7 + \frac{1}{2}F_4F_4 &= 0
\end{aligned}
$$

which can be interpreted as the Bianchi identity and equation of motion for the 11D SUGRA 3-form C_3, with the identification $dC = G$ and $F_7 = \star F_4 = \star G_4$.

Finally, we identify (infinitesimal) gauge transformations as automorphisms of the graded Poisson structure on E. For a infinitesimal automorphism X, it should be generated by Ξ, *i.e.*, $X = \{\Xi, \cdot\}_P$, while

$$\Xi = A^a(x)p_a + B^a{}_b(x)\psi^b\gamma_a + C_3(x)\xi\psi^3 + D_6(x)\psi^6$$

On a generalized vector $V \in \Gamma(E)$, the action of X is

$$
\begin{aligned}
X(V) = \{\Xi, V\}_P =& - B^a{}_b v^b\gamma_a + \frac{1}{2}(2B^a_z\tau_{aq} + v^a C_{azq})\xi\psi^z\psi^q - \\
&(5B^z_a\sigma_{zbcde} + 2C_{abc}\tau_{de} + v^z D_{zabcde})\frac{1}{5}\psi^a\psi^b\psi^c\psi^d\psi^e
\end{aligned}
$$

Hence generating the transformation of a subgroup $GL(d)$ of the local E_d action, along with shifts by 3-forms C and a 6-forms D. With the constraint $X(\Theta) = 0$,

$$X(\Theta) = -\frac{1}{3!}\partial_z C_{abc}\xi\psi^z\psi^a\psi^b\psi^c + \frac{1}{6!}\partial_z D_{a_1\ldots a_6}\psi^{a_1}\ldots\psi^{a_6}$$

the 3- and 6-forms are closed, hence automorphisms of the Dorfman derivative generate shifts by closed 3- and 6-forms (along with the action of $GL(d)$). The F_4 and F_7 then enter the Dorfman derivative correctly.

Now we can proceed to the construction of 7-dimensional σ-model.

8.2 σ-model and Wess-Zumino term

The AKSZ construction with the symplectic form of degree 6 gives a 7d Chern-Simons theory, and from its boundary one obtains the topological sector, *i.e.* the WZW term of the (Abelian) M5-brane action. Since we have already determined the form of the cohomological function Θ, the σ-model can be immediately written

$$S = \int_M -p_a dx^a + \frac{1}{6}(\psi^a d\gamma_a + 5\gamma_a d\psi^a) - (p_a\psi^a - F_7(x)\psi^7 + F_4(x)\xi\psi^4)$$

With the help of the equations of motions for p and ψ, the boundary term

$$\int_{\partial M} -p_a dx^a - \frac{1}{6}(\psi^a d\gamma_a + 5\gamma_a d\psi^a) - \frac{1}{2}\xi d\xi$$

becomes

$$\int_{\partial M} \frac{1}{6}d(-\gamma_a dx^a) - \iota_{dx}F_7 + \xi\iota_{dx}F_4 - \frac{1}{2}\xi d\xi.$$

With the help of the identities

$$dC_3 = F_4$$
$$dC_6 = F_7 + \frac{1}{2}C_3F_4$$

one gets another set of identities

$$\iota_{\delta x}F_7 = \iota_{\delta x}dC_6 - \frac{1}{2}\iota_{\delta x}(C_3F_4)$$
$$\iota_{\delta x}dC_6 = dC_6 - d\iota_{\delta x}C_6$$
$$\iota_{\delta x}(C_3F_4) = C_3\delta C_3 - 2C_3\iota_{\delta x}F_4 + d(C_3\iota_{\delta x}C_3)$$

Considering the equation of motion for ξ

$$d\xi = \iota^* F_4$$

on ∂M the general solution is

$$\xi = C_3 + G$$

where G is a closed 3-form.

The integrand in the boundary term becomes

$$-\delta[C_6 - \frac{1}{2}GC_3] + d\left(\iota_{\delta x}C_6 + \frac{1}{2}C_3\iota_{\delta x}C_3 - G\iota_{\delta x}C_3\right) - \frac{1}{2}GdG.$$

The last term should vanish taking into account the fact that for consistency ∂M should be in a Lagrangian submanifold of M and Θ vanishes there, *cf.* Theorem 4.1.1. Invoking a Lorentzian metric on ∂M and imposing an (anti) self-duality condition, $\star_{\partial M}h = \pm h$, we have, when on shell

$$\delta\left(S + \int_{\partial M}\left[C_6 - \frac{1}{2}hC_3\right]\right) = 0$$

which gives the M5-brane Wess-Zumino term

$$S_{WZ} = \int_{\partial M}\left[C_6 - \frac{1}{2}hC_3\right]$$

This is indeed the Wess-Zumino term of M5-brane action, see for example [48], equation (3.15), identifying h here with $-H_3$ there.

Chapter 9

Discussion, Conclusions and Outlook

Through combining the power of exceptional generalized geometry in geometrizing the dynamics of 11D SUGRA, the power of AKSZ construction in producing higher Chern-Simons like σ-models in Batalin-Vilkovisky formalism, and the observation that Green-Schwarz action functionals of p-branes are some sort of higher WZW models which can be seen as, in some sense, dual to higher Chern-Simons theories, we have shown that when only considering non-coincidental, *i.e.* single p-branes, the Wess-Zumino terms of the Abelian M2-brane and M5-brane action can be obtained from the boundary of a AKSZ σ-model action.

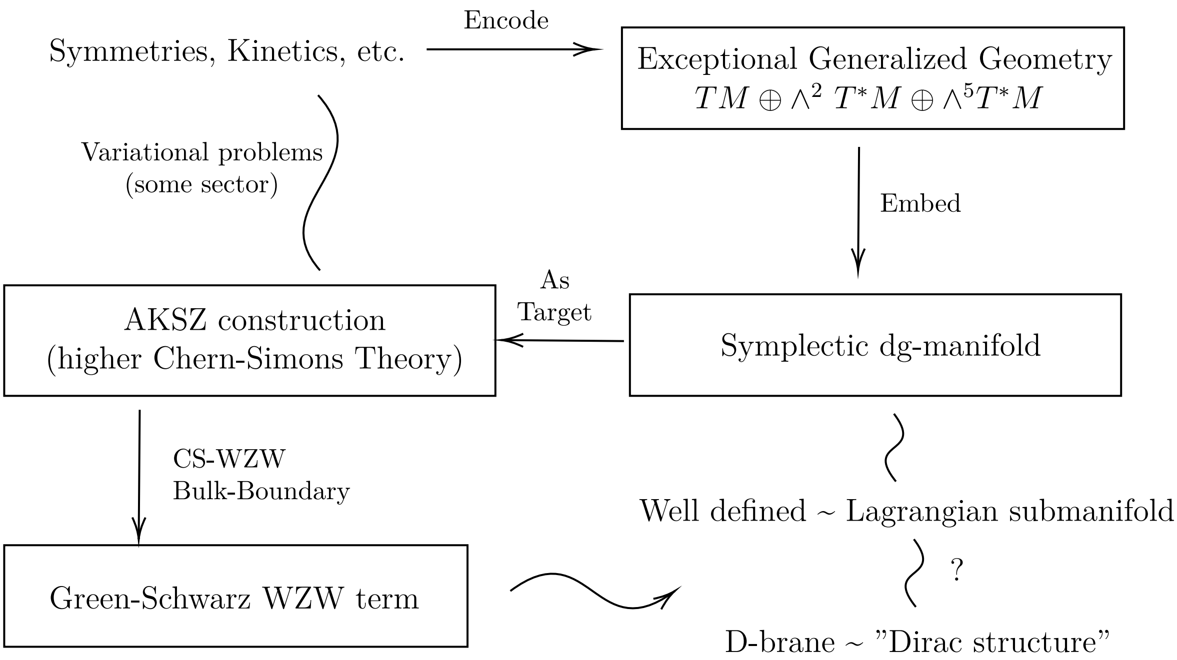

Figure 9.1: Full strategy and logic of this thesis

The AKSZ σ-model construction is through the embedding of the generalized tangent bundle into a symplectic dg-manifold. For the M5-brane, the embedding is through $E \simeq TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \hookrightarrow T^*[6]T[1]M \times \mathbb{R}[3]$, and for the M2-brane, the embedding is through $E \simeq TM \oplus \wedge^2 T^*M \hookrightarrow T^*[3]T[1]M$. The meaning of the embedding can be made clear from the Hamiltonian formalism we discussed in Section 4.4, the symmetries and dynamics of a theory is encoded in a algebroid structure, and the structure is embedded into a symplectic dg-manifold, which give rise to a formulation Hamiltonian mechanics for a extended object. The AKSZ procedure captures the informations encoded inside the transgression of characteristic classes associated to the underlying symplectic dg-manifold. On the boundary, one finds the associated Green-Schwarz WZW term. The full strategy and overall logic is concisely expressed in Figure 9.1.

We repeat with some refinement the speculated three-fold coincidence we have made,

1. After embedding a transitive (equipped with a surjective anchor $\rho : E \to TM$) algebroid structure inside a symplectic dg-manifold, the Lagrangian submanifold of the latter is where D-branes live in the associated variational problem.

2. For consistency, the boundary of a AKSZ σ-model must lie in a Lagrangian submanifold.

3. The boundary terms of p-dimensional AKSZ σ-models, which are at the same time WZW terms of $p-1$-dimensional Green-Schwarz action functinonals, are well defined in the sense of [35].

where the second one is a theorem and the first and the third statements are merely speculations. We give arguments and some more discussions regarding these speculations.

We give an argument for the first statement. The surjective anchor to the tangent bundle translates to that the form of the symplectic dg-manifold where the algebroid structure is embedded should be $T^*[n]T[1]M$[1]. The crucial thing is the only appearing shifted tangent bundle should be of grade 1, hence relating the σ-model to a variational problem, as discussed in Section 4.4. The explicit splitting of tangent bundle with cotangent bundles is natural since two TM's in the generalized tangent bundle (equivalently, the shift of the tangent bundle in the dg-manifold by a degree greater than 1) means that there are two spacetime metrics in the theory, or in the associated Hamiltonian formalism there are two velocities associated to a point $x \in M$. If the algebroid is a higher Courant algebroid isomorphic to $TM \oplus \wedge^n T^*M$, a Lagrangian submanifold corresponds to a higher Dirac structure in the sense of [33], and even though slightly different in their definitions, Dirac structures are related to Dirichlet boundary conditions.

For the third statement, we have given a crude argument that since AKSZ σ-models are transgressions of characteristic classes associated to the underlying symplectic dg-manifold which encodes the symmetries and dynamics of the theory, on the reduction to the boundary the characteristic classes force the σ-model boundary to lie in a Lagrangian submanifold. The encoded symmetries are not lost, and are forced to stay in a submanifold where the symmetries cannot be violated under deformation. Also we have seen that anomaly-free conditions in various cases are related to Dirac structures [11, 12] and Lagrangian submanifolds [14, 15]. This also might be related to the recent work [13], which derives anomalies of WZW terms on a manifold Σ from a higher-dimensional Chern-Simons theory on a manifold M with $\partial M = \Sigma$, strongly resembling the setting we have taken. The cancelled anomaly can only be the conformal anomaly, which results in the central charge in the Virasoro algebra that can make the WZW term not well defined, since it was studied in detail that chiral and gauge anomalies needs other terms to cancel [49]. Moreover in [35] it was made clear that under some reasonable hypothesis the M5-brane Wess-Zumino term we derived is well defined.

One cruial thing is to clarify what precisely are the relations between various notions of higher Dirac structures[2] and their relations to Lagrangian submanifolds of symplectic

[1] Possibly times $\mathbb{R}[n]$, and so on.

[2] And not yet studied analogues of Dirac structure for algebroids such as $TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$.

dg-manifolds, where they might be embedded.

Finally, from what we have learned, we give some outlooks. The physics encoded in the geometry of the target symplectic dg-manifold, with the algebroid structure embbedded, is complicated, and needs further investigation.

1. The behaviour of the σ-models constructed throught the AKSZ formalism on the boundary might shed lights on the nature of AdS/CFT correspondence in some particular sectors.

2. On the boundary of the σ-models constructed one gets the Wess-Zumino-Witten term of the boundary brane action, which implies that these topological $p+1$-brane actions are related to the Green-Schwarz actions of p-brane, in particular its chiral part. It is known that chiral part of any 2d conformal field theory defines a 3d topological quantum field theory [39], the observation could also shed light to the higher analogues of this fact.

3. The Hamiltonian formalism describing extended object propagating with the mapping between a shifted tangent bundle and a symplectic dg-manifold is nearly identical to that of the AKSZ construction, and the actions describing the propagation of extended objects are those of Nambu-Goto type. However, plain Nambu-Goto type actions fail to reproduce the dynamics of the corresponding RNS action, and a modification with the addition of a WZW term is required, resulting in the Green-Schwarz action functionals. Endowing the symplectic dg-manifold with a super structure might improve the understanding about the necessarity and the nature of the WZW term, in a purely mechanical perspective.

4. As it was speculated above, the relation between the condition of anomaly-freeness and Lagrangian submanifolds of the target symplectic dg-manifold might improve the understanding of anomalies.

5. It would be interesting to know the meaning of Lagrangian submanifolds (or Dirac structures and their analogues) and its preservation under Lie derivatives (or Dorfman derivatives and analogues) in the symplectic dg-geometric picture (or in the algebroid picture), from the perspective of anomaly cancellation of current algebra, and from the perspective of defining D-branes, or similar objects: how are the symmetries and dynamics encoded inside the geometry?

Appendix

The proofs to the three statements in Section 3.2 are given here.

Proposition 9.0.1. *There is a full and faithful embedding of symplectic dg-manifolds of grade n into symplectic Lie n-algebroids.*

Proof. The symplectic form $\omega \in \Omega^2(M)$ is closed, and its Lie derivative along the cohomological vector field vanishes, so $(d + \mathcal{L}_v)\omega = 0$. Identifying $\Omega^\bullet(M) \simeq \mathrm{W}(\mathfrak{a})$, ω is an invariant polynomial on $\mathfrak{a}$. The form ω is of grade n, so it has no components in elements of grade $> n$, non-degeneracy implies that all such elements vanish, so $\mathfrak{a}$ is a Lie n-algebroid. $\square$

Proposition 9.0.2. *A symplectic Lie $n-$algebroid $(\mathfrak{S}, \omega)$ given by the embedding of a symplectic dg-manifold carries a canonical L_∞-algebroid cocycle π, and is the Hamiltonian of $d_{\mathrm{CE}(\mathfrak{S})}$. $\pi = \frac{1}{n+1}\iota_\epsilon \iota_v \omega$.*

Proof. We have that

$$\begin{aligned} \mathbf{d}\iota_v\iota_\epsilon\omega &= \mathbf{d}\iota_v\iota_\epsilon\omega \\ &= (\iota_v\mathbf{d} - \mathcal{L}_v)\iota_\epsilon\omega\iota_v\mathcal{L}_\epsilon\omega - \iota_{[v,\epsilon]}\omega \\ &= (n+1)\iota_v\omega. \end{aligned}$$

$\square$

Proposition 9.0.3. *The cocycle $\frac{1}{n}\pi$ is in transgression with ω, with the Chern-Simons element being $\mathrm{cs} = \frac{1}{n}(\iota_\epsilon\omega + \pi)$.*

Proof. Under the projection i^* the first term vanishes. $d_{\mathrm{W}}\pi = \mathbf{d}\pi$ (π is a cocycle), and $d_W\iota_\epsilon\omega = [\mathbf{d} + \mathcal{L}_v, \iota_\epsilon]\omega = n\omega - \mathbf{d}\omega$. $\square$

The proof to the is given here.

Proposition 9.0.4. *For $(\mathfrak{S}, \omega)$ a symplectic Lie $n-$algebroid associated to a symplectic dg-manifold of positive grade n, with global Darboux chart, the action functional induced by the Chern-Simons element cs$= \frac{1}{n}(\iota_\epsilon\omega + \pi)$ where $\pi = \frac{1}{n+1}\iota_\epsilon\iota_v\omega \in \mathrm{W}(\mathfrak{S})$, is the AKSZ action, i.e., $\int_\Sigma \mathrm{cs} = \int_\Sigma L_{AKSZ}$*

Proof. The Chern-Simons element in the local form, with Darboux coordinates $\{x^a\}$ chosen such that $\omega = \frac{1}{2}\omega_{ab}\mathbf{d}x^a \wedge \mathbf{d}x^b$, is

$$\mathrm{cs} = \frac{1}{n}\left(\deg(x^a)\omega_{ab}x^a \wedge d_{\mathrm{W}}x^b - n\pi\right)$$

so for a degree 1 $\mathfrak{G}$-valued differential form $A : \mathrm{W}(\mathfrak{G}) \to \Omega^\bullet(\Sigma)$ on Σ, with $A(d_{\mathrm{W}} x^b = d_{\mathrm{dR}} A^b)$, we have

$$\begin{aligned}
\int_\Sigma \mathrm{cs}(A) &= \frac{1}{n} \int_\Sigma \left[\deg(x^a)\omega_{ab} A^a \wedge d_{\mathrm{dR}} A^b - n\pi(A)\right] \\
&= \frac{1}{2n} \int \left[\deg(x^a)\omega_{ab} A^a \wedge (d_{\mathrm{dR}} A^b) + \deg(x^b)\omega_{ab} A^a \wedge (d_{\mathrm{dR}} A^b) - 2n\pi(A)\right] \\
&= \int \left[\frac{1}{2}\omega_{ab} A^a \wedge (d_{dR} A^b) - \pi(A)\right]
\end{aligned}$$

Now substitute A for the symbol ϕ.

$\square$

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