Branes on Generalized Geometry

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(Dated: January 27, 2025)

English: Generalized geometry is a recent framework for geometric structures that naturally contains B-fields appearing in bosonic sectors of superstring theories. We explain how the notion of generalized complex submanifold extends the well-known A-type and B-type branes in topological string theories, thus suggesting the existence of other types of branes for even-dimensional manifolds. On the other hand, B_n-generalized geometry offers an extension of generalized geometry to manifolds of any dimension. In this work, we define a new notion of B_n-generalized submanifolds and we discuss the conditions that arise in B_n-generalized complex submanifolds.

Català: La geometria generalitzada és un formalisme recent per estructures geomètriques que conté naturalment els B-camps presents als sectors bosònics de teories de supercordes. Expliquem com la idea de subvarietats generalitzades complexes estén la noció de branes de tipus A i de tipus B de les teories topològiques de cordes, suggerint així l'existència d'altres tipus de branes en varietats de dimensió parella. Per una altra banda, la geometria generalitzada de tipus B_n ofereix una extensió de la geometria generalitzada per a varietats de qualsevol dimensió. En aquest treball, definim una nova noció de subvarietats generalitzades de tipus B_n i estudiem l'estructura d'unes subvarietats complexes generalitzades de tipus B_n en particular.

Keywords: Brane, generalized geometry, generalized complex submanifold, B_n-generalized geometry.

I. INTRODUCTION

Quantum Field Theory, which combines quantum mechanics with special relativity, is currently the best theory describing particle physics. Its success relies on the fact that, in most scenarios, the gravitational interaction, not included in the classical Quantum Field Theory, is extremely weak compared to the strong and electroweak interactions. The addition of gravity makes it impossible to have a static space-time background, which is where Quantum Field Theories are built. Moreover, the Coleman-Mandula theorem [1] states that in a relativistic theory of interacting point-like particles, the Poincaré and internal group symmetries cannot mix.

Branes in string theories

String theory allows to mix space-time symmetries with internal ones because it is a quantum, interacting and relativistic theory of, instead of point-like particles, one-dimensional objects (thus avoiding the hypothesis of Coleman-Mandula theorem). These objects, called strings, propagate through $d+1$-dimensional space-time along world-sheets instead of the usual world-lines for point-like particles. Closed strings are topologically a circle and their world-sheets are deformed cylinders, while the world-sheets of open strings are two-dimensional surfaces inside the background space-time. However, open strings need boundary conditions that can be given in terms of the derivatives of the position (Neumann boundary conditions) or in terms of the position itself (Dirichlet boundary conditions). When the boundary conditions for $d-p$ spatial coordinates are of Dirichlet type, the endpoints of open strings are fixed to some $p+1$-dimensional submanifold, as represented in figure 1. These manifolds are given the name of branes, or p-branes if we want to specify its dimension. If all the coordinates have Neumann boundary conditions, the endpoints of open strings are free, which suggests the existence of space-filling d-branes. Branes generalize point-like particle world-lines and string world-sheets because the former is a 0-brane and the latter a 1-branes.

Superstring theories are 10-dimensional supersymmetric versions of string theories that prevent the appearance of tachyonic particles otherwise present in other string theories. In these supersymmetric theories, p-branes become infinitely massive as the string coupling tends to 0, with the only exception of fundamental strings, which are 1-branes. This is why branes are not present at the perturbative level, although they must be considered when studying superstring theories non-perturbatively. Different superstring theories have different bosonic and fermionic fields, although they all contain the massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. The first one is the metric tensor, the second one is an antisymmetric 2-tensor and the third one is the scalar dilaton (see, e.g., [2, Ch.

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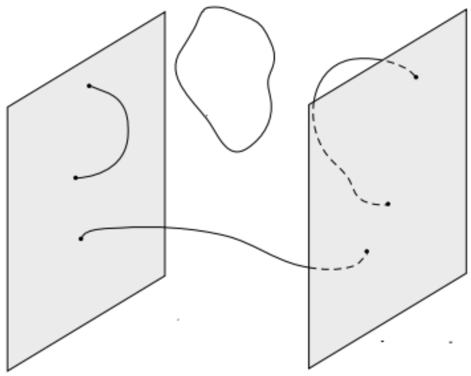

Figure 1: Open and closed strings in presence of branes.

2]). The 2-tensor $B_{\mu\nu}$ is also called the Kalb-Ramond B-field and is the generalization of the electromagnetic potential to the 2-dimensional world-sheet. Similarly to how the electromagnetic field strength is defined by $F = \mathrm{d}A$, the field $H = \mathrm{d}B$ is the Neveu-Schwarz 3-form field strength. Both the Neveu-Schwarz field strength H and Kalb-Ramond B-fields arise naturally in the framework offered by generalized geometry, a recent unifying approach to geometric structures.

Currently, superstring theory has some problems to describe reality because, for instance, supersymmetry has not yet been found in LHC experiments despite the expectation to discover it at the beginning. However, superstring theories (and branes therein) seem to be good candidates to help relating different fields in theoretical physics. In fact, AdS/CFT duality provides a link between superstring theories in AdS background geometries and (quantum) conformal field theories [3, Ch. 1]. An example of this duality is that N overlapping 3-branes and the open strings that end there in the 10-dimensional type IIB superstring theory are related to the 4-dimensional $\mathsf{SU}(N)$ Yang-Mills conformal theory with $\mathcal{N} = 4$ supersymmetry [4]. This example links the formalism of quantum conformal field theories to a possible description of quantum gravity in terms of superstring theories in AdS backgrounds. For instance, black hole and solitonic solutions for the background geometry metric on N overlapping 4-branes allow to explain the transition between confined and deconfined states of a 5-dimensional $\mathsf{SU}(N)$ supersymmetric Yang-Mills theory. A similar transition has been observed in nature, although the Standard Model is neither 5-dimensional nor supersymmetric. However, some analogous relation between the strong sector of the Standard Model and geometric solutions of the metric on branes might help understanding the phase transition.

On the other hand, topological string theories are a type of string theories obtained from a topological twist of a Calabi-Yau manifold, which is the target space of these types of string theories. This topological twist can be done in two different ways, leading to the A-model and the B-model. The former is related to the symplectic structure of the Calabi-Yau manifold and the latter to its complex structure. Therefore, boundary conditions for open strings that are consistent with the topological twist give rise to A-type and B-type branes (see, e.g., [5, Ch. 2]). In the topological B-model, branes turn out to be complex submanifolds of the target Calabi-Yau manifold. In the A-model, branes were initially considered to be Lagrangian submanifolds of the target manifold. However, A. Kapustin and D. Orlov showed in [6] that more general co-isotropic A-branes must be allowed in order for the mirror symmetry conjecture to be true. Again, symplectic and complex structures arise naturally in generalized geometry as particular cases of generalized complex structures, and in a way that generalized submanifolds allow to describe both A- and B-type branes.

Aim of this work

In this work we introduce the framework of generalized geometry, first introduced in [7] and developed in [8]. It turns out that B-fields arise naturally in this framework as orthogonal symmetries, and H-fields as a twisted analog of the Lie bracket. Using structures that generalize complex and symplectic structures, we focus on submanifolds [8, Ch. 7] that allow us to describe A-type and B-type branes of topological string theories. It turns out that these generalized structures can only exist in even-dimensional manifolds. Therefore, generalized geometry does not allow us to build an analogous notion of a brane in 11-dimensional M-theory. Due to this fact, we present an extension of generalized geometry that does not have this restriction on the dimension. This extension is called B_n-generalized geometry and was first developed in [9]. Finally, we propose, as an original contribution, a B_n version of the submanifolds that describe topological branes, and discuss the conditions arising from some B_n-generalized complex submanifolds.

Structure

Sections II and III are a brief summary of generalized geometry. First, section II deals with the linear structure on each generalized tangent space [10, Ch. 2]. Then, section III extends the previous point-wise linear structures to the corresponding global ones [10, Ch. 4]. Generalized submanifolds [8, Ch. 7] are defined in section IV and section V introduces B_n-generalized geometry [9]. Finally, section VI presents a proposal for the B_n-generalized version of the submanifolds introduced in section IV. Finally, we also include an appendix describing an alternative description of generalized complex structures that invokes spinors. We give some proofs in the text, with no proof environment, in order to keep the text lighter.

Notation and conventions

We work on the smooth category and assume familiarity with basic differential geometry (see, e.g., [11] or [12, Ch. 5]). Throughout the work M will denote a real n-dimensional smooth manifold, $x \in M$ a point on the manifold, TM its tangent bundle and T^*M its cotangent bundle. Tangent vectors (elements of TM) and vector fields (sections of TM) will be denoted by X or Y. Similarly, ξ or η will denote elements or sections of T^*M. Real numbers or real-valued functions over M will be denoted by λ or μ. The interior product or contraction by X will be denoted by ι_X, and d will denote the exterior derivative. The inclusion of a subspace $U \subseteq W$ in W will be denoted by $\imath : U \to W$. The dual or pull-back map of a linear or smooth map h will be denoted by h^*, and the push-forward or differential map by h_*. Finally, sections of a bundle E will be denoted by $\Gamma(E)$ and for a shorthand notation, $\Omega^r(M) = \Gamma(\wedge^r(TM))$.

II. GENERALIZED LINEAR ALGEBRA

Differential geometry is the study of geometric structures on M such as metrics, symplectic or complex structures. These structures are studied via the tangent bundle. Generalized geometry, instead, studies structures on its tangent plus cotangent bundle $\mathbb{T}M := TM \oplus T^*M$. Elements or sections of $\mathbb{T}M$ are of the form $\mathbb{X} = X + \xi$ or $\mathbb{Y} = Y + \eta$. In this section we study the linear algebra on the generalized tangent space $\mathbb{T}_xM$ at $x \in M$.

A. Symmetric pairing and isotropic subspaces

At $\mathbb{T}_xM$ there is a natural symmetric pairing $\langle \cdot, \cdot \rangle$ given by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi)\,.$$

This pairing has signature (n, n) and allows us to define the orthogonal complement of a subspace $W \subseteq \mathbb{T}_xM$ by

$$W^\perp = \{\mathbb{X} \in \mathbb{T}_xM \,:\, \langle \mathbb{X}, \mathbb{Y} \rangle = 0 \text{ for all } \mathbb{Y} \in W\}\,.$$

The subspace W is called isotropic if $W \subseteq W^\perp$, and co-isotropic if $W^\perp \subseteq W$.An isotropic subspace is said to be maximally isotropic if it is not strictly contained in another isotropic subspace.

It can be shown that for a non-degenerate symmetric bilinear form of signature (m_1, m_2), all maximally isotropic subspaces have dimension $\min\{m_1, m_2\}$, which in the case of $\mathbb{T}_xM$ implies that W is maximally isotropic if and only if it is isotropic and $\dim W = n$.

In fact, given a subspace $W \subseteq T_xM$ and an antisymmetric 2-form $\omega \in \wedge^2 W$,

$$L(W, \omega) = \{X + \xi \in \mathbb{T}_xM \,:\, X \in W \text{ and } \xi|_W = \iota_X \omega\} \quad (1)$$

is maximally isotropic and any maximally isotropic subspace of T_xM is of this form [8, Prop. 2.6].

B. Orthogonal symmetries and B-transforms

Orthogonal symmetries of $\mathbb{T}_xM$ are defined as linear automorphisms of $\mathbb{T}_xM$ that preserve the pairing, similarly to how Lorentz symmetries of spacetime preserve the Minkowski metric at each point. A type of these symmetries extends the usual symmetries of T_xM, and are given by $X + \xi \mapsto A^{-1}X + A^*\xi$ for $A^{-1} \in \mathsf{GL}(T_xM)$. Indeed, in this case

$$\begin{aligned}\langle A^{-1}X + A^*\xi, A^{-1}X + A^*\xi \rangle &= \iota_{A^{-1}X}(A^*\xi) \\ &= \xi(AA^{-1}X) = \xi(X) \\ &= \langle X + \xi, X + \xi \rangle\,.\end{aligned}$$

These are the only symmetries that preserve the components T_xM and T_x^*M. However, there are other symmetries that mix these components and therefore are a new feature of the generalized setup. Letting $B \in \wedge^2(T_x^*M)$ an antisymmetric 2-form and $\beta \in \wedge^2(T_xM)$ an antisymmetric 2-vector, the linear automorphisms that send $X + \xi$ to $X + \xi + \iota_X B$ and to $X + \iota_\xi \beta + \xi$ are also symmetries of $\mathbb{T}_xM$. Indeed, for the first case,

$$\begin{aligned}\langle X + \xi + \iota_X B, X + \xi + \iota_X B \rangle &= \iota_X(\xi + \iota_X B) \\ &= \iota_X \xi = \langle X + \xi, X + \xi \rangle\end{aligned}$$

by antisymmetry of B, and similarly for the β case. These types of symmetries are called B-transforms and β-transforms, respectively, and are denoted by e^B and e^β.

Note that the image of a maximally isotropic subspace of $\mathbb{T}_xM$ under a symmetry is again maximally isotropic because the pairing is preserved and the dimension is still n. In fact, one gets that

$$\mathrm{e}^B L(W, \omega) = L(W, \omega + \imath^* B)\,.$$

C. Generalized linear complex structures

By analogy with complex structures, a generalized linear complex structure on $\mathbb{T}_xM$ is defined as an endomorphism $\mathcal{J} \in \mathrm{End}(\mathbb{T}_xM)$ such that $\mathcal{J}^2 = -\mathbb{1}$ and $\langle \mathcal{J}\mathbb{X}, \mathbb{X}\rangle = 0$ for all $\mathbb{X} \in \mathbb{T}_xM$. A necessary and sufficient condition for a generalized linear complex structure on T_xM to exist is that n is even [8, Prop. 4.5]. In an even-dimensional vector space one can also consider a usual complex structure $J \in \mathrm{End}(T_xM)$ with $J^2 = -\mathbb{1}$, which gives the generalized complex structure $\mathcal{J}_J$ on $\mathbb{T}_xM$ that sends $X + \xi$ to $-JX + J^*\xi$. Even-dimensional vector spaces also admit a symplectic structure, i.e., a non-degenerate $\omega \in \wedge^2(T_x^*M)$, and it gives the generalized complex structure $\mathcal{J}_\omega$ on $\mathbb{T}_xM$ that sends $X + \xi$ to $-\iota_\xi\omega^{-1} + \iota_X\omega$. Here, ω^{-1} is the 2-vector defined by $\iota_\xi\omega^{-1} = X$ whenever $\iota_X\omega = \xi$, which always exists due to the non-degeneracy of ω. Thus, generalized complex structures put on the same footing complex and symplectic structures in T_xM.

Generalized linear complex structures can be formulated in terms of the isotropic subspaces introduced above. Indeed, the fact that $\mathcal{J}^2 = -\mathbb{1}$ yields an eigenspace decomposition of the complexification $\mathbb{T}_xM_\mathbb{C} := \{\mathbb{X} + i\mathbb{Y} : \mathbb{X}, \mathbb{Y} \in \mathbb{T}_xM\}$. Its $\pm i$-eigenspaces are

$$L = \{\mathbb{X} - i\mathcal{J}\mathbb{X} : \mathbb{X} \in \mathbb{T}_xM\}$$
$$\bar{L} = \{\mathbb{X} + i\mathcal{J}\mathbb{X} : \mathbb{X} \in \mathbb{T}_xM\}\,.$$

Since $\mathcal{J}$ is a real endomorphism, $L \cap \bar{L} = \{0\}$ and, by dimension, L and $\bar{L}$ are maximally isotropic subspaces of $\mathbb{T}_xM_\mathbb{C}$. Conversely, a maximally isotropic subspace $L \subseteq \mathbb{T}_xM_\mathbb{C}$ such that $L \cap \bar{L} = \{0\}$ is equivalent to a generalized complex structure on $\mathbb{T}_xM$ [8, Prop. 4.3].

Given a complex structure J on T_xM, we denote its $\pm i$-eigenspaces as $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$, respectively. Then, the $+i$-eigenspace of $\mathcal{J}_J$ is just $T_x^{(0,1)}M \oplus (T_x^{(1,0)}M)^*$. For the case of $\mathcal{J}_\omega$, the elements of its $+i$-eigenspace are of the form $X_\mathbb{C} - i\iota_{X_\mathbb{C}}\omega$ for $X_\mathbb{C} \in T_xM_\mathbb{C}$.

An arbitrary linear generalized complex structure $\mathcal{J}$ on $\mathbb{T}_xM$ determines disjoint symplectic $n-2r$-dimensional and complex $2r$-dimensional subspaces of $\mathbb{T}_xM$ for some r called the type of $\mathcal{J}$ [8, Thm. 4.35].

III. GENERALIZED GEOMETRY

All the structures on $\mathbb{T}_xM$ can be smoothly promoted to structures on $\mathbb{T}M$. In this sense, the pairing becomes a $C^\infty(M)$-bilinear symmetric map

$$\Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to C^\infty(M)$$
$$(\mathbb{X}, \mathbb{Y}) \mapsto (x \mapsto \langle \mathbb{X}_x, \mathbb{Y}_x\rangle)\,.$$

Analogously, maximally isotropic subbundles are bundles where the sub-fibres are maximally isotropic. Orthogonal symmetries of $\mathbb{T}M$ are linear automorphisms on each fibre. For example, $f \in \mathrm{Diff}(M)$ induces an orthogonal symmetry $f_* \in \Gamma(\mathsf{GL}(M))$ and $B \in \Omega^2(M)$ encodes a global B-transform. Finally, a linear generalized complex structure on each fibre gives an almost generalized complex structure since, as we will see, we have to add an integrability condition.

A. Dorfman bracket and Courant algebroids

Recall that a Lie bracket is an antisymmetric bilinear map on $\Gamma(TM)$ satisfying the Jacobi identity and obeying the Leibniz rule $[X, fY] = f[X, Y] + X(f)Y$. The Dorfman bracket on $\Gamma(\mathbb{T}M)$ is defined as

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \iota_Y \mathrm{d}\xi\,,$$

where the bracket on vector fields is just the Lie bracket and $\mathcal{L}_X$ is the Lie derivative with respect to the field X.

It can be proven that $[\mathbb{X}, \cdot]$ is a derivation of the Dorfman bracket and of the pairing $\langle\cdot,\cdot\rangle$. Moreover, using Cartan's formula $\mathcal{L}_X = \{\mathrm{d}, \iota_X\}$ on forms and antisymmetry of the Lie bracket,

$$[X + \xi, X + \xi] = [X, X] + \mathcal{L}_X\xi - \iota_X\mathrm{d}\xi$$
$$= \mathrm{d}i_X\xi = \mathrm{d}\langle X + \xi, X + \xi\rangle.$$

Let $\pi : \mathbb{T}M \to TM$ be the canonical projection. Then, the data $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ is an example of what is called a Courant algebroid. A Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho)$ is a vector bundle E over M with a bundle map $\rho : E \to TM$ called the anchor map, a symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on E and a bilinear bracket $[\cdot,\cdot]$ on $\Gamma(E)$ such that for $\mathbb{X} \in \Gamma(E)$, $[\mathbb{X}, \cdot]$ is a derivation of the form and the bracket, and $[\mathbb{X}, \mathbb{X}] = D\langle \mathbb{X}, \mathbb{X}\rangle$, where $D = (2\langle\cdot,\cdot\rangle)^{-1}\pi^*\mathrm{d}$.

It can be shown that in a Courant algebroid $\rho([\mathbb{X}, \mathbb{Y}]) = [\rho(\mathbb{X}), \rho(\mathbb{Y})]$ and that the Leibniz rule $[\mathbb{X}, \lambda\mathbb{Y}] = \lambda[\mathbb{X}, \mathbb{Y}] + \rho(\mathbb{X})(\lambda)\mathbb{Y}$ holds, for $\mathbb{X}, \mathbb{Y} \in \Gamma(E)$ and $\lambda \in C^\infty(M)$.

The Courant algebroid $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ is, moreover, a so-called exact Courant algebroid, that is, the anchor map π is surjective and $\mathrm{rk}\, E = 2n$. In fact [13, Letter 1], it turns out that any exact Courant algebroid is isomorphic to $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H, \pi)$ for an H-twisted Dorfman bracket

$$[\mathbb{X},\mathbb{Y}]_H = [\mathbb{X},\mathbb{Y}] + \iota_{\pi(\mathbb{X})}\iota_{\pi(\mathbb{Y})}H\,,$$

where H is a closed 3-form. Antisymmetry of H ensures the twisted bracket $[\mathbb{X},\cdot]_H$ is a derivation of $\langle\cdot,\cdot\rangle$, and the closure of H is the necessary and sufficient condition for it to be a derivation of $[\cdot,\cdot]_H$.

B. Generalized diffeomorphisms

Recall that a diffeomorphism f of M fulfils that $[f_*X, f_*Y] = f_*[X,Y]$ for the Lie bracket. Analogously, a generalized diffeomorphism of M is defined as a diffeomorphism $F:\mathbb{T}M\to\mathbb{T}M$ that is a symmetry of $\mathbb{T}M$ and additionally satisfies $[F(\mathbb{X}),F(\mathbb{Y})]_H = F([\mathbb{X},\mathbb{Y}]_H)$. Let $f\in\mathrm{Diff}(M)$ and $B\in\Omega^2(M)$, which makes both f_* and e^B symmetries of $\mathbb{T}M$. Then the composition $f_*\mathrm{e}^B$ that sends $X+\xi$ to $f_*X + (f^{-1})^*(\xi+\iota_X B)$ can be computed to obey

$$\begin{aligned}[f_*\mathrm{e}^B(\mathbb{X}), f_*\mathrm{e}^B(\mathbb{Y})]_H &= f_*\mathrm{e}^B([\mathbb{X},\mathbb{Y}]_H)\\ &\quad + (f^{-1})^*\iota_{\pi(\mathbb{Y})}\iota_{\pi(\mathbb{X})}(\mathrm{d}B - (f^*H - H))\,,\end{aligned}$$

which is a generalized diffeomorphism if and only if $\mathrm{d}B = f^*H - H$. In fact, by [8, Prop. 3.24], any generalized diffeomorphism is of this form. Computing the action of $f_*\mathrm{e}^B\circ g_*\mathrm{e}^{B'}$ on an element $X+\xi$ we get that it equals

$$\begin{aligned}&(f_*\mathrm{e}^B)(g_*X + (g^{-1})^*(\xi+\iota_X B'))\\ &\quad = (f\circ g)_*X + ((f\circ g)^{-1})^*(\xi + \iota_X B' + \iota_X(g^*B))\,.\end{aligned}$$

Thus, the composition rule for generalized diffeomorphisms is $f_*\mathrm{e}^B\circ g_*\mathrm{e}^{B'} = (f\circ g)_*\mathrm{e}^{g^*B+B'}$, and forms the group $\mathrm{GDiff}_H(M) = \left\{f_*\mathrm{e}^B : f^*H - H = \mathrm{d}B\right\}$ of generalized diffeomorphisms on M. In particular, B-transforms for B closed are generalized diffeomorphisms taking f the identity. In fact, $\mathrm{GDiff}_H(M)$ is an extension of the subgroup of $\mathrm{Diff}(M)$ preserving the cohomology class of H by the group of closed 2-forms.

C. Generalized complex structures

On the one hand, almost complex structures and non-degenerate 2-forms become complex and symplectic structures if an additional integrability condition is satisfied. In the generalized geometry case, the integrability condition is defined in terms of invariance under the Dorfman bracket, in analogy to the integrability condition of a complex structure in terms of the Lie bracket. A subbundle $L\subseteq\mathbb{T}M$ is said to be integrable if for all $\mathbb{X},\mathbb{Y}\in\Gamma(L)$, $[\mathbb{X},\mathbb{Y}]_H\in\Gamma(L)$. Then, a generalized almost complex structure $\mathcal{J}$ on $\mathbb{T}M$ is said to be integrable if its $+i$-eigenbundle is integrable. These structures are called generalized complex structures.

In the particular case of $\mathcal{J}_J$, the $+i$-eigenspace is integrable if and only if J is integrable on M and

$$\mathcal{L}_X\eta - \iota_Y\mathrm{d}\xi + \iota_X\iota_Y H \tag{2}$$

is in $(\mathbb{T}^{(1,0)}M)^*$. In particular, we can decompose $\mathrm{d} = \partial+\bar\partial$ and noting that η is of complex type $(1,0)$ and $X\in\mathbb{T}^{(0,1)}M$, $\mathcal{L}_X\eta = \iota_X\mathrm{d}\eta = \iota_X\bar\partial\eta$. Moreover, being $\bar\partial\eta$ of complex type $(1,1)$, $J^*\iota_X\bar\partial\eta = \iota_{-JX}J^*\bar\partial\eta = i\iota_X\bar\partial\eta$ and similarly for $\iota_Y\mathrm{d}\xi$. Therefore, the first two terms in (2) already belong to $(\mathbb{T}^{(1,0)}M)^*$. Therefore, we only need that $J^*\iota_X\iota_Y H = i\iota_X\iota_Y H$, where only the components of complex types $(1,2)$ and $(0,3)$ of H do not vanish automatically. However, $\iota_X\iota_Y H^{(0,3)}$ is of complex type $(0,1)$, so it must be 0. Finally, H must be of complex type $(2,1)+(1,2)$ since it is a real form.

For the case of $\mathcal{J}_\omega$, the $+i$-eigenspace is integrable if and only if $[X - i\iota_X\omega, Y - i\iota_Y\omega]_H$ is of the form $Z - i\iota_Z\omega$, which is equivalent to

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathrm{d}\iota_X\omega + i\iota_X\iota_Y H\,.$$

This condition can be rewritten as $\iota_Y\iota_X(\mathrm{d}\omega + iH) = 0$, which implies that ω is closed and $H = 0$.

Note that the generalized versions $\mathcal{J}_J$ and $\mathcal{J}_\omega$ are integrable only if the complex or symplectic structures are integrable. Then, generalized complex geometry is a manifestly good formalism to study both complex and symplectic structures.

IV. BRANES AS GENERALIZED SUBMANIFOLDS

Recall, from the introduction, that a p-brane is a $p+1$-dimensional submanifold Σ of the target space M. In order for branes to be physical gauge-invariant objects, they need to carry an additional gauge potential $A\in\Omega^1(\Sigma)$. Under a gauge transformation by $\Lambda\in\Omega^1(M)$, the new gauge potential A and the background Kalb-Ramond B-field transform as

$$\begin{aligned}B &\mapsto B + \mathrm{d}\Lambda\\ A &\mapsto A - \imath^*\Lambda\,,\end{aligned}$$

where $\imath$ is the inclusion $T\Sigma\subseteq TM$. This allows us to build an intrinsic gauge invariant 2-form field $C = \imath^*B + \mathrm{d}A$ related to the brane. Therefore, a physical brane consists of a pair (Σ, C), where $\mathrm{d}C$ coincides with the Neveu-Schwarz field strength on Σ.

When M. Gualtieri was searching for the right notion of generalized submanifold in [8, Ch. 7], A. Kapustin emphasized that the gauge invariance above should be important in order that these submanifolds describe branes. This motivated M. Gualtieri to give the following definitions.

Given a closed $H \in \Omega^3(M)$, the pair (Σ, C) is said to be a **generalized submanifold** of (M, H) if $\mathrm{d}C = H|_{T\Sigma}$, where $\Sigma \subseteq M$ is a submanifold and $C \in \Omega^2(\Sigma)$ a 2-form. We can characterize branes in the language of generalized geometry if we add, to each element $X \in T\Sigma$, 1-forms that extend $\iota_X C \in \Omega^1(\Sigma)$. This is done with the **generalized tangent bundle** of the brane, defined as

$$\tau_\Sigma^C \coloneqq \{X + \xi \in \mathbb{T}M|_\Sigma \,:\, X \in T\Sigma \text{ and } \xi|_{T\Sigma} = \iota_X C\}\,.$$

Note that the fibres of τ_Σ^C at $x \in \Sigma$ are the maximally isotropic subspaces $L(T_x\Sigma, C_x)$ of (1), so that τ_Σ^C is a maximally isotropic subbundle of $\mathbb{T}M|_\Sigma$. The image of τ_Σ^C under $f_*^{-1}\mathrm{e}^B$ is the generalized tangent bundle $\tau_{\tilde\Sigma}^{\tilde C}$ for $\tilde\Sigma = f^{-1}\Sigma$ and $\tilde C = f^*(C + \imath^* B)$. Indeed, for $X + \xi \in \tau_\Sigma^C$,

$$f_*^{-1}\mathrm{e}^B(X+\xi) = f_*^{-1}X + f^*(\xi + \iota_X B) \in \mathbb{T}M|_{\tilde\Sigma}\,,$$

where $f_*^{-1}X \in T\tilde\Sigma$ and

$$\begin{aligned} f^*(\xi + \iota_X B)|_{T\tilde\Sigma} &= f^*\iota_X(C + B|_{T\Sigma}) \\ &= \iota_{f_*^{-1}X} f^*(C + \imath^* B)\,. \end{aligned}$$

Moreover, $(\tilde\Sigma, \tilde C)$ is again a generalized submanifold if $f_*^{-1}\mathrm{e}^B \in \mathrm{GDiff}_H(M)$, because

$$\mathrm{d}\tilde C = f^*(\mathrm{d}C + \imath^*\,\mathrm{d}B) = f^*(H|_{T\Sigma} + \mathrm{d}B\,|_{T\Sigma})$$

and $H|_{T\tilde\Sigma} = f^*((f^{-1})^* H|_{T\Sigma})$. This gives infinitely many generalized submanifolds: in particular, $(\Sigma, C + \imath^* B)$ for closed B and $f = \mathrm{id}$.

The generalized tangent bundle τ_Σ^C is always integrable because the restriction of the Lie bracket on any submanifold is again the Lie bracket and

$$\begin{aligned} &\mathcal{L}_X\iota_Y C - \iota_Y \mathrm{d}\iota_X C + \iota_X\iota_Y H|_{T\Sigma} \\ &\qquad = \mathcal{L}_X\iota_Y C - \iota_Y\mathcal{L}_X C = \iota_{[X,Y]}C\,. \end{aligned}$$

A-type and B-type branes, which appear in topological string theories, are branes compatible with, respectively, an underlying symplectic or complex structure. Recall that these structures are particular cases of generalized complex structures. Therefore, a generalized submanifold (Σ, C) is said to be invariant under a generalized (almost) complex structure $\mathcal{J}$ if $\mathcal{J}\tau_\Sigma^C = \tau_\Sigma^C$.

Having an almost complex structure J on M, a generalized submanifold (Σ, C) is invariant under $\mathcal{J}_J$ if and only if $-JX + J^*\xi \in \tau_\Sigma^C$ for $X + \xi \in \tau_\Sigma^C$. For this, $-JX$ needs to belong to $T\Sigma$ again, which implies that $J|_\Sigma$ is an almost complex structure on Σ. The second condition implies that $J^*\iota_X C = \iota_{-JX}C$, which is equivalent to $C(X,Y) = C(JX, JY)$ for all $X, Y \in T\Sigma$. In other words, C must be of complex type $(1,1)$ on Σ. Moreover, if J is integrable, the $+i$-eigenbundle of J on M is invariant under the H-twisted bracket, and therefore $J|_\Sigma$ is also integrable. Finally, since $H|_{T\Sigma} = \mathrm{d}C = \partial C + \bar\partial C$ and C is of complex type $(1,1)$, $H|_{T\Sigma}$ must be of complex type $(2,1)+(1,2)$. In the end, the complex structure $\mathcal{J}_J|_{T\Sigma}$ on $\mathbb{T}\Sigma$ is integrable on its own. In the case that $H|_{T\Sigma} = 0$, the complex submanifold Σ is endowed with a holomorphic line bundle [8, Ex. 7.7], which is the description of B-type branes (see, e.g., [6, Sec. 2]).

For an almost symplectic structure ω on M, let $\omega|_{T\Sigma}^{-1} : T^*\Sigma \to TM/T^\omega\Sigma$ be the linear map defined by $\omega|_{T\Sigma}^{-1}(\eta) = \left[\iota_\xi\omega^{-1}\right]$ whenever $\xi|_{T\Sigma} = \eta$. This map is well defined because if $\xi|_{T\Sigma} = 0$, then $\iota_\xi\omega^{-1} \in T^\omega\Sigma$. Here, $[\cdot]$ denotes the class of an element in the quotient vector space. A generalized submanifold (Σ, C) is invariant under $\mathcal{J}_\omega$ if and only if $-\iota_\xi\omega^{-1} + \iota_X\omega \in \tau_\Sigma^C$ for $X + \xi \in \tau_\Sigma^C$. This is equivalent to the linear map from $T\Sigma/T^\omega\Sigma$ to itself $[X] \mapsto -\omega|_{T\Sigma}^{-1}(\iota_X C)$ being well defined and an almost complex structure. In particular, $T^\omega\Sigma \subseteq T\Sigma$ and Σ must be a co-isotropic submanifold. We prove this equivalence in the next two paragraphs.

Assume that for $X + \xi \in \tau_\Sigma^C$, $\mathcal{J}_\omega(X+\xi) \in \tau_\Sigma^C$, which means that $Y_\xi = -\iota_\xi\omega^{-1} \in T\Sigma$ and $\iota_X\omega|_{T\Sigma} = \iota_{Y_\xi}C$. Let $Z \in T^\omega\Sigma$, so that $\iota_Z\omega|_{T\Sigma} = 0$. Then $\xi = 0 + \iota_Z\omega \in \tau_\Sigma^C$, and the conditions for these elements read $Y_\xi = -Z \in T\Sigma$ and $\iota_Z C = 0$. This implies that $T^\omega\Sigma \subseteq T\Sigma$ and that C descends to the quotient $T\Sigma/T^\omega\Sigma$. Therefore, $[\pi(\mathcal{J}_\omega\mathbb{X})] = \left[-\iota_\xi\omega^{-1}\right] = -\omega|_{T\Sigma}^{-1}(\iota_X C)$ only depends on $[X]$ and the linear map $[\pi(\mathbb{X})] \mapsto [\pi(\mathcal{J}_\omega\mathbb{X})]$ is a well defined endomorphism of $T\Sigma/T^\omega\Sigma$. Finally, since $\mathcal{J}_\omega$ is a generalized almost complex structure, the endomorphism is automatically an almost complex structure on $T\Sigma/T^\omega\Sigma$.

Conversely, suppose that the linear endomorphism $[X] \mapsto -\omega|_{T\Sigma}^{-1}(\iota_X C)$ of $T\Sigma/T^\omega\Sigma$ is well defined and an almost complex structure. For $X + \xi \in \tau_\Sigma^C$, we have that $\left[\iota_\xi\omega^{-1}\right] = \omega|_{T\Sigma}^{-1}(\iota_X C) \in T\Sigma/T^\omega\Sigma$, so $\iota_\xi\omega^{-1} \in T\Sigma$. Being an almost complex structure, the class of $Y = -\iota_\xi\omega^{-1} \in T\Sigma$ must map to $-\omega|_{T\Sigma}^{-1}(\iota_Y C)$ and to $-[X]$ at the same time. This can only happen if $X + Z = \iota_\eta\omega^{-1}$ for some $Z \in T^\omega\Sigma$ and η such that $\eta|_{T\Sigma} = \iota_Y C$, which implies that $\iota_X\omega|_{T\Sigma} = \iota_{X+Z}\omega|_{T\Sigma} = \iota_Y C$. Then, $\mathcal{J}_\omega(X+\xi) = Y + \iota_X\omega \in \tau_\Sigma^C$ and the generalized tangent bundle is invariant under $\mathcal{J}_\omega$.

In the case that $C = 0$, we have that $T\Sigma/T^\omega\Sigma$ must be trivial, and Σ becomes a Lagrangian submanifold with a flat line bundle [8, Ex. 7.8], which was the usual description of A-type brane. However, more general co-isotropic submanifolds seem to be another option for characterizing A-type branes. In fact, giving string theory arguments, [6] proves that co-isotropic A-type branes must also be allowed. In conclusion, generalized submanifolds allow us to describe A-type and B-type branes under particular cases of $\mathcal{J}$. Considering a general $\mathcal{J}$, this formalism suggests [14, Sec. 7.2] that a topological brane should be more generally considered as a generalized submanifold compatible with any complex structure $\mathcal{J}$.

V. B_n-GENERALIZED GEOMETRY

Recall that generalized geometry allows us to study all exact Courant algebroids, which are necessarily of even rank. A Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho)$ is said to be odd exact if ρ is surjective and $\operatorname{rk} E = 2n+1$. Similarly to how exact Courant algebroids are isomorphic to a twisted structure on $\mathbb{T}M$, B_n-generalized geometry allows us to study all odd exact Courant algebroids. In B_n-generalized geometry we add an additional trivial rank-1 bundle $\mathbb{R} \times M$ to $\mathbb{T}M$, so that it is extended to $E' \coloneqq TM \oplus \mathbb{R} \times M \oplus T^*M$. Elements or sections of E' are of the form $u = X + \lambda + \xi$ for $\lambda \in \mathbb{R}$ or $\lambda \in C^\infty(M)$. The pairing is defined as

$$\langle X + \lambda + \xi, Y + \mu + \eta\rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi) + \lambda\mu \,,$$

which has signature $(n+1, n)$.

Given a subspace $W \subseteq T_xM$, $\omega \in \wedge^2 W$ and $\delta \in W^*$,

$$\begin{aligned} L(W, \delta, \omega) = \{X + \iota_X\delta + \xi \in E'_x \,:\, X \in W \\ \text{and } \xi|_W = \iota_X\omega - \iota_X\delta \cdot \delta\} \end{aligned} \tag{3}$$

is a maximally isotropic subspace of the fibre E'_x, and any such subspace is of this form [9, Prop. 1.2].

Orthogonal symmetries include the previous symmetries of generalized geometry (diffeomorphisms and B-transforms) and also an additional symmetry for $A \in T_x^*M$ sending $u = X + \lambda + \xi$ to $u + \iota_X A - (2\lambda + \iota_X A)A$ called an A-transform. It turns out that B- and A-transforms commute. We denote the composition of both of them by (B, A) and call it a (B, A)-transform. Being orthogonal symmetries, (B, A)-transforms of maximally isotropic subspaces are also maximally isotropic. In fact,

$$(B, A)L(W, \delta, \omega) = L(W, \delta + \imath^*A, \omega + \imath^*A \wedge \delta + \imath^*B)\,,$$

where $\imath : W \to T_xM$ is the inclusion map.

In analogy to the standard case, a B_n-generalized almost complex structure is a maximally isotropic subbundle $L \subseteq E'_{\mathbb{C}}$ such that $L \cap \bar{L} = \{0\}$. Since $E'_{\mathbb{C}}$ has complex rank $2n+1$, $L \oplus \bar{L}$ can be enlarged [9, Sec. 4.1.2] by the rank-1 real subbundle $U = L^\perp \cap \bar{L}^\perp$ so that $E'_{\mathbb{C}} = L \oplus \bar{L} \oplus U$. The analogous endomorphism of an almost complex structure J can be defined on $E'_{\mathbb{C}}$ by setting L to be the $+i$-eigenbundle, $\bar{L}$ the $-i$-eigenbundle and U the 0-eigenbundle. This endomorphism $\mathcal{F}$, fulfils $\mathcal{F}^3 + \mathcal{F} = 0$ and has maximal rank, that is, $2n$. It can be proven that isotropy of L (and $\bar{L}$) is equivalent to antisymmetry of $\mathcal{F}$ with respect to the pairing. In conclusion, an endomorphism $\mathcal{F}$ of E' such that $\mathcal{F}^3 + \mathcal{F} = 0$, has maximal rank at each point and $\langle \mathcal{F}v, v\rangle = 0$, called an $\mathcal{F}$-operator, equivalently defines a B_n-generalized almost complex structure.

For n even, given an almost complex structure J and a 1-form σ such that $\sigma \wedge J^*\sigma = 0$, the map

$$\begin{aligned} &\mathcal{F}_{J,\sigma} : E' \to E' \\ &X + \lambda + \xi \mapsto \mathcal{J}_J(X + \xi) - 2\lambda\sigma + \iota_X\sigma \end{aligned}$$

is an $\mathcal{F}$-operator that extends $\mathcal{J}_J$. On the other hand, given an almost symplectic structure ω and a vector field Z, the map

$$\begin{aligned} &\mathcal{F}_{\omega,Z} : E' \to E' \\ &X + \lambda + \xi \mapsto \mathcal{J}_\omega(X + \xi) + \iota_Z\iota_X\omega \cdot (1 - Z) \\ &\qquad\qquad + (2\lambda - \iota_Z\xi)\iota_Z\omega \end{aligned}$$

is an $\mathcal{F}$-operator that extends $\mathcal{J}_\omega$. For n odd, there exist contact and cosymplectic structures, which resemble the complex and symplectic cases in even-dimensional manifolds.

An almost contact structure on M is defined by a field Z, a 1-form σ and an endomorphism ϕ of TM such that $\iota_Z\sigma = 1$ and $\phi^2(X) = -X + \iota_X\sigma \cdot Z$. Note that in this case, $\phi^2(Z) = 0$, which implies that $\phi(Z) = \iota_{\phi(Z)}\sigma \cdot Z$, and applying again ϕ we conclude that $\phi(Z) = 0$. Similarly, $\phi^*(\sigma) = 0$, and if we let $\mathbb{S} = Z + \sigma$, the map

$$\begin{aligned} &\mathcal{F}_{\phi,\mathbb{S}} : E' \to E' \\ &X + \lambda + \xi \mapsto -\phi X + \phi^*\xi - \lambda\mathbb{S} + \langle X + \xi, \mathbb{S}\rangle \end{aligned} \tag{4}$$

turns out to be an $\mathcal{F}$-operator. An almost cosymplectic structure is defined by a 1-form σ and a 2-form ω such that $\sigma \wedge \omega^{\frac{n-1}{2}}$ is a volume form. In this case, there exists a vector field Z defined by $\iota_Z\omega = 0$ and normalized as $\iota_Z\sigma = 1$, and $\iota_{TM}\omega$ has maximal rank, that is, $n-1$. Let $\mathbb{S} = Z + \sigma$ and let P be the linear map from $T^*M = \langle\sigma\rangle \oplus \iota_{TM}\omega$ to TM given by $P(\sigma) = 0$ and $P(\iota_X\omega) = -X + \iota_X\sigma \cdot Z$. Then, the map

$$\begin{aligned} &\mathcal{F}_{\omega,\mathbb{S}} : E' \to E' \\ &X + \lambda + \xi \mapsto P(\xi) + \iota_X\omega - \lambda\mathbb{S} + \langle X + \xi, \mathbb{S}\rangle \end{aligned} \tag{5}$$

turns out to be an $\mathcal{F}$-operator.

On sections of E', the Dorfman bracket is modified as

$$[X+\lambda+\xi, Y+\mu+\eta] = [X+\xi, Y+\eta]+\mathcal{L}_X\mu-\iota_Y \mathrm{d}\lambda+2\mu \mathrm{d}\lambda,$$

which also makes $(E', \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ a Courant algebroid for the projection $\pi : E' \to TM$. The (F, H)-twisted Dorfman bracket is defined, for a closed $F \in \Omega^2(M)$ and $H \in \Omega^3(M)$ such that $\mathrm{d}H + F^2 = 0$, as

$$[u, v]_{F,H} = [u, v] + \iota_X\iota_Y(F + H) + 2(\mu\iota_X F - \lambda\iota_Y F)\,,$$

where $u = X+\lambda+\xi$ and $v = Y+\mu+\eta$. Again, $[u, \cdot]_{F,H}$ is a derivation of $\langle\cdot,\cdot\rangle$ due to the antisymmetry of H and F, and being a derivation of $[\cdot,\cdot]_{F,H}$ is equivalent to closure of F and $\mathrm{d}H + F^2 = 0$. In fact, any odd exact Courant algebroid is isomorphic to $(E', \langle\cdot,\cdot\rangle, [\cdot,\cdot]_{F,H}, \pi)$ for some F and H [15, Sec. 5].

Generalized diffeomorphisms now include some A-transforms in addition to the elements in $\mathrm{GDiff}_H(M)$. Letting $f \in \mathrm{Diff}(M)$, $B \in \Omega^2(M)$ and $A \in \Omega^1(M)$, it can be proven that, for $u = X+\lambda+\xi$ and $v = Y+\mu+\eta$,

$$[f_*(B, A)(u), f_*(B, A)(v)]_{F,H} - f_*(B, A)[u, v]_{F,H} \quad (6)$$

has a term $2(f^{-1})^*((\lambda\iota_Y - \mu\iota_X)(\mathrm{d}A - f^*F + F))$ that is the only one depending on λ and μ. Therefore, $\mathrm{d}A = f^*F - F$, and in this case, the remaining term of (6) is

$$(f^{-1})^*\iota_Y\iota_X(\mathrm{d}B - A \wedge \mathrm{d}A - 2A \wedge F - f^*H + H)\,.$$

In conclusion, the group of generalized diffeomorphisms $\mathrm{GDiff}_{F,H}(M)$ is

$$\{f_*(B, A) \,:\, f^*F - F = \mathrm{d}A \text{ and } f^*H - H = \mathrm{d}B - A \wedge \mathrm{d}A - 2A \wedge F\}\,,$$

with the composition rule being $f_*(B, A) \circ g_*(B', A') = (f \circ g)_*(g^*B + B' + g^*A \wedge A', g^*A + A')$ [16, Prop. 2.2]. In particular, (B, A)-transforms are generalized diffeomorphisms for A closed and B such that $\mathrm{d}B = 2A \wedge F$.

A B_n-generalized almost complex structure L is said to be integrable if $[u, v]_{F,H} \in \Gamma(L)$ for all $u, v \in \Gamma(L)$, and it is called a B_n-generalized complex structure.

VI. B_n-GENERALIZED SUBMANIFOLDS

Here we propose, as an original contribution, the definition of B_n-generalized submanifolds. In section IV, we saw that generalized submanifolds of (M, H) are pairs (Σ, C) consisting of a submanifold $\Sigma \subseteq M$ and a 2-form C such that $\mathrm{d}C = H|_{T\Sigma}$. In B_n-generalized geometry, in addition to a 3-form H, which is not necessarily closed, we have a closed 2-form F such that $\mathrm{d}H + F^2 = 0$. In analogy, we define a B_n-generalized submanifold of (M, F, H) as a triple (Σ, C, α) consisting of a submanifold $\Sigma \subseteq M$, and forms $C \in \Omega^2(\Sigma)$ and $\alpha \in \Omega^1(\Sigma)$ satisfying some conditions related to F and H. By considering equal degree on forms, these relations should be of the form

$$F|_{T\Sigma} = a\,\mathrm{d}\alpha + bC$$
$$H|_{T\Sigma} = \tilde{a}\,\mathrm{d}C + \tilde{b}\alpha \wedge C + \tilde{c}\alpha \wedge \mathrm{d}\alpha\ ,$$

and imposing that the case $F = \alpha = 0$ recovers the condition $\mathrm{d}C = H|_{T\Sigma}$, we get that $b = 0$ and $\tilde{a} = 1$. Then, by a redefinition of α we might set $a = 1$ and, recalling that $\mathrm{d}H + F^2 = 0$, we get that

$$\tilde{b}(\mathrm{d}\alpha \wedge C - \alpha \wedge \mathrm{d}C) + \tilde{c}\,\mathrm{d}\alpha^2 + \mathrm{d}\alpha^2 = 0\,.$$

Assuming that this has to be true for any F and H in a manifold with dimension $n > 3$, we deduce that $\tilde{b} = 0$ from the case $F = 0$ and that $\tilde{c} = -1$ from the case $F^2 \neq 0$.

Consider a closed $F \in \Omega^2(M)$ and $H \in \Omega^3(M)$ such that $\mathrm{d}H + F^2 = 0$. Given a submanifold $\Sigma \subseteq M$, $\alpha \in \Omega^1(\Sigma)$ and $C \in \Omega^2(M)$, we define the triple (Σ, α, C) to be a B_n**-generalized submanifold** of (M, F, H) if

$$F|_{T\Sigma} = \mathrm{d}\alpha \text{ and}$$
$$H|_{T\Sigma} = \mathrm{d}C - \alpha \wedge \mathrm{d}\alpha\ .$$

Recall that the fibres of τ_Σ^C were the maximal isotropic subspaces $L(T_x\Sigma, C_x)$ defined in (1), so the additional form α suggests that we set the fibres of its B_n version to be $L(T_x\Sigma, \alpha_x, C_x)$ defined in (3). Therefore, we define the B_n**-generalized tangent bundle** as

$$\tau_\Sigma^{\alpha,C} := \{X + \iota_X\alpha + \xi \in E'|_\Sigma \,:\, X \in T\Sigma \text{ and } \xi|_{T\Sigma} = \iota_X C - \iota_X\alpha \cdot \alpha\}\,.$$

Just as in the standard case, it is a maximally isotropic subbundle of $E'|_\Sigma$ and it can be proven to be integrable. Indeed, the Lie bracket on M restricts to the Lie bracket on Σ, the function part of $[X + \iota_X\alpha + \xi, Y + \iota_Y\alpha + \eta]_{F,H}$ is $\mathcal{L}_X\iota_Y\alpha - \iota_Y \mathrm{d}\iota_X\alpha + \iota_X\iota_Y F|_{T\Sigma} = \iota_{[X,Y]}\alpha$ and the form part restricts to

$$\mathcal{L}_X\eta|_{T\Sigma} - \iota_Y \mathrm{d}\xi|_{T\Sigma} + 2\iota_Y\alpha \cdot \mathrm{d}\iota_X\alpha + \iota_X\iota_Y H|_{T\Sigma} + 2(\iota_Y\alpha\cdot\iota_X F|_{T\Sigma} - \iota_X\alpha\cdot\iota_Y F|_{T\Sigma}) = \iota_{[X,Y]}C - \iota_{[X,Y]}\alpha\cdot\alpha\,.$$

Let us study what happens with the image of $X + \iota_X\alpha + \xi \in \tau_\Sigma^{\alpha,C}$ under $f_*^{-1}(B, A)|_\Sigma$. First of all, letting

$Y+\mu+\eta = f_*^{-1}(B,A)|_\Sigma(X+\iota_X\alpha+\xi)$, we get that $Y=f_*^{-1}X$, so $Y\in T(f^{-1}\Sigma)$. Letting $\imath: T\Sigma\to TM$, for the function part we get that

$$\mu = f^*(\iota_X\alpha+\iota_X A|_\Sigma) = \iota_{f_*^{-1}X} f^*(\alpha+\imath^* A)\,,$$

and, for the form part, $\eta|_{T(f^{-1}\Sigma)}$ equals

$$\begin{aligned} &f^*(\xi|_{T\Sigma}+\iota_X B|_{T\Sigma} - (2\iota_X\alpha+\iota_X A|_\Sigma)\cdot A|_{T\Sigma}) \\ &\quad = \iota_{f_*^{-1}X}(f^*(C+\imath^*B)) - \iota_{f_*^{-1}X}(f^*\alpha)\cdot f^*\alpha \\ &\qquad - \iota_{f_*^{-1}X}(f^*(2\alpha+\imath^*A))\cdot f^*\imath^*A\,. \quad (7)\end{aligned}$$

From the function part we get that $f_*^{-1}(B,A)\tau_\Sigma^{\alpha,C}$ is a generalized tangent bundle $\tau_{f^{-1}\Sigma}^{\tilde\alpha,\tilde C}$ if and only if $\tilde\alpha = f^*(\alpha+\imath^*A)$. However, equation (7) needs some more work to deduce what $\tilde C$ should be. The last two terms of (7) equal

$$\begin{aligned} &-\iota_{f_*^{-1}X}(f^*\alpha)\cdot f^*\alpha - \iota_{f_*^{-1}X}(f^*\alpha+\tilde\alpha)\cdot(\tilde\alpha - f^*\alpha) \\ &\quad = -\iota_{f_*^{-1}X}(f^*\alpha+\tilde\alpha)\cdot\tilde\alpha + \iota_{f_*^{-1}X}\tilde\alpha\cdot f^*\alpha \\ &\qquad = -\iota_{f_*^{-1}X}\tilde\alpha\cdot\tilde\alpha + \iota_{f_*^{-1}X}(\tilde\alpha\wedge f^*\alpha)\,,\end{aligned}$$

and noting that $\tilde\alpha\wedge f^*\alpha = f^*(\imath^*A\wedge\alpha)$, it is clear that $\tilde C = f^*(C+\imath^*B+\imath^*A\wedge\alpha)$. In this case again, $(f^{-1}\Sigma,\tilde\alpha,\tilde C)$ is a B_n-generalized submanifold if $f^{-1}(B,A)\in \mathrm{GDiff}_{F,H}(M)$. Indeed,

$$\begin{aligned} \mathrm{d}\tilde\alpha &= f^*(\mathrm{d}\alpha+\imath^*\,\mathrm{d}A) = f^*(F|_{T\Sigma}+\mathrm{d}A\,|_{T\Sigma}) \\ &= f^*((f^{-1})^*F|_{T\Sigma}) = F|_{T(f^{-1}\Sigma)}\,,\end{aligned}$$

and

$$\begin{aligned} &\mathrm{d}\tilde C - \tilde\alpha\wedge\mathrm{d}\tilde\alpha \\ &= f^*(\mathrm{d}C+\imath^*\mathrm{d}B-\alpha\wedge\mathrm{d}\alpha - 2\imath^*A\wedge\mathrm{d}\alpha - \imath^*A\wedge\imath^*\mathrm{d}A) \\ &= f^*(H|_{T\Sigma}+\mathrm{d}B|_{T\Sigma} - 2A|_{T\Sigma}\wedge F|_{T\Sigma} - A|_{T\Sigma}\wedge\mathrm{d}A|_{T\Sigma}) \\ &\qquad = f^*((f^{-1})^*H|_{T\Sigma}) = H|_{T(f^{-1}\Sigma)}\,.\end{aligned}$$

Given an $\mathcal{F}$-operator, the B_n-generalized submanifold (Σ,α,C) is said to be invariant under $\mathcal{F}$ if $\mathcal{F}\tau_\Sigma^{\alpha,C}\subseteq\tau_\Sigma^{\alpha,C}$. Let us study what are the conditions for particular $\mathcal{F}$-operators introduced in section V.

For $\mathcal{F}_{\phi,\mathbb{S}}$ given in (4), recall that $\phi Z=0$, $\phi^*\sigma=0$, $\phi^2X=-X+\iota_X\sigma\cdot Z$ and $(\phi^*)^2\xi = -\xi+\iota_Z\xi\cdot\sigma$. Then, if $u=X+\iota_X\alpha+\xi\in\tau_\Sigma^{\alpha,C}$, we get that

$$\begin{aligned} \mathcal{F}_{\phi,\mathbb{S}}(u) &= -\phi X+\phi^*\xi - \iota_X\alpha\cdot\mathbb{S}+\langle X+\xi,\mathbb{S}\rangle\,, \\ \mathcal{F}^2_{\phi,\mathbb{S}}(u) &= \phi^2X+(\phi^*)^2\xi - \langle X+\xi,\mathbb{S}\rangle\cdot\mathbb{S} \\ &\quad + \langle -\phi X+\phi^*\xi-\iota_X\alpha\cdot\mathbb{S},\mathbb{S}\rangle \\ &= -X+\iota_X\sigma\cdot Z - \xi+\iota_Z\xi\cdot\sigma \\ &\quad - \langle X+\xi,\mathbb{S}\rangle\cdot\mathbb{S} - \iota_X\alpha\,.\end{aligned}$$

Letting $\tilde{\mathbb{S}} = -Z+\sigma$, note that $\iota_X\sigma - \langle X+\xi,\mathbb{S}\rangle = \langle X+\xi,\tilde{\mathbb{S}}\rangle$ and $\iota_Z\xi - \langle X+\xi,\mathbb{S}\rangle = -\langle X+\xi,\tilde{\mathbb{S}}\rangle$. Then, imposing $\mathcal{F}_{\phi,\mathbb{S}}(u),\mathcal{F}^2_{\phi,\mathbb{S}}(u)\in\tau_\Sigma^{\alpha,C}$, invariance under $\mathcal{F}_{\phi,\mathbb{S}}$ is equivalent to the conditions

$$\begin{aligned} &-\phi X - \iota_X\alpha\cdot Z, \langle X+\xi,\tilde{\mathbb{S}}\rangle\cdot Z\in T\Sigma \\ &\langle X+\xi,\mathbb{S}\rangle = \iota_{-\phi X-\iota_X\alpha\cdot Z}\alpha \\ &\iota_{\langle X+\xi,\tilde{\mathbb{S}}\rangle\cdot Z}\alpha = 0 \\ &(\phi^*\xi - \iota_X\alpha\sigma)|_{T\Sigma} = \iota_{-\phi X-\iota_X\alpha Z}C - \langle X+\xi,\mathbb{S}\rangle\alpha \\ &\iota_{\langle X+\xi,\tilde{\mathbb{S}}\rangle\cdot Z}C = -\langle X+\xi,\tilde{\mathbb{S}}\rangle\cdot\sigma|_{T\Sigma}\,.\end{aligned} \quad (8)$$

First of all, if $\langle X+\xi,\tilde{\mathbb{S}}\rangle\neq 0$ for some element, from the first equation in (8) we get that $Z\in T\Sigma$, but then from the last one in (8), contracting with Z we get that $\langle X+\xi,\tilde{\mathbb{S}}\rangle=0$, which is impossible. Then, we must have $\langle X+\xi,\tilde{\mathbb{S}}\rangle = 0$ for all elements, so that $\langle X+\xi,\mathbb{S}\rangle = \iota_X\sigma=\iota_Z\xi$ and the third and last equations in (8) become trivial. In fact, in this case $\mathcal{F}^2_{\phi,\mathbb{S}} = -\mathbb{1}$ on $\tau_\Sigma^{\alpha,C}$ and $\tilde{\mathbb{S}}$ is orthogonal to $\tau_\Sigma^{\alpha,C}$.

The first condition in (8) is equivalent to the fact that the linear map $\tilde\phi$ defined by $\tilde\phi X = -\phi X-\iota_X\alpha\cdot Z$ is an endomorphism of $T\Sigma$. The second condition in (8) is equivalent to $\tilde\phi^*\alpha = \sigma|_{T\Sigma}$ because we have that $\iota_X\sigma = \langle X+\xi,\mathbb{S}\rangle = \iota_{\tilde\phi X}\alpha$ for all $X\in T\Sigma$. This last condition is equivalent to $\tilde\phi$ being an almost complex structure. Note that, in particular, Σ must be even-dimensional. The fourth condition in (8) is equivalent to C having complex type $(1,1)$ with respect to $\tilde\phi$. Indeed, for $Y\in T\Sigma$, contracting the fourth condition in (8) with $\tilde\phi Y\in T\Sigma$ we get that

$$\iota_{\tilde\phi Y}(\phi^*\xi) - \iota_X\alpha\cdot\iota_{\tilde\phi Y}\sigma = \iota_{\tilde\phi Y}\iota_{\tilde\phi X}C - \iota_{\tilde\phi X}\alpha\cdot\iota_{\tilde\phi Y}\alpha\,. \quad (9)$$

Since $\xi|_{T\Sigma} = \iota_XC - \iota_X\alpha\cdot\alpha$ and $\phi\tilde\phi Y = Y-\iota_Y\sigma\cdot Z$, the first term in the left-hand side equals

$$\begin{aligned} \iota_{\phi\tilde\phi Y}\xi &= \iota_{Y-\iota_Y\sigma\cdot Z}\xi = \iota_Y\xi - \iota_Y\sigma\cdot\iota_Z\xi \\ &= \iota_Y\iota_XC - \iota_X\alpha\cdot\iota_Y\alpha - \iota_Y\sigma\cdot\iota_X\sigma\,,\end{aligned}$$

where we have used that $\iota_Z\xi = \iota_X\sigma$ and that $\xi|_{T\Sigma} = \iota_XC - \iota_X\alpha\cdot\alpha$. Recalling that $\tilde\phi^*\alpha = \sigma|_{T\Sigma}$, (9) becomes $\iota_Y\iota_XC = \iota_{\tilde\phi Y}\iota_{\tilde\phi X}C$, that is, C is of complex type $(1,1)$ with respect to $\tilde\phi$.

For $\mathcal{F}_{\omega,\mathbb{S}}$ given in (5), recall that $\iota_Z\omega=0$ and $P(\sigma)=0$. Then, if $u=X+\iota_X\alpha+\xi\in\tau_\Sigma^{\alpha,C}$, we get that

$$\begin{aligned} \mathcal{F}_{\omega,\mathbb{S}}(u) &= P(\xi)+\iota_X\omega - \iota_X\alpha\cdot\mathbb{S}+\langle X+\xi,\mathbb{S}\rangle\,, \\ \mathcal{F}^2_{\omega,\mathbb{S}}(u) &= P(\iota_X\omega)+\iota_{P(\xi)}\omega - \langle X+\xi,\mathbb{S}\rangle\cdot\mathbb{S} \\ &\quad + \langle P(\xi)+\iota_X\omega-\iota_X\alpha\cdot\mathbb{S},\mathbb{S}\rangle \\ &= -X+\iota_X\sigma\cdot Z-\xi+\iota_Z\xi\cdot\sigma \\ &\quad - \langle X+\xi,\mathbb{S}\rangle\cdot\mathbb{S} - \iota_X\alpha\,,\end{aligned}$$

where we have used that $\xi = \iota_Z\xi \cdot \sigma + \iota_Y\omega$ for some $Y \in TM$, so that $\iota_{P(\xi)}\sigma = \iota_{-Y+\iota_Y\sigma\cdot Z}\sigma = 0$ and $\iota_{P(\xi)}\omega = \iota_{-Y+\iota_Y\sigma\cdot Z}\omega = -\iota_Y\omega = -\xi + \iota_Z\xi\cdot\sigma$. Note that the expression for $\mathcal{F}^2_{\omega,\mathbb{S}}(u)$ is identical to the $\mathcal{F}^2_{\phi,\mathbb{S}}(u)$ case. Then, imposing $\mathcal{F}_{\omega,\mathbb{S}}(u), \mathcal{F}^2_{\omega,\mathbb{S}}(u) \in \tau^{\alpha,C}_\Sigma$, invariance under $\mathcal{F}_{\omega,\mathbb{S}}$ is equivalent to

$$\begin{aligned} &P(\xi) - \iota_X\alpha\cdot Z \in T\Sigma \\ &\langle X+\xi, \mathbb{S}\rangle = \iota_{P(\xi)-\iota_X\alpha\cdot Z}\alpha \\ &(\iota_X\omega - \iota_X\alpha\sigma)|_{T\Sigma} = \iota_{P(\xi)-\iota_X\alpha Z}C - \langle X+\xi,\mathbb{S}\rangle\alpha\,, \end{aligned} \tag{10}$$

in addition to the previous conditions $\langle X+\xi, \tilde{\mathbb{S}}\rangle\cdot Z \in T\Sigma$ and the third and last equations in (8). Reasoning as before, we conclude that $\iota_X\sigma = \iota_Z\xi$ for all elements of $\tau^{\alpha,C}_\Sigma$ (i.e., $\tilde{\mathbb{S}}$ is orthogonal to $\tau^{\alpha,C}_\Sigma$) and $\mathcal{F}^2_{\phi,\mathbb{S}} = -\mathbb{1}$ at $\tau^{\alpha,C}_\Sigma$.

Let $T^{\sigma,\omega}\Sigma = \ker\sigma \cap T^\omega\Sigma$. Note that for $X \in T^{\sigma,\omega}\Sigma$, $\xi = \iota_X\omega \in \tau^{\alpha,C}_\Sigma$ and for this case the conditions read $P(\iota_X\omega) = -X \in T\Sigma$, $\iota_X\alpha = 0$ and $\iota_X C = 0$. Then, $T^{\sigma,\omega}\Sigma \subseteq T\Sigma$ and α and C descend to the quotient $T\Sigma/T^{\sigma,\omega}\Sigma$. Let us see that the linear endomorphism of $T\Sigma/T^{\sigma,\omega}\Sigma$ given by $[X] \mapsto [P(\xi) - \iota_X\sigma\cdot Z]$ whenever $X + \iota_X\alpha + \xi \in \tau^{\alpha,C}_\Sigma$ is well defined. If ξ' is another 1-form such that $X+\iota_X\alpha+\xi' \in \tau^{\alpha,C}_\Sigma$, we must have $\xi - \xi' \in \tau^{\alpha,C}_\Sigma$. This implies that $\iota_Z(\xi-\xi') = \iota_0\sigma$ and $\xi|_{T\Sigma} = \xi'|_{T\Sigma}$, so that

$$\iota_{P(\xi-\xi')}\omega|_{T\Sigma} = (\xi' - \xi + \iota_Z(\xi-\xi')\cdot\sigma)|_{T\Sigma} = 0\,,$$

which means that $P(\xi-\xi') \in T^{\sigma,\omega}\Sigma$ and the endomorphism is independent from the 1-form chosen. Moreover, since α and C descend to the quotient, $T^{\sigma,\omega}\Sigma \subseteq \tau^{\alpha,C}_\Sigma$, which implies that $[P(\xi) - \iota_X\alpha\cdot Z]$ only depends on $[X]$. Finally, since the endomorphism is $[\pi(u)] \mapsto [\pi(\mathcal{F}_{\omega,\mathbb{S}}(u))]$, it is automatically an almost complex structure.

Conversely, assume that $u = X + \iota_X\alpha + \xi \in \tau^{\alpha,C}_\Sigma$, that $-Z+\sigma$ is orthogonal to $\tau^{\alpha,C}_\Sigma$ and that the endomorphism of $T\Sigma/T^{\sigma,\omega}\Sigma$ given by $[X] \mapsto [P(\xi) - \iota_X\alpha\cdot Z]$ is well defined and an almost complex structure. Let $Y = P(\xi) - \iota_X\alpha\cdot Z$, which lies in $T\Sigma$ by hypothesis, and note that $\iota_Y\sigma = -\iota_X\alpha$. Let $\eta \in \Omega^1(M)$ extend $\iota_Y C - \iota_Y\alpha\cdot\alpha$ so that $Y + \iota_Y\alpha + \eta \in \tau^{\alpha,C}_\Sigma$. Since the endomorphism is an almost complex structure, $[Y]$ must map to $[P(\eta) - \iota_Y\alpha\cdot Z]$ and to $-[X]$ at the same time, which implies that $X + \tilde{X} = \iota_Y\alpha\cdot Z - P(\eta)$ for some $\tilde{X} \in T^{\sigma,\omega}\Sigma$. Then, contracting with σ we get that $\iota_X\sigma = \iota_Y\alpha$, and contracting with ω,

$$\iota_X\omega|_{T\Sigma} = -\iota_{P(\eta)}\omega|_{T\Sigma} = (\eta - \iota_Z\eta\cdot\sigma)|_{T\Sigma}\,.$$

Using that $\iota_Z\eta = \iota_Y\sigma = -\iota_X\alpha$ we conclude that $\mathcal{F}_{\omega,\mathbb{S}}(u) = Y + \iota_X\sigma + \iota_X\omega - \iota_X\alpha\cdot\sigma \in \tau^{\alpha,C}_\Sigma$ also, so $\tau^{\alpha,C}_\Sigma$ is invariant under $\mathcal{F}_{\omega,\mathbb{S}}$.

In summary, a B_n-generalized submanifold (Σ,α,C) is invariant under $\mathcal{F}_{\phi,\mathbb{S}}$ if and only if $\langle\tilde{\mathbb{S}}, \tau^{\alpha,C}_\Sigma\rangle = 0$, $\tilde{\phi} \in \mathrm{End}(T\Sigma)$ given by $\tilde{\phi}X = -\phi X - \iota_X\alpha\cdot Z$ is an almost complex structure and C is of complex type $(1,1)$ with respect to $\tilde{\phi}$. On the other hand, (Σ,α,C) is invariant under $\mathcal{F}_{\omega,\mathbb{S}}$ if and only if $\langle\tilde{\mathbb{S}}, \tau^{\alpha,C}_\Sigma\rangle = 0$ and the linear endomorphism of $T\Sigma/T^{\sigma,\omega}\Sigma$ that sends $[X]$ to $[P(\xi) - \iota_X\alpha\cdot Z]$ whenever $X + \iota_X\alpha + \xi \in \tau^{\alpha,C}_\Sigma$ is well defined and an almost complex structure.

VII. CONCLUSION

We have seen how generalized geometry serves as a good mathematical framework to work with Kalb-Ramond B-fields and the Neveu-Schwarz 3-form field H in string theories. We have focused on how physical branes of topological string theories are described in terms of generalized submanifolds invariant under particular generalized complex structures. Then, motivated by the fact that generalized complex structures only exist in even-dimensional manifolds, we have introduced the B_n version of generalized complex structures that exist in odd-dimensional manifolds. Finally, we have studied the conditions that B_n-generalized submanifolds need to fulfil in order to be invariant under two types of B_n-generalized complex structures. This opens the way for other possible branes that contain an additional gauge invariant 1-form. On the one hand, the almost contact structure case studied in section VI suggests that even-dimensional branes embedded in odd-dimensional background geometries can also be described in terms of B_n-generalized submanifolds. On the other hand, the almost cosymplectic structure case suggests that there might exist other branes similar to the co-isotropic branes in odd-dimensional background geometries.

ACKNOWLEDGMENTS

I would like to thank my supervisor Roberto Rubio for introducing me to the very interesting topic of generalized geometry and for his continuous support and patience during the work. I would also like to thank David Mateos for his help in technical details of the introduction.

Appendix: A third point of view on generalized complex structures

We describe here a third way to define generalized complex structures that shows its interaction with the theory of Clifford algebras and spinors. We will denote the sum of forms of arbitrary degree by $\wedge^\bullet(TM)$ and its sections by $\Omega^\bullet(M)$.

1. Forms and Clifford action

Consider a form $\varphi \in \wedge^\bullet(T_x^*M)$. The natural action of an element of T_xM is the interior product, and the natural action of an element of T_x^*M is the exterior wedge product. Thus, it might be natural to consider the $\mathbb{R}$-linear action of $\mathbb{T}_xM$ on $\wedge^\bullet(T_x^*M)$ given by $(X+\xi)\cdot\varphi = \iota_X\varphi + \xi\wedge\varphi$. Recalling that the interior product is antisymmetric on elements of $\wedge^\bullet(T_x^*M)$, one easily computes that for $\mathbb{X} = X+\xi$,

$$\begin{aligned}\mathbb{X}\cdot(\mathbb{X}\cdot\varphi) &= \mathbb{X}\cdot(\iota_X\varphi + \xi\wedge\varphi) = \iota_x(\xi\wedge\varphi) + \xi\wedge\iota_X\varphi \\ &= (\iota_X\xi)\varphi = \langle\mathbb{X},\mathbb{X}\rangle\varphi\,.\end{aligned}$$

This action is called the Clifford action since bilinearity of the action implies that $\mathbb{X}\cdot\mathbb{Y}+\mathbb{Y}\cdot\mathbb{X}$ acts on $\wedge^\bullet(T_x^*M)$ as the scalar $2\langle\mathbb{X},\mathbb{Y}\rangle$, which is the defining property of a Clifford algebra. Being this the standard spin representation, elements on the exterior algebra $\wedge^\bullet(T_x^*M)$ are called spinors. Given a non-zero spinor $\varphi\in\wedge^\bullet(T_x^*M)$, consider its annihilator: the subspace $\mathrm{Ann}(\varphi) = \{\mathbb{X}\in\mathbb{T}_xM : \mathbb{X}\cdot\varphi = 0\}$. This subspace is isotropic because $0 = \mathbb{X}\cdot(\mathbb{X}\cdot\varphi) = \langle\mathbb{X},\mathbb{X}\rangle\varphi$, which implies $\langle\mathbb{X},\mathbb{X}\rangle = 0$ since $\varphi\neq 0$. Denoting by $e^B = \sum_{j=0}^\infty \frac{1}{j!}B^j$ where exponentiation is understood as repeated wedge product with itself, $e^{-B}\wedge\varphi$ is said to be the B-action on φ and one can prove that the B-transform of $\mathrm{Ann}(\varphi)$ is the annihilator of the B-action on φ. A spinor φ is said to be pure if $\mathrm{Ann}(\varphi)$ is maximally isotropic, which, as studied in [17], happens if and only if φ is of the form $e^B\wedge\psi_{\mathrm{d}}$ for $B\in\wedge^2(T_x^*M)$ and a decomposable $\psi_{\mathrm{d}}\in\wedge^\bullet(T_x^*M)\setminus\{0\}$. Moreover, it is proven in [17, III.1.2] that any maximally isotropic subspace of T_xM is the annihilator of a pure spinor line in $\wedge^\bullet(T_x^*M)$. The degree of ψ_{d}, which equals the degree of the lowest non-vanishing term of φ is said to be the type of the pure spinor φ and, by extension, the type of $\mathrm{Ann}(\varphi)$.

Let σ be the operator on $\wedge^\bullet(T_x^*M)$ that extends linearly the index-reversing operator $\sigma(\xi_1\wedge\cdots\wedge\xi_k) = \xi_k\wedge\cdots\wedge\xi_1$. Then, one defines the Chevalley pairing of two spinors φ,ψ to be

$$(\varphi,\psi) = (\sigma(\varphi)\wedge\psi)\,|_{\mathrm{top}}\,,$$

where $|_{\mathrm{top}}$ projects to the top-degree component of the spinor.

2. Generalized linear complex structures

When it comes to characterizing a generalized complex structure in terms of pure spinors, one can always choose $\varphi\in\wedge^\bullet(T_xM_{\mathbb{C}})$ such that the $+i$-eigenspace of $\mathcal{J}$ at x is $L = \mathrm{Ann}(\varphi)$ and $L\cap\bar{L} = \{0\}$ implies that $(\varphi,\bar\varphi)\neq 0$. Moreover, it can be proven that such a pure spinor, up to a complex non-zero multiple, equivalently gives a generalized complex structure. In the particular cases of $\mathcal{J}_J$ and $\mathcal{J}_\omega$, the corresponding pure spinors are a volume form of $T_x^{(1,0)}M$ and $e^{i\omega}$, respectively.

3. Generalized geometry

The Clifford action of $\mathbb{X}\in\Gamma(M)$ on $\varphi\in\Omega^\bullet(M)$ is extended to the function $x\mapsto\mathbb{X}_x\cdot\varphi_x$ on M.

Letting $\mathrm{d}^H = \mathrm{d}+H\wedge$ denote the H-twisted differential, it is proven in [8, Prop. 3.44] that a maximally isotropic subbundle $L = \mathrm{Ann}(\varphi)$ is integrable if and only if $\mathrm{d}^H\varphi = \mathbb{X}\cdot\varphi$ for some $\mathbb{X}\in\Gamma(\mathbb{T}M_{\mathbb{C}})$.

Imposing that the $+i$-eigenbundle of $\mathcal{J}$ is integrable can be seen to be equivalent to the vanishing of the H-twisted Nijenhuis tensor

$$N_{\mathcal{J}}^H(\mathbb{X},\mathbb{Y}) = [\mathcal{J}\mathbb{X},\mathcal{J}\mathbb{Y}]_H - \mathcal{J}[\mathcal{J}\mathbb{X},\mathbb{Y}]_H - \mathcal{J}[\mathbb{X},\mathcal{J}\mathbb{Y}]_H - [\mathbb{X},\mathbb{Y}]_H\,.$$

4. B_n-generalized geometry

Denoting by τ the operation that sends an r-degree form to $(-1)^r$ times itself, the Clifford action of $X+\lambda+\xi\in E'_x$ on $\varphi\in\wedge^\bullet(T_x^*M)$ is $\iota_X\varphi+\lambda\tau\varphi+\xi\wedge\varphi$. Similarly to the case in generalized geometry, $u\cdot(v\cdot\varphi)+v\cdot(u\cdot\varphi) = 2\langle u,v\rangle\varphi$, so $\mathrm{Ann}(\varphi)$ is again an isotropic subspace of E'_x. The (B,A)-action on φ is defined as $e^{-(B+A\tau)}\wedge\varphi$, and it can be proven that the (B,A)-transform of $\mathrm{Ann}(\varphi)$ is the annihilator of the (B,A)-action on φ. Analogously, $\mathrm{Ann}(\varphi)$ is maximally isotropic if and only if φ equals a (B,A)-action on a non-zero decomposable form, and φ is called a pure spinor. The Chevalley pairing depends on the parity of n, since $(\varphi,\psi) = (\sigma(\varphi)\wedge\tau^n\psi)|_{\mathrm{top}}$.

Similarly, given a pure spinor $\varphi\in\Omega^\bullet(M,\mathbb{C})$, $\mathrm{Ann}(\varphi)$ is a B_n-generalized almost complex structure if and only if $(\varphi,\bar\varphi)\neq 0$, and a B_n-generalized almost complex structure is locally equivalent to a complex line of pure spinors.

The (F,H)-twisted differential $\mathrm{d}^{F,H} = \mathrm{d}+F\wedge\tau+H\wedge$ allows to characterize integrability of $\mathrm{Ann}(\varphi)$ in terms of the pure spinor φ. A B_n-generalized almost complex structure given locally by $\mathrm{Ann}(\varphi)$ is integrable if and only if $\mathrm{d}^{F,H}\varphi = u\cdot\varphi$ for some $u\in\Gamma(E'_{\mathbb{C}})$ [9, Prop. 4.15]. The spinorial viewpoint, although not used for generalized submanifolds, is a very good source of examples in the general theory.

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