CAPACITIES ASSOCIATED WITH CALDERÓN-ZYGMUND KERNELS

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Abstract. Analytic capacity is associated with the Cauchy kernel $1/z$ and the L^∞-norm. For $n \in \mathbb{N}$, one has likewise capacities related to the kernels $K_i(x) = x_i^{2n-1}/|x|^{2n}$, $1 \leq i \leq 2$, $x = (x_1, x_2) \in \mathbb{R}^2$. The main result of this paper states that the capacities associated with the vectorial kernel (K_1, K_2) are comparable to analytic capacity.

The analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup |f'(\infty)|$$

where the supremum is taken over those analytic functions in $\mathbb{C} \setminus E$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{C} \setminus E$. Sets of zero analytic capacity are exactly the removable sets for bounded analytic functions, as shown by Ahlfors, and thus $\gamma(E)$ quantifies the non-removability of E. Early work on analytic capacity used basically one complex variable methods (see, e.g., [A], [Ga1] and [Vi]). Analytic capacity may be written as

$$\gamma(E) = \sup |\langle T, 1\rangle|, \tag{1}$$

where the supremum is taken over all complex distributions T supported on E whose Cauchy potential $f = 1/z * T$ is in the closed unit ball of $L^\infty(\mathbb{C})$. Expression (1) shows that analytic capacity is formally an analogue of classical logarithmic capacity, in which the logarithmic kernel has been replaced by the complex kernel $1/z$. This suggests that real variables techniques could help in studying analytic capacity, in spite of the fact that the Cauchy kernel is complex. In fact, significant progress in the understanding of analytic capacity was achieved when real variables methods were systematically used ([C], [D], [MaMeV], [MTV], [T2] and [T4]), in particular the Calderón-Zygmund theory of the Cauchy singular integral.

Recall that for a Borel set E with finite length, $0 < \mathcal{H}^1(E) < \infty$, David and Léger (see [Lé]) proved that the $L^2(\mathcal{H}^1|E)-$boundedness of the singular integral associated with the Cauchy kernel (or even with one of its coordinate parts $x_1/|x|^2$, $x_2/|x|^2$, $x = (x_1, x_2) \in \mathbb{R}^2$) implies that E is rectifiable. We recall that a set in $\mathbb{R}^2$ is rectifiable if it is contained, up to an $\mathcal{H}^1$-negligible set, in a countable union of 1-dimensional Lipschitz graphs. In [ChMPT] we extended this result to any kernel of the form $x_i^{2n-1}/|x|^{2n}$, $i = 1, 2$, $n \in \mathbb{N}$, providing the first non-trivial examples of operators not directly related to the Cauchy transform whose L^2-boundedness implies rectifiability.

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In this paper we introduce capacities associated with these kernels. For $n \geq 1$, write $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the kernels

$$K_1(x) = x_1^{2n-1}/|x|^{2n} \text{ and } K_2(x) = x_2^{2n-1}/|x|^{2n}. \tag{2}$$

For compact sets $E \subset \mathbb{R}^2$, we define

$$\gamma_n(E) = \sup |\langle T, 1\rangle|,$$

the supremum taken over those real distributions T supported on E such that for $i = 1, 2$, the potentials $K_i * T$ are in the unit ball of $L^\infty(\mathbb{R}^2)$.

We will show that the above defined capacity is comparable to analytic capacity, that is,

Theorem 1. *There exists some positive constant C such that for all compact sets $E \subset \mathbb{R}^2$,*

$$C^{-1}\gamma_n(E) \leq \gamma(E) \leq C\gamma_n(E).$$

The main motivation to study these capacities is getting a better understanding of the relation between the operators whose L^2-boundedness implies rectifiability and the comparability of analytic capacity and the capacities related to the kernels of such operators. It is worth to mention that if one considers the kernel $k(x_1, x_2) = x_1 x_2^2/|x|^4$, then the comparability result between analytic capacity and the capacity related to the kernel k does not hold. See Section 1.1 for more details.

For our second main result, we turn to the higher dimensional setting. Motivated by the paper [MPrVe], we set $n = 1$ and consider capacities in $\mathbb{R}^d$, associated with the kernels $x_i/|x|^2$, $1 \leq i \leq d$.

For a compact $E \subset \mathbb{R}^d$ set

$$\Gamma(E) = \sup \{|\langle T, 1\rangle|\},$$

where the supremum is taken over those real distributions T supported on E such that the vector field $\dfrac{x}{|x|^2} * T$ is in the unit ball of $L^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Notice that, for $d = 2$, due to [T2], $\Gamma(E)$ is comparable to the analytic capacity $\gamma(E)$. Finally, for $1 \leq k \leq d$, set

$$\Gamma_{\hat{k}}(E) = \sup \left\{ |\langle T, 1\rangle| : \left\| \frac{x_i}{|x|^2} * T \right\|_\infty \leq 1,\ 1 \leq i \leq d,\ i \neq k \right\}. \tag{3}$$

Thus we require the boundedness of $d-1$ components of the vector valued potential $x/|x|^2 * T$ with Riesz kernel of homogeneity -1.

In the plane, an easy complex argument (see [MPrVe]) shows that

$$\gamma(E) \approx \Gamma_{\hat{k}}(E), \quad k = 1, 2. \tag{4}$$

However in higher dimensions, this is an open question and indeed very little is known about these capacities $\Gamma_{\hat{k}}$. The reason why $\Gamma_{\hat{k}}$ is difficult to understand in higher dimensions is that boundedness of $d-1$ potentials does not provide any linear growth condition on the distribution T. Concretely, it is not true that boundedness of $x_i/|x|^2 * T$, $1 \leq i \leq d-1$, implies that for each cube Q one has

$$|\langle T, \varphi_Q\rangle| \leq Cl(Q), \tag{5}$$

for each test function $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying $\|\varphi_Q\|_\infty \leq 1$ and $\|\nabla\varphi_Q\|_\infty \leq l(Q)^{-1}$. See Section 5 of [MPrVe] for some examples of such phenomenon. Here $l(Q)$ stands for the side length of Q.

In [MPrVe] it was shown that the capacities $\Gamma_{\hat{k}}(E)$ are finite. Moreover, the following higher dimensional version of (4) was also shown: for $d \geq 3$,

$$\Gamma(E) \approx \Gamma_{\hat{k}}(E), \quad 1 \leq k \leq d, \tag{6}$$

assuming an extra growth condition on the definition of the capacities $\Gamma_{\hat{k}}(E)$. Naturally, the following open question appeared: is it true that (6) holds without any growth condition on the definition of $\Gamma_{\hat{k}}(E)$?

Our next result deals with this question and answers it in the affirmative sense, replacing the capacity $\Gamma_{\hat{k}}$, $1 \leq k \leq d$, by the capacity $\Gamma_{\hat{k},+}$, which is a version of $\Gamma_{\hat{k}}$ in the sense that one replaces the real distributions in (3) by positive measures. It is defined as follows, given a compact set $E \subset \mathbb{R}^d$,

$$\Gamma_{\hat{k},+}(E) = \sup \mu(E),$$

the supremum taken over those positive measures μ supported on E such that the potentials $\mu * x_i/|x|^2$, $1 \leq i \leq d$, $i \neq k$, are in the unit ball of $L^\infty(\mathbb{R}^d)$.

Theorem 2. *There exists some positive constant C such that for all compact sets $E \subset \mathbb{R}^d$*

$$C^{-1}\Gamma_{\hat{k},+}(E) \leq \Gamma(E) \leq C\Gamma_{\hat{k},+}(E).$$

The paper is organized as follows, Section 1.1 contains some examples of capacities that are not comparable to analytic capacity. In Section 1.2 we present a sketch of the proof of Theorem 1. It becomes clear that the proof depends on two facts: the close relationship between the quantities one obtains after symmetrization of the kernels $1/z$ and $x_i^{2n-1}/|x|^{2n}$, $i = 1, 2$, and a localization L^∞ estimate for the scalar kernels $x_i^{2n-1}/|x|^{2n}$. In Section 2 we deal with the symmetrization issue and in Section 3 with the localization estimate. In Section 4 we show an exterior regularity property of γ_n needed for the proof of Theorem 1. In Section 5 we prove Theorem 2. Finally, in Section 6 we present various additional results.

1. Preliminaries

1.1. **Some capacities that are not comparable to analytic capacity.** Let K be some Calderón-Zygmund kernel of homogeneity -1 and consider its associated capacity γ_K which is defined as follows: for a compact set $E \subset \mathbb{R}^2$,

$$\gamma_K(E) = \sup\{|\langle T, 1\rangle|\},$$

the supremum taken over all distributions supported on the set E and such that $K * T$ is an $L^\infty-$ function with $\|K * T\|_\infty \leq 1$.

As we already stated in the Introduction, we are interested in characterizing which are the homogeneous Calderón-Zygmund kernels whose related capacity is comparable to the analytic capacity γ. We are as well interested in the open problem of fully characterizing the homogeneous Calderón-Zygmund operators whose boundedness in $L^2(\mathcal{H}^1|E)$ implies the rectifiability of E (see [MaMeV], [Lé] and [ChMPT]). We think that both characterizations are deep problems in the area as even the candidate classe of "reasonable" kernels for the problems is far from clear. The relation between the two problems is illustrated in the Proposition 3 below. As a consequence, Corollary 5 shows that for some Calderón-Zygmund kernels, the capacities related to them are not comparable to analytic capacity.

Proposition 3. *Let $E \subset \mathbb{R}^2$ be a compact set with $\mathcal{H}^1(E) < \infty$. Let K be some Calderón-Zygmund kernel of homogeneity -1 and S_K its associated Calderón-Zygmund operator. If $\gamma_K(E) \approx \gamma(E)$ and $S_K : L^2(\mathcal{H}^1|E) \to L^2(\mathcal{H}^1|E)$, then E is not purely unrectifiable.*

Proof. Let $F \subset E$ be such that $\mathcal{H}^1(F) > 0$ and $\mathcal{H}^1|F$ has linear growth. Set $\mu = \mathcal{H}^1|F$. From the $L^2(\mu)-$boundedness of S_K, we get that each S_K is of weak type $(1,1)$ with respect to μ. This follows from the standard Calderón-Zygmund theory if the measure is doubling and by an argument from [NTV1] in the general case. By a standard dualization process (see [DØ], [Ch, Theorem 23], [U] and [MP]) we get that for each compact set $G \subset F$ with $0 < \mu(G) < \infty$, there exists a function h supported on G, $0 \leq h \leq 1$, such that $\displaystyle\int_G h d\mu \geq C\mu(G)$ and $\|S_K(h d\mu)\|_\infty = \|K * h d\mu\|_\infty \leq 1$. Therefore $\gamma_K(E) > 0$ and $\gamma(E) > 0$ as well. Then by [D], E is not purely unrectifiable (recall that a set E is purely unrectifiable if the intersection of E with any curve of finite length has zero 1-dimensional Hausdorff measure). □

From Proposition 3 we obtain the following corollary :

Corollary 4. *Let K be some Calderón-Zygmund kernel of homogeneity -1 and S_K its associated Calderón-Zygmund operator. Suppose $\gamma_K \approx \gamma$. If $E \subset \mathbb{R}^2$ is a compact set with $\mathcal{H}^1(E) < \infty$ and S_K is bounded in $L^2(\mathcal{H}^1|E)$, then E is rectifiable.*

Proof. If E were not rectifiable, then taking a purely unrectifiable compact subset $F \subset E$ with $\mathcal{H}^1(F) > 0$ and using that, by Proposition 3, $\gamma_K(F) \approx \gamma(F)$, we would get that F is not purely unrectifiable, a contradiction. □

In [H], it is shown that there exist homogeneous kernels, such as $H(x_1, x_2) = \frac{x_1 x_2^2}{|x|^4}$, $x = (x_1, x_2) \in \mathbb{R}^2$, whose corresponding singular integrals are L^2-bounded on purely unrectifiable sets. We consider now the capacity related to this kernel H, namely γ_H. As a consequence of Proposition 3 we obtain the following corollary

Corollary 5. *There exists some compact set $E \subset \mathbb{R}^2$ with $\gamma(E) = 0$ and $\gamma_H(E) > 0$.*

It is worth saying that Huovinen's method does not work for the kernels we are considering in (2), namely his construction does not give a purely unrectifiable set when changing the kernel H by the kernels in (2).

1.2. **Sketch of the proof of Theorem 1.** In this section we will sketch the proof of the two inequalities appearing in the statement of Theorem 1. The first one is the following, for a compact set $E \subset \mathbb{R}^2$,

$$\gamma_n(E) \leq C\,\gamma(E). \tag{7}$$

For the proof of this inequality we need to introduce the Cauchy transform with respect to an underlying positive Radon measure μ satisfying the linear growth condition

$$\mu(B(x,r)) \leq C\,r, \quad x \in \mathbb{R}^2, \quad r \geq 0. \tag{8}$$

Given $\epsilon > 0$ we define the truncated Cauchy transform at level ϵ as

$$C_\epsilon(f\,\mu)(z) = \int_{|w-z|>\epsilon} \frac{f(w)}{w-z}\,d\mu(w), \quad z \in \mathbb{R}^2, \tag{9}$$

for $f \in L^2(\mu)$. For a finite measure μ, the growth condition on μ insures that each C_ϵ is a bounded operator on $L^2(\mu)$ with operator norm $\|C_\epsilon\|_{L^2(\mu)}$ possibly depending on ϵ. We say

that the Cauchy transform is bounded on $L^2(\mu)$ when the truncated Cauchy transforms are uniformly bounded on $L^2(\mu)$. Call $L(E)$ the set of positive Radon measures supported on E which satisfy (8) with $C = 1$. One defines the capacities $\gamma_{\rm op}(E)$ and $\gamma_+(E)$ by

$$\gamma_{\rm op}(E) = \sup\{\mu(E) : \mu \in L(E) \quad \text{and} \quad \|C\|_{L^2(\mu)} \leq 1\},$$

$$\gamma_+(E) = \sup\{\mu(E) : \mu \in L(E) \quad \text{and} \quad \|\frac{1}{z} * \mu\|_\infty \leq 1\}.$$

Clearly $\gamma_+(E) \leq \gamma(E)$. The deep result in [T2] asserts that in fact $\gamma_+(E)$ is comparable to the anality capacity of E. In [T1], it was proved that the capacitiy $\gamma_+(E)$ is comparable to $\gamma_{\rm op}(E)$, that is, for some positive constant C one has

$$C^{-1}\,\gamma_{\rm op}(E) \leq \gamma_+(E) \leq C\,\gamma_{\rm op}(E), \tag{10}$$

for each compact set $E \subset \mathbb{R}^2$. We remind the reader that the first inequality in (10) depends on a simple but ingenious duality argument due to Davie and Øksendal (see [DØ, p.139], [Ch, Theorem 23, p.107] and [V, Lemma 4.2]).

From the first inequality in (10) we get that for some constant C and all compact sets E,

$$\gamma_{\rm op}(E) \leq C\,\gamma(E).$$

To prove (7) we will estimate $\gamma_n(E)$ by a constant times $\gamma_{\rm op}(E)$. The natural way to perform that is to introduce the capacity $\gamma_{n,{\rm op}}(E)$ and check the validity of the two estimates

$$\gamma_n(E) \leq C\,\gamma_{n,{\rm op}}(E) \tag{11}$$

and

$$\gamma_{n,{\rm op}}(E) \leq C\,\gamma_{\rm op}(E). \tag{12}$$

To define $\gamma_{n,{\rm op}}$, first we introduce the truncated transform $S_{n,\varepsilon}(f\,\mu)(x)$ associated with the vectorial kernel $K = (K_1, K_2)$ with $K_i(x) = x_i^{2n-1}/|x|^{2n}$, $i = 1, 2$, as in (9), but with the Cauchy kernel replaced by the vector valued kernel K just defined. We also set

$$\|S_n\|_{L^2(\mu)} = \sup_{\epsilon>0} \|S_{n,\varepsilon}\|_{L^2(\mu)},$$

and

$$\gamma_{n,{\rm op}}(E) = \sup\{\mu(E) : \mu \in L(E) \quad \text{and} \quad \|S_n\|_{L^2(\mu)} \leq 1\,\}.$$

One proves (12) by checking that the symmetrization of the Cauchy kernel is controlled by the symmetrization of kernel K (see Lemma 6 and Corollary 9). In fact, we prove in Corollary 8 that for a positive measure μ having linear growth, the $L^2(\mu)$ boundedness of the Cauchy transform is equivalent to the $L^2(\mu)$ boundedness of the operators S_n. Therefore, the capacities $\gamma_{n,{\rm op}}(E)$ and $\gamma_{\rm op}(E)$ are comparable. Here the fact that we are dealing with kernels of homogeneity -1 plays a key role, because, as it is shown by Farag in [F], they enjoy a special positivity property which is missing in general. See Section 2 for complete details.

The proof of (11) depends on Tolsa's proof of $\gamma(E) \leq C\,\gamma_{\rm op}(E)$. One of the technical points that we need to prove in our setting is a localization result for the potentials we deal with in this case, namely for the potentials associated with the kernels K_i, $i = 1, 2$. Specifically, in Section 3 we prove that there exists a positive constant C such that, for each compactly supported distribution T and for each coordinate i, we have

$$\left\| \frac{x_i^{2n-1}}{|x|^{2n}} * \varphi_Q T \right\|_\infty \leq C \left(\left\| \frac{x_i^{2n-1}}{|x|^{2n}} * T \right\|_\infty + G(T) \right) \tag{13}$$

for each square Q and each $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying $\|\varphi_Q\|_\infty \leq 1$ and $\|\nabla \varphi_Q\|_\infty \leq l(Q)^{-1}$. Here $G(T)$ is some constant related to the linear growth of T (see Section 3 for a definition).

Once (13) is at our disposition, we claim that inequality (11) can be proved by adapting the scheme of the proof of Theorems 1.1 in [T2] and 7.1 in [T3]. As Lemma 16 shows, the capacities γ_n, $n \in \mathbb{N}$, enjoy the exterior regularity property. This is also true for the capacities $\gamma_{n,+}$, defined by

$$\gamma_{n,+}(E) = \sup\left\{\mu(E) : \left\|\frac{x_j^{2n-1}}{|x|^{2n}} * \mu\right\|_\infty \leq 1, j = 1, 2\right\},$$

just by the weak $\star$ compactness of the set of positive measures with total variation not exceeding 1. Therefore we can approximate a general compact set E by sets which are finite unions of squares of the same side length in such a way that the capacities γ_n and $\gamma_{n,+}$ of the approximating sets are as close as we wish to those of E. As in (10), one has, using the Davie-Øksendal Lemma for several operators [MP, Lemma 4.2],

$$C^{-1}\,\gamma_{n,\mathrm{op}}(E) \leq \gamma_{n,+}(E) \leq C\,\gamma_{n,\mathrm{op}}(E).$$

Thus we can assume, without loss of generality, that E is a finite union of squares of the same size. This will allow to implement an induction argument on the size of certain rectangles. The first step involves rectangles of diameter comparable to the side length of the squares whose union is E.

The starting point of the general inductive step in [T2] and [T3] consists in the construction of a positive Radon measure μ supported on a compact set F which approximates E in an appropriate sense. The set F is defined as the union of a special family of squares $\{Q_i\}_{i=1}^N$ that cover the set E and approximate E at an appropriate intermediate scale. One then sets

$$F = \bigcup_{i=1}^N Q_i.$$

The construction of the approximating set F implies that $\gamma_{n,+}(F) \leq C\,\gamma_{n,+}(E)$. This part of the proof extends without any obstruction to our case because of the positivity properties of the symmetrization of our kernels (see Section 2). To construct the measure μ, observe that the definition of $\gamma_n(E)$ gives us a real distribution S_0 supported on E such that

(1) $\gamma_n(E) \leq 2|\langle S_0, 1\rangle|$.

(2) $\left\|\frac{x_j^{2n-1}}{|x|^{2n}} * S_0\right\|_\infty \leq 1, \quad 1 \leq j \leq 2.$

Consider now functions $\varphi_i \in \mathcal{C}_0^\infty(2Q_i)$, $0 \leq \varphi_i \leq 1$, $\|\varphi_i\|_\infty \leq 1$ and $\|\nabla \varphi_i\|_\infty \leq l(Q_i)^{-1}$ and $\sum_{i=1}^N \varphi_i = 1$ on $\bigcup_i Q_i$. We define now simultaneously the measure μ and an auxiliary measure ν, which should be viewed as a model for S_0 adapted to the family of squares $\{Q_i\}_{i=1}^N$. For each square Q_i take a concentric segment Σ_i of length a small fixed fraction of $\gamma_n(E \cap 2Q_i)$ and set

$$\mu = \sum_{i=1}^N \mathcal{H}^1_{|\Sigma_i} \quad \text{and} \quad \nu = \sum_{i=1}^N \frac{\langle S_0, \varphi_i\rangle}{\mathcal{H}^1(\Sigma_i)} \mathcal{H}^1_{|\Sigma_i}.$$

We have $d\nu = b d\mu$, with $b = \dfrac{\langle \varphi_i, \nu_0 \rangle}{\mathcal{H}^1(\Sigma_i)}$ on Σ_i. At this point we need to show that our function b is bounded, to apply later a suitable $T(b)$ Theorem. To estimate $\|b\|_\infty$ we use the localization inequalities (13). Thus, $|\langle S_0, \varphi_i \rangle| \leq C\, \gamma_n(2Q_i \cap E)$, for $1 \leq i \leq N$. It is now easy to see that $\gamma_n(E) \leq C\, \mu(F)$:

$$\gamma_n(E) \leq 2\, |\langle S_0, 1 \rangle| = 2 \left| \sum_{i=1}^N \langle S_0, \varphi_i \rangle \right| \leq C \sum_{i=1}^N \gamma_n(2Q_i \cap E) = C\, \mu(F).$$

Notice that the construction of F and μ gives readily that $\gamma_n(E) \leq C\, \mu(F)$, and $\gamma_{n,+}(F) \leq C\, \gamma_{n,+}(E)$, which tells us that F is not too small but also not too big. However, one cannot expect the operator S_n to be bounded on $L^2(\mu)$. One has to carefully look for a compact subset G of F such that $\mu(F) \leq C\, \mu(G)$, the restriction μ_G of μ to G has linear growth and S_n is bounded on $L^2(\mu_G)$ with dimensional constants. This completes the proof because then

$$\begin{aligned} \gamma_n(E) &\leq C\, \mu(F) \leq C\, \mu(G) \leq C\, \gamma_{n,\mathrm{op}}(G) \leq C\, \gamma_{n,\mathrm{op}}(F) \\ &\leq C\, \gamma_{n,+}(F) \leq C\, \gamma_{n,+}(E) \leq C\, \gamma_{n,\mathrm{op}}(E). \end{aligned}$$

We do not insist in summarizing the intricate details, which can be found in [T2] and [T3], of the definition of the set G and of the application of the $T(b)$ Theorem of [NTV2].

The second inequality in Theorem 1 is

$$\gamma(E) \leq C \gamma_n(E). \tag{14}$$

Since by [T2], $\gamma(E) \approx \gamma_{\mathrm{op}}(E)$, and as we mentioned above we have

$$\gamma_{\mathrm{op}}(E) \leq C \gamma_{n,\mathrm{op}}(E), \tag{15}$$

we get that $\gamma(E) \leq C\gamma_{\mathrm{op}}(E) \leq C\gamma_{n,\mathrm{op}}(E)$. The duality arguments used to prove the first inequality in (10) can also be used in our setting, therefore $\gamma_{n,\mathrm{op}}(E) \leq C\gamma_{n,+}(E)$ holds. Finally, by definition, $\gamma_{n,+}(E) \leq \gamma_n(E)$. This shows how (14) in Theorem 1 can be proved.

2. Symmetrization process and L^2-boundedness

The symmetrization process for the Cauchy kernel introduced in [Me] has been successfully applied to many problems of analytic capacity and L^2 boundedness of the Cauchy integral operator (see [MeV], [MaMeV], [T2], and the book [P], for example). In the recent paper [ChMPT], the symmetrization method was also used to give the first non-trivial examples of operators not directly related to the Cauchy transform whose L^2-boundedness implies rectifiability.

Given three distinct points in the plane, z_1, z_2 and z_3, one finds out, by an elementary computation that

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} \tag{16}$$

where the sum is taken over the permutations of the set $\{1, 2, 3\}$ and $c(z_1, z_2, z_3)$ is *Menger curvature*, that is, the inverse of the radius of the circle through z_1, z_2 and z_3. In particular (16) shows that the sum on the right hand side is a non-negative quantity.

In $\mathbb{R}^2$ and for $1 \leq i \leq 2$ the quantity

$$p_i(z_1, z_2, z_3) = K_i(z_1 - z_2)\, K_i(z_1 - z_3) + K_i(z_2 - z_1)\, K_i(z_2 - z_3) + K_i(z_3 - z_1)\, K_i(z_3 - z_2),$$

is the obvious analogue of the right hand side of (16) for the kernel $K_i(x) = x_i^{2n-1}/|x|^{2n}$. In [ChMPT] it was shown that for any three distinct points $z_1, z_2, z_3 \in \mathbb{R}^2$, the quantities $p_i(z_1, z_2, z_3)$, $1 \leq i \leq 2$, are also non negative and they vanish if and only if the three points are colinear.

The relationship between the quantity $p_i(z_1, z_2, z_3)$, $1 \leq i \leq 2$, and the L^2 estimates of the operator with kernel $x_i^{2n-1}/|x|^{2n}$ is as follows. Take a compactly supported positive Radon measure μ in $\mathbb{R}^2$ with linear growth. Given $\varepsilon > 0$ consider the truncated transform $T_\varepsilon^i(\mu)$ of μ associated with the kernel K_i, as in Section 1.2. Then we have (see in [MeV] the argument for the Cauchy integral operator)

$$\left| \int |T_\varepsilon^i(\mu)(x)|^2 \, d\mu(x) - \frac{1}{3} p_{i,\varepsilon}(\mu) \right| \leq C\|\mu\|,$$

C being a positive constant depending only on n and the linear growth constant of μ, and

$$p_{i,\varepsilon}(\mu) = \iiint_{S_\varepsilon} p_i(x, y, z) \, d\mu(x) \, d\mu(y) \, d\mu(z),$$

with

$$S_\varepsilon = \{(x, y, z) : |x - y| > \varepsilon, \, |x - z| > \varepsilon \text{ and } |y - z| > \varepsilon\}.$$

It is worth saying now that for $n = 1$ and $i = 1, 2$, $p_i(z_1, z_2, z_3) = \frac{1}{2} c(z_1, z_2, z_3)^2$. For $n > 1$, it is in general not true that $p_i(z_1, z_2, z_3)$, $i = 1, 2$, is comparable to Menger curvature $c(z_1, z_2, z_3)^2$. The next two lemmas show that the sum of the above defined permutations, $p_1(z_1, z_2, z_3) + p_2(z_1, z_2, z_3)$ is comparable to Menger curvature, $c(z_1, z_2, z_3)^2$.

Lemma 6. *There exists a constant $c_1 = c_1(n)$, such that for all distinct points $z_1, z_2, z_3 \in \mathbb{R}^2$,*

$$p_1(z_1, z_2, z_3) + p_2(z_1, z_2, z_3) \geq c_1 c(z_1, z_2, z_3)^2.$$

Proof. It suffices to prove the claim for any triple $(0, z, w), z \neq w \in \mathbb{R}^2 \setminus \{0\}$. For any line L denote by $\theta_V(L)$ and $\theta_H(L)$ the smallest angle that L forms with the vertical and horizontal axes respectively. Then at least two of the angles,

$$\theta_V(L_{0,z}), \quad \theta_V(L_{0,w}), \quad \theta_V(L_{z,w})$$

or at least two of the angles

$$\theta_H(L_{0,z}), \quad \theta_H(L_{0,w}), \quad \theta_H(L_{z,w})$$

are greater or equal than $\pi/4$. Without loss of generality we can assume that

$$\theta_V(L_{0,z}) \geq \frac{\pi}{4} \quad \text{and} \quad \theta_V(L_{0,w}) \geq \frac{\pi}{4}. \tag{17}$$

Now let $\theta = \theta_V(L_{z,w})$. Let c be some very small constant, depending on n, that will be chosen later.

Case 1: $\theta \geq c$.

As in Lemma 2.3 in [ChMPT], we have that for $z = (x, y)$ and $w = (a, b)$,

$$p_1(0, z, w) \geq n \left(\frac{|x|}{|z|} \right)^{2n-2} \left(\frac{|a|}{|w|} \right)^{2n-2} \left(\frac{|x - a|}{|z - w|} \right)^{2n-2} \frac{\sin^2(z, w)}{|z - w|^2}. \tag{18}$$

By (17) we have that

$$\frac{|x|}{|z|} > \frac{1}{2}, \qquad \frac{|a|}{|w|} > \frac{1}{2} \tag{19}$$

and by the assumption in this case,

$$\frac{|x-a|}{|z-w|} \geq \sin c.$$

Furthermore,

$$c(0,z,w) = \frac{2\sin(z,w)}{|z-w|}.$$

By (18),

$$p_1(0,z,w) \geq c_1 c(0,z,w)^2,$$

for some positive constant c_1 depending on n.

Case 2: $\theta < c$.

In this case, notice that by (19),

$$\begin{aligned}\big||x|-|a|\big| \leq |x-a| = |z-w|\sin\theta &\leq |z|\sin\theta + |w|\sin\theta \\ &\leq 2|x|\sin\theta + 2|a|\sin\theta.\end{aligned}$$

Hence,

$$\frac{1-2\sin\theta}{1+2\sin\theta}|a| \leq |x| \leq \frac{1+2\sin\theta}{1-2\sin\theta}|a|$$

and since $\theta < c$ and c will be chosen very small, it follows that

$$\frac{|a|}{2} \leq |x| \leq 2|a|. \tag{20}$$

Combining (20) and (19) we obtain that

$$\frac{|w|}{4} \leq |z| \leq 4|w|. \tag{21}$$

Expanding $p_1(0,z,w)$ we get

$$\begin{aligned}p_1(0,z,w) &= \frac{x^{2n-1}a^{2n-1}}{|z|^{2n}|w|^{2n}} + \frac{(x-a)^{2n-1}}{|z-w|^{2n}}\left(\frac{x^{2n-1}}{|z|^{2n}} - \frac{a^{2n-1}}{|w|^{2n}}\right) \\ &= A + B,\end{aligned}$$

where the last equality is a definition for A and B. Since

$$|x^{2n-1} - a^{2n-1}| \leq |x-a|\left(|x|^{2n-2} + |x|^{2n-3}|a| + \cdots + |x||a|^{2n-3} + |a|^{2n-2}\right),$$

then by (20),

$$|x^{2n-1} - a^{2n-1}| \leq (2n-1)2^{2n-2}|x-a||x|^{2n-2}. \tag{22}$$

Arguing in the same way and using (21) we obtain

$$\frac{\big||w|^{2n} - |z|^{2n}\big|}{|z|^{2n}|w|^{2n}} \leq 8n\ 4^{2n-1}\frac{\big||z|-|w|\big|}{|w|^{2n+1}}. \tag{23}$$

Notice that

$$\frac{x^{2n-1}}{|z|^{2n}} - \frac{a^{2n-1}}{|w|^{2n}} = \frac{x^{2n-1}-a^{2n-1}}{|z|^{2n}} + a^{2n-1}\left(\frac{1}{|z|^{2n}} - \frac{1}{|w|^{2n}}\right).$$

Therefore from (22) and (23) we get

$$|B| \leq \frac{(\sin\theta)^{2n-1}}{|z-w|}\left(\frac{(2n-1)2^{2n-2}|x-a||x|^{2n-2}}{|z|^{2n}} + \frac{8n\ 4^{2n-1}\big||z|-|w|\big||a|^{2n-1}}{|w|^{2n+1}}\right)$$
$$\leq (\sin\theta)^{2n-1}\left(\frac{(2n-1)2^{2n-2}}{|z|^2} + \frac{8n\ 4^{2n-1}}{|w|^2}\right) \leq (\sin\theta)^{2n-1}\left(\frac{16n4^{2n-1}}{|w|^2}\right).$$

On the other hand, by (19) and (21),

$$|A| = \left(\frac{|x|}{|z|}\right)^{2n-1}\left(\frac{|a|}{|w|}\right)^{2n-1}\frac{1}{|w||z|} \geq \left(\frac{1}{4}\right)^{2n}\frac{1}{|w|^2}$$

Therefore choosing $c \leq \dfrac{1}{10^4 n}$ we obtain that

$$p_1(0,z,w) \geq \frac{1}{2}\left(\frac{1}{4}\right)^{2n}\frac{1}{|w|^2}.$$

Since it follows easily that $c(0,z,w) \leq \dfrac{2}{|w|}$, the proof is complete. □

Lemma 7. *There exists a positive constant $C = C(n)$ such that for all distinct points $z_1, z_2, z_3 \in \mathbb{R}^2$,*

$$p_i(z_1,z_2,z_3) \leq C\ c(z_1,z_2,z_3)^2, \ \ 1 \leq i \leq 2.$$

Proof. Without loss of generality fix $i = 1$. Since p_1 is translation invariant, it is enough to estimate the permutations $p_1(0,z,w)$ for any two distinct points $z = (x,y)$, $w = (a,b) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$|z| \leq |z-w| \quad \text{and} \quad |w| \leq |z-w|. \tag{24}$$

As shown in Proposition 2.1 in [ChMPT],

$$p_1(0,z,w) = \frac{A(z,w)}{|z|^{2n}|w|^{2n}|z-w|^{2n}}, \tag{25}$$

where

$$A(z,w) = \sum_{k=1}^{n}\binom{n}{k}x^{2(n-k)}a^{2(n-k)}(x-a)^{2(n-k)}F_k(z,w) \tag{26}$$

and

$$F_k(z,w) = x^{2k-1}a^{2k-1}(y-b)^{2k} + x^{2k-1}(x-a)^{2k-1}b^{2k} - a^{2k-1}(x-a)^{2k-1}y^{2k}.$$

Notice also that

$$(xb-ay)^2 = |z|^2|w|^2\sin^2(z,w) = \frac{1}{4}|z|^2|w|^2|z-w|^2c(0,z,w)^2. \tag{27}$$

Case 1: $a = 0$.

In this case, notice that $F_n(z,w) = x^{4n-2}b^{2n}$ and all sumands in (26) are zero, apart from the last one. Therefore, using (27) and (24),

$$\begin{aligned} p_1(0,z,w) &= \frac{x^{4n-4}b^{2n-2}}{|z|^{2n}|w|^{2n}|z-w|^{2n}}x^2b^2 \\ &= \frac{1}{4}\frac{|z|^2|w|^2|z-w|^2x^{4n-4}b^{2n-2}}{|z|^{2n}|w|^{2n}|z-w|^{2n}}c(0,z,w)^2 \\ &\leq \frac{1}{4}\frac{|x|^{2n-2}}{|z-w|^{2n-2}}c(0,z,w)^2 \leq \frac{1}{4}c(0,z,w)^2. \end{aligned}$$

Case 2: $a \neq 0$ and $b \neq 0$.

Let $t = x/a$ and $s = y/b$. Then F_k can be rewritten as follows

$$\begin{aligned} \frac{F_k(z,w)}{a^{4k-2}b^{2k}} &= \left(\frac{x}{a}\right)^{2k-1}\left(\frac{y}{b}-1\right)^{2k} + \left(\frac{x}{a}\right)^{2k-1}\left(\frac{x}{a}-1\right)^{2k-1} - \left(\frac{x}{a}-1\right)^{2k-1}\left(\frac{y}{b}\right)^{2k} \\ &= t^{2k-1}(s-1)^{2k} + t^{2k-1}(t-1)^{2k-1} - (t-1)^{2k-1}s^{2k} \\ &= P(s,t), \end{aligned}$$

the last identity being the definition of the polynomial $P(s,t)$. Then, for some polynomial $Q(s,t)$,

$$P(s,t) = (s-t)^2Q(s,t),$$

because if we consider P as a polynomial of the variable s with parameter t , i.e. $P_t(s) := P(s,t)$, we obtain easily that

$$P_t(t) = P_t'(t) = 0.$$

It is also immediate to check that the degree of P is $4k-2$ and the smallest degree of the monomials of P is $2k$.

Therefore

$$Q(s,t) = \sum_{l+l'=2k-2}^{4k-4} c_{l,l'}t^l s^{l'}.$$

By (27) and (24), for each $1 \leq k \leq n$,

$$\begin{aligned}
|F_k(z,w)| &= \left|a^{4k-4}b^{2k-2}(xb-ay)^2 Q\left(\frac{x}{a},\frac{y}{b}\right)\right| \\
&= \frac{1}{4}|a|^{4k-4}|b|^{2k-2}|z|^2|w|^2|z-w|^2 c(0,z,w)^2 \ \left|Q\left(\frac{x}{a},\frac{y}{b}\right)\right| \\
&\leq C(n)|a|^{4k-4}|b|^{2k-2}\ |z|^2|w|^2|z-w|^2\ c(0,z,w)^2 \sum_{l+l'=2k-2}^{4k-4} \left|\frac{x}{a}\right|^l \left|\frac{y}{b}\right|^{l'} \\
&= C(n)|z|^2|w|^2|z-w|^2 c(0,z,w)^2 \sum_{l+l'=2k-2}^{4k-4} |a|^{4k-4-l}|b|^{2k-2-l'}|x|^l|y|^{l'} \\
&\leq C(n)|z|^2|w|^2|z-w|^2 c(0,z,w)^2 \sum_{l+l'=2k-2}^{4k-4} |w|^{6k-6-(l+l')}|z|^{l+l'} \\
&= C(n)|z|^{2k}|w|^{2k}|z-w|^2\ c(0,z,w)^2 \sum_{l+l'=2k-2}^{4k-4} |w|^{4k-4-(l+l')}|z|^{l+l'-2k+2} \\
&\leq C(n)|z|^{2k}|w|^{2k}|z-w|^2\ c(0,z,w)^2 \sum_{l+l'=2k-2}^{4k-4} |z-w|^{4k-4-(l+l')+l+l'-2k+2} \\
&\leq C(n)|z|^{2k}|w|^{2k}|z-w|^{2k}\ c(0,z,w)^2.
\end{aligned}$$

Then, from (25) we conclude that

$$\begin{aligned}
p_1(0,z,w) &= \sum_{k=1}^{n} \binom{n}{k} \frac{x^{2(n-k)}a^{2(n-k)}(x-a)^{2(n-k)}}{|z|^{2n}|w|^{2n}|z-w|^{2n}} F_k(z,w) \\
&\leq C(n) \left(\frac{|x|}{|z|}\right)^{2n} \left(\frac{|a|}{|w|}\right)^{2n} \left(\frac{|x-a|}{|z-w|}\right)^{2n} c(0,z,w)^2 \sum_{k=1}^{n} \binom{n}{k} \\
&\leq C(n)\ c(0,z,w)^2.
\end{aligned}$$

Case 3: $b = 0$.
In this case $F_k(z,w) = a^{2k-1}y^{2k}(x^{2k-1} - (x-a)^{2k-1})$. Hence by (27)

$$F_1(z,w) = a^2y^2 = \frac{1}{4}|z|^2|w|^2|z-w|^2 c(0,z,w)^2. \tag{28}$$

For $1 < k \leq n$, by using (27) again,

$$\begin{aligned}
F_k(z,w) &= a^{2k-1}y^{2k}(x^{2k-1} - (x-a)^{2k-1}) \\
&= a^2y^2a^{2k-3}y^{2k-2}(x^{2k-1} - (x-a)^{2k-1}) \\
&= \frac{1}{4}|z|^2|w|^2|z-w|^2 c(0,z,w)^2 a^{2k-3}y^{2k-2} \sum_{j=0}^{2k-2} \binom{2k-1}{j} x^j a^{2k-1-j}.
\end{aligned}$$

And using (24) we estimate,

$$\begin{aligned}|F_k(z,w)| &\le C(n)c(0,z,w)^2|z|^2|w|^2|z-w|^2|w|^{2k-3}|z|^{2k-2}\sum_j^{2k-2}|z|^j|w|^{2k-1-j}\\ &\le C(n)c(0,z,w)^2|z|^{2k}|w|^{2k}|z-w|^2\sum_{j=0}^{2k-1}|z|^j|w|^{2k-2-j}\\ &= C(n)c(0,z,w)^2|z|^{2k}|w|^{2k}|z-w|^2\sum_{j=0}^{2k-1}|z-w|^j|z-w|^{2k-2-j}\\ &\le C(n)c(0,z,w)^2|z|^{2k}|w|^{2k}|z-w|^{2k}.\end{aligned}$$

The previous estimate combined with (28) implies that for $1 \le k \le n$

$$|F_k(z,w)| \le C(n)c(0,z,w)^2|z|^{2k}|w|^{2k}|z-w|^{2k}.$$

Therefore, from (25) we derive that

$$p_1(0,z,w) \le C(n)\ c(0,z,w)^2$$

in an identical manner to case 1.

□

From these two lemmas and the relationship between the symmetrization method and the L^2-norm we obtain the following:

Corollary 8. *Let S_n be the operator associated with the vectorial kernel $K = (K_1, K_2)$, with $K_i = x_i^{2n-1}/|x|^{2n}$, $n \in \mathbb{N}$ and $1 \le i \le 2$. If μ is a compactly supported positive measure in the plane having linear growth, the Cauchy transform of μ is bounded on $L^2(\mu)$ if and only if S_n is bounded on $L^2(\mu)$.*

We state now inequalities (12) and (15), because they are immediate consequences of the preceding corollary.

Corollary 9. *There exists a positive constant C such that for any compact set $E \subset \mathbb{R}^2$,*

$$C^{-1}\ \gamma_{\rm op}(E) \le \gamma_{n,\rm op}(E) \le C\,\gamma_{\rm op}(E).$$

It is worth to mention that for $n = 1$, it was proven in [MPrVe] that corollary 8 remains valid if the operator S_1 is replaced by one of its coordinates, S_1^1 or S_1^2, (here S_1^i is the operator with kernel $x_i/|x|^2$, $i = 1, 2$).

3. Growth conditions and localization

We need the following reproduction formula for the kernels $K_i(x) = x_i^{2n-1}/|x|^{2n}$:

Lemma 10. *If a function $f(x)$ has continuous derivatives up to order one, then it is representable in the form*

$$f(x) = (\varphi_1 * K_1)(x) + (\varphi_2 * K_2)(x),\ x \in \mathbb{R}^2, \tag{29}$$

where for $i = 1, 2$,

$$\varphi_i = S_i(\partial_i f) := c\partial_i f + \widetilde{S}_i(\partial_i f), \tag{30}$$

for some constant c and Calderón-Zygmund operators $\widetilde{S}_1$ and $\widetilde{S}_2$.

The proof of Lemma 10 is a consequence of the following two lemmas:

Lemma 11. *For $m \geq 0$,*

$$\sum_{k=0}^{m} \frac{(-1)^k\, 2^{2k}\, k!}{(2k+1)!(m-k)!} = \frac{1}{(2m+1)\, m!}.$$

Proof. We will show that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_k = \frac{1}{2m+1}, \tag{31}$$

where

$$a_k = \frac{(2^k\, k!)^2}{(2k+1)!}.$$

Notice that (31) is equivalent to saying that the binomial transform of the sequence a_k is $1/(2m+1)$ (see [GKP]). Since the binomial transform is an involution of sequences, (31) is equivalent to regaining the original sequence a_m by the inversion formula

$$a_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{2k+1}. \tag{32}$$

To prove this identity, consider the Newton binomial formula

$$(1-x)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^k$$

and multiply on both sides by $x^{-1/2}$. Integration between 0 and 1 gives now

$$\int_0^1 (1-x)^m x^{-1/2} dx = 2 \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{2k+1}.$$

Recall that

$$\int_0^1 (1-x)^m x^{-1/2} dx = B\left(\frac{1}{2}, m+1\right),$$

$B(x,y)$ being the beta function. Since it is easily seen that

$$B\left(\frac{1}{2}, m+1\right) = 2\frac{(2^m\, m!)^2}{(2m+1)!} = 2a_m,$$

(32) follows. □

The next lemma computes the Fourier transform of the kernel $K_i = x_i^{2n-1}/|x|^{2n}$, $1 \leq i \leq 2$, $n \geq 1$, by using Lemma 11.

Lemma 12. *For $n \geq 1$, $1 \leq i \leq 2$,*

$$\widehat{K_i}(\xi) = c\frac{\xi_i}{|\xi|^{2n}} p(\xi_1, \xi_2), \tag{33}$$

where $p(\xi_1, \xi_2)$ is a homogeneous polynomial of degree $2n-2$ with no non-vanishing zeros.

Proof. Without loss of generality fix $i = 1$. For $n \geq 1$, let E_n be the fundamental solution of the $n-$th power Δ^n of the Laplacean in the plane, that is

$$E_n(x) = |x|^{-(2-2n)}(\alpha + \beta \log |x|^2), \tag{34}$$

for some positive constants α and β depending on n (see [ACrL]). Notice that, since $\Delta^n E_n = \delta_0$, then

$$\widehat{(\partial_1^{2n-1} E_n)}(\xi) = c\frac{\xi_1^{2n-1}}{|\xi|^{2n}}$$

for some constant c. We will show that for some positive coefficients b_{2m}, $0 \le m \le n-1$,

$$(\partial_1^{2n-1} E_n)(x) = c\frac{x_1}{|x|^{2n}} \sum_{m=0}^{n-1} b_{2m} x_1^{2m} x_2^{2(n-1-m)}. \tag{35}$$

Notice that (33) follows from this fact.

To compute $\partial_1^{2n-1} E_n$, we will use the following formula from [LZ]:

$$L(\partial)E_n = \sum_{\nu=0}^{n-1} \frac{1}{2^\nu\ \nu!}\ \Delta^\nu L(x) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{2n-1-\nu} E_n(r), \tag{36}$$

where $r = |x|$ and $L(x) = x_1^{2n-1}$. First notice that for $0 \le \nu \le n-1$, we have

$$\Delta^\nu(x_1^{2n-1}) = \binom{2n-1}{2\nu}(2\nu)!\ x_1^{2n-2\nu-1},$$

and for $0 \le k \le n-1$, one can check

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n+k} E_n(r) = 2^n(n-1)!\ \frac{(-1)^k\ 2^k\ k!}{r^{2+2k}}.$$

Plugging these, with $k = n-1-\nu$, into equation (36) we get

$$\partial_1^{2n-1} E_n(x) = 2^{2n-1}(n-1)!\ \frac{x_1}{r^{2n}} \sum_{\nu=0}^{n-1} a_\nu\ x_1^{2(n-\nu-1)}\ r^{2\nu}, \tag{37}$$

where

$$a_\nu = \frac{(2\nu)!}{2^{2\nu}\ \nu!}\binom{2n-1}{2\nu}(-1)^{n-\nu-1}\ (n-1-\nu)!.$$

We claim that the homogeneous polinomial of degree $2n-2$ appearing in (37),

$$p(x_1, x_2) = \sum_{\nu=0}^{n-1} a_\nu\ x_1^{2(n-\nu-1)}\ r^{2\nu}, \tag{38}$$

has positive coefficients. To prove this, write $r^2 = x_1^2 + x_2^2$. Then

$$\begin{aligned} p(x) &= \sum_{\nu=0}^{n-1} a_\nu\ x_1^{2(n-\nu-1)}\ (x_1^2 + x_2^2)^\nu \\ &= \sum_{\nu=0}^{n-1}\sum_{k=0}^{\nu} a_\nu \binom{\nu}{k} x_1^{2(n-\nu+k-1)} x_2^{2(\nu-k)} = \sum_{m=0}^{n-1} b_{2m} x_1^{2m} x_2^{2(n-1-m)}, \end{aligned}$$

where for $0 \leq m \leq n-1$,

$$b_{2m} = \sum_{k=1}^{m+1} a_{n-k} \binom{n-k}{m+1-k}$$

$$= \frac{(2n-1)!}{2^{2n}\,(n-m-1)!} \sum_{k=0}^{m} \frac{(-1)^k\, 2^{2k}\, k!}{(2k+1)!\,(m-k)!}.$$

Applying now Lemma 11, we get that for $0 \leq m \leq n-1$,

$$b_{2m} = \frac{(2n-1)!}{2^{2n}\,(n-m-1)!} \frac{1}{(2m+1)\,m!} > 0,$$

which completes the proof of (35) and the lemma.

□

Proof of Lemma 10. By Lemma 12, taking the Fourier transform in (29) is equivalent to

$$\widehat{f}(\xi) = \widehat{\varphi}_1(\xi) \frac{\xi_1}{|\xi|^2} \frac{p(\xi_1,\xi_2)}{|\xi|^{2n-2}} + \widehat{\varphi}_2(\xi) \frac{\xi_2}{|\xi|^2} \frac{p(\xi_2,\xi_1)}{|\xi|^{2n-2}},$$

where p is some homogeneous polynomial of degree $2n-2$ with no non-vanishing zeros. Define the operator R_1 associated with the kernel

$$\widehat{r}_1(\xi_1,\xi_2) = \frac{p(\xi_1,\xi_2)}{|\xi|^{2n-2}}.$$

One defines also R_2, associated with r_2, where r_2 is given by $\widehat{r}_2(\xi_1,\xi_2) = \widehat{r}_1(\xi_2,\xi_1)$. Since p is a homogeneous polynomial of degree $2n-2$, it can be decomposed as

$$p(\xi_1,\xi_2) = \sum_{j=0}^{n-1} p_{2j}(\xi_1,\xi_2)|\xi|^{2n-2-2j},$$

where p_{2j} are homogeneous harmonic polynomials of degree $2j$ (see [St, 3.1.2 p. 69]). Therefore, the operators R_i, $1 \leq i \leq 2$, can be written in the form

$$R_i f = af + \text{p. v.}\, \frac{\Omega(x/|x|)}{|x|^2} * f, \tag{39}$$

for some constant a and $\Omega \in \mathcal{C}^\infty(S^1)$ with zero average. Consequently, by [Du, Theorem 4.15, p.82], the operators R_i, $1 \leq i \leq 2$, are invertible and the inverse operators, say S_i, $1 \leq i \leq 2$, have the same form, namely the operators S_i, associated with the kernels s_i, $1 \leq i \leq 2$, defined by

$$\widehat{s_1}(\xi) = \frac{|\xi|^{2n-2}}{p(\xi_1,\xi_2)} \quad \text{and} \quad \widehat{s_2}(\xi) = \frac{|\xi|^{2n-2}}{p(\xi_2,\xi_1)},$$

can be written as in (39), too. Therefore, setting

$$\varphi_i = S_i(\partial_i f),$$

for $1 \leq i \leq 2$, finishes the proof of Lemma 10. □

Observe that for a compactly supported distribution T with bounded Cauchy potential

$$|\langle T,\varphi_Q\rangle| = \left|\left\langle T, \frac{1}{\pi z} * \overline{\partial}\varphi_Q\right\rangle\right| = \left|\left\langle \frac{1}{\pi z} * T, \overline{\partial}\varphi_Q\right\rangle\right|$$
$$\leq \frac{1}{\pi}\left\|\frac{1}{z} * T\right\|_\infty \|\overline{\partial}\varphi_Q\|_{L^1(Q)} \leq \frac{1}{\pi}\left\|\frac{1}{z} * T\right\|_\infty l(Q),$$

whenever φ_Q satisfies $\|\overline{\partial}\varphi_Q\|_{L^1(Q)} \leq l(Q)$.

In our present case we do have a similar growth condition: if T is a compactly supported distribution with bounded potentials $K_1 * T$ and $K_2 * T$, then by Lemma 10

$$\begin{aligned} |\langle T,\varphi_Q\rangle| &= |\langle T, K_1 * S_1(\partial_1\varphi_Q) + K_2 * S_2(\partial_2\varphi_Q)\rangle| \\ &\leq |\langle K_1 * T, S_1(\partial_1\varphi_Q)\rangle| + |\langle K_2 * T, S_2(\partial_2\varphi_Q)\rangle| \\ &\leq \|K_1 * T\|_\infty \, \|S_1(\partial_1\varphi_Q)\|_{L^1(\mathbb{R}^2)} + \|K_2 * T\|_\infty \, \|S_2(\partial_2\varphi_Q)\|_{L^1(\mathbb{R}^2)} \\ &\leq \left(\|K_1 * T\|_\infty + \|K_2 * T\|_\infty\right) l(Q), \end{aligned} \tag{40}$$

whenever φ_Q satisfies

$$\|S_i(\partial_i\varphi_Q)\|_{L^1(\mathbb{R}^2)} \leq l(Q), \text{ for } i = 1, 2. \tag{41}$$

The next lemma states a sufficient condition for a test function to satisfy conditions (41).

Lemma 13. *Let $1 < q_0 < \infty$ and assume that f_Q is a test function supported on the square Q satisfying,*

$$\|\partial_i f_Q\|_{L^{q_0}(Q)} \leq l(Q)^{2/q_0 - 1}, \quad \textit{for } 1 \leq i \leq 2.$$

Then,

$$\|S_i(\partial_i f_Q)\|_{L^1(\mathbb{R}^2)} \leq Cl(Q) \quad \textit{for } 1 \leq i \leq 2.$$

Proof. Without loss of generality fix $i = 1$. Let p_0 be the dual exponent to q_0. By Hölder's inequality and the fact that the operator S_1 is bounded in $L^{q_0}(\mathbb{R}^2)$, $1 < q_0 < \infty$, we get

$$\begin{aligned} \|S_1(\partial_1 f_Q)\|_{L^1(2Q)} &\leq Cl(Q)^{2/p_0}\|S_1(\partial_1 f_Q)\|_{L^{q_0}(\mathbb{R}^2)} \\ &\leq Cl(Q)^{2/p_0}\|\partial_1 f_Q\|_{L^{q_0}(Q)} \\ &\leq Cl(Q). \end{aligned}$$

To estimate the L^1 norm outside $2Q$, notice first that since $\partial_1 f_Q$ is supported on Q, by (30),

$$\|S_1(\partial_1 f_Q)\|_{L^1((2Q)^c)} = \|\widetilde{S}_1(\partial_1 f_Q)\|_{L^1((2Q)^c)}.$$

Integrating by parts to take one derivative to the kernel K of $\widetilde{S}_1$ and then using Fubini we obtain

$$\begin{aligned}\|S_1(\partial_1 f_Q)\|_{L^1((2Q)^c)} &= C\int_{(2Q)^c} |\int_Q \partial_1 f_Q(z)K(z-y)\,dz|\,dy\\ &= C\int_{(2Q)^c} |\int_Q f_Q(z)\partial_1 K(z-y)\,dz|\,dy\\ &\leq C\|f_Q\|_{L^1(Q)}\, l(Q)^{-1}\\ &\leq C\int |\nabla f_Q|,\end{aligned}$$

the last estimate coming from the Cauchy-Schwarz inequality, together with a well known result of Maz'ya (see [MzS, 1.1.4, p. 15] and [MzS, 1.2.2, p. 24]) stating that

$$\|f_Q\|_2 \leq C\int |\nabla f_Q|.$$

Now Hölder's inequality together with $\|\partial_i f_Q\|_{L^{q_0}(Q)} \leq l(Q)^{2/q_0-1}$, $1 \leq i \leq 2$, gives the desired estimate, namely

$$\|S_1(\partial_1 f_Q)\|_{L^1((2Q)^c)} \leq Cl(Q).$$

□

Fix $1 < q_0 < 2$. We say that a distribution T has linear growth if

$$G(T) = \sup_{\varphi_Q} \frac{|\langle T, \varphi_Q\rangle|}{l(Q)} < \infty,$$

where the supremum is taken over all $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfying the normalization inequalities

$$\|\partial_i \varphi_Q\|_{L^{q_0}(Q)} \leq 1, \quad \text{for } 1 \leq i \leq 2. \tag{42}$$

Notice that from (40) and Lemma 13, if T is a compactly supported distribution with bounded potentials $k_1 * T$ and $k_2 * T$, then T has linear growth.

We now state the localization lemma we need.

Lemma 14. *Let T be a compactly supported distribution in $\mathbb{R}^2$ such that $(x_i^{2n-1}/|x|^{2n}) * T$ is in $L^\infty(\mathbb{R}^2)$ for some $n \in \mathbb{N}$ and $1 \leq i \leq 2$. Let Q be a square and assume that $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfies $\|\varphi_Q\|_\infty \leq C$ and $\|\nabla\varphi_Q\|_\infty \leq l(Q)^{-1}$. Then $(x_i^{2n-1}/|x|^{2n}) * \varphi_Q T$ is in $L^\infty(\mathbb{R}^2)$ and*

$$\left\| \frac{x_i^{2n-1}}{|x|^{2n}} * \varphi_Q T \right\|_\infty \leq C\left(\left\| \frac{x_i^{2n-1}}{|x|^{2n}} * T \right\|_\infty + G(T)\right),$$

for some positive constant C.

For the proof we need the following result.

Lemma 15. *Let T be a compactly supported distribution in $\mathbb{R}^2$ with linear growth and assume that Q is a square and $\varphi_Q \in \mathcal{C}_0^\infty(Q)$ satisfies $\|\varphi_Q\|_\infty \leq 1$ and $\|\nabla\varphi_Q\|_\infty \leq l(Q)^{-1}$. Then, for each coordinate i, the distribution $(x_i^{2n-1}/|x|^{2n}) * \varphi_Q T$ is an integrable function in the interior of $\frac{1}{4}Q$ and*

$$\int_{\frac{1}{4}Q} \left| \left(\frac{x_i^{2n-1}}{|x|^{2n}} * \varphi_Q T \right)(y) \right| dy \leq C\, G(T)\, l(Q)^2,$$

where C is a positive constant.

Proof of Lemma 15. The proof of this lemma follows the lines of Lemma 13 in [MPrVe], although now the growth conditions we have are different from the ones in [MPrVe] (see (42)). We write the proof for the sake of completeness.

Without loss of generality set $i = 1$ and write $K_1(x) = x_1^{2n-1}/|x|^{2n}$. We will prove that $K_1 * \varphi_Q T$ is in $L^{p_0}(2Q)$ where p_0 is the dual exponent of q_0 (see (42)). Therefore we need to estimate the action of $K_1 * \varphi_Q T$ on functions $\psi \in \mathcal{C}_0^\infty(2Q)$ in terms of $\|\psi\|_{q_0}$. We clearly have

$$\langle K_1 * \varphi_Q T, \psi\rangle = \langle T, \varphi_Q (K_1 * \psi)\rangle.$$

We claim that, for an appropriate positive constant C, the test function

$$\frac{\varphi_Q (K_1 * \psi)}{C\, l(Q)^{\frac{2}{p_0}-1} \|\psi\|_{q_0}} \tag{43}$$

satisfies the normalization inequalities (42) in the definition of $G(T)$. Once this is proved, by the definition of $G(T)$ we get that $|\langle K_1 * \varphi_Q T, \psi\rangle| \le C\, l(Q)^{\frac{2}{p_0}} \|\psi\|_{q_0}\, G(T)$, and therefore $\|K_1 * \varphi_Q T\|_{L^{p_0}(2Q)} \le C\, l(Q)^{\frac{2}{p_0}} G(T)$. Hence

$$\begin{aligned}
\frac{1}{|\frac{1}{4}Q|} \int_{\frac{1}{4}Q} |(K_1 * \varphi_Q T)(x)|\, dx &\le 16 \frac{1}{|Q|} \int_Q |(K_1 * \varphi_Q T)(x)|\, dx \\
&\le 16 \left(\frac{1}{|Q|} \int_Q |(K_1 * \varphi_Q T)(x)|^{p_0}\, dx \right)^{\frac{1}{p_0}} \\
&\le C\, G(T),
\end{aligned}$$

which proves Lemma 15.

By Lemma 13, to prove the claim we only have to show that for $1 \le i \le 2$,

$$\|\partial_i \left(\varphi_Q \left(K_1 * \psi\right)\right)\|_{L^{q_0}(Q)} \le C\, \|\psi\|_{q_0}.$$

Clearly, for $1 \le i \le 2$, we have

$$\partial_i \left(\varphi_Q \left(K_1 * \psi\right)\right) = \varphi_Q\, \partial_i (K_1 * \psi) + \partial_i \varphi_Q \left(K_1 * \psi\right) = A + B,$$

where the last identity is the definition of A and B.

To estimate the L^{q_0}-norm of B we recall that $|K_1(x)| \le C\, |x|^{-1}$. Hence, for $1 \le i \le 2$,

$$\|\partial_i \varphi_Q \left(K_1 * \psi\right)\|_{L^{q_0}(Q)} \le C\, \|\partial_i \varphi_Q\|_\infty \left(\int_Q \left(\int_{2Q} \frac{|\psi(y)|}{|x-y|} dy \right)^{q_0} dx \right)^{1/q_0} \le C \|\psi\|_{q_0},$$

where the last inequality comes from Schur's Lemma applied to the operator with kernel $K(x,y) = |x-y|^{-1} \chi_{2Q}(x) \chi_{2Q}(y)$ and the fact that $\|\partial_i \varphi_Q\|_\infty \le l(Q)^{-1}$, $1 \le i \le 2$. We therefore conclude that $\|B\|_{q_0} \le C \|\partial_i \varphi_Q \left(K_1 * \psi\right)\|_{q_0} \le C\, \|\psi\|_{q_0}$.

We turn now to the term A. We remark that, for $1 \le i \le 2$,

$$\partial_i K_1 * \psi = c\, \psi + S(\psi), \tag{44}$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator and c some constant. This can be seen by computing the Fourier transform of $\partial_i K_1$ and then using that each homogeneous polynomial can be decomposed in terms of homogeneous harmonic polynomials of lower degrees (see [St, 3.1.2 p. 69]). Since Calderón-Zygmund operators are bounded in $L^{q_0}(\mathbb{R}^2)$, $1 < q_0 < \infty$, and $\|\varphi_Q\|_\infty \le C$, we get that $\|A\|_{q_0} \le C\, \|\psi\|_{q_0}$. This completes the estimate of term A and the proof of (43). □

Proof of Lemma 14. Here we argue as in Lemma 12 in [MPrVe]. We write the proof for the sake of completeness. Without loss of generality take $i = 1$. Let $x \in \mathbb{R}^2 \setminus \frac{3}{2}Q$. Then $K_1(x-y)\,\varphi_Q(y)$ is in $\mathcal{C}_0^\infty(Q)$ as a function of y. Since for all $y \in \mathbb{R}^2$, $|\partial_i(K_1(x-y)\,\varphi_Q(y))| \leq C\,l(Q)^{-2}$, $1 \leq i \leq 2$, the function $c\ l(Q)\,K_1(x-y)\,\varphi_Q(y)$ satisfies the normalization conditions (42) for some small constant c. Therefore

$$|(K_1 * \varphi_Q T)(x)| = |\langle T, K_1(x-\cdot)\,\varphi_Q\rangle| \leq c^{-1}\,G(T),$$

for all $x \in \mathbb{R}^2 \setminus \frac{3}{2}Q$. We are now left with the case $x \in \frac{3}{2}Q$. Since $K_1 * T$ and φ_Q are bounded functions, we can write

$$|(K_1 * \varphi_Q T)(x)| \leq |(K_1 * \varphi_Q T)(x) - \varphi_Q(x)(K_1 * T)(x)| + \|\varphi_Q\|_\infty \|K_1 * T\|_\infty.$$

Let $\psi_Q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be such that $\psi_Q \equiv 1$ in $2Q$, $\psi_Q \equiv 0$ in $(4Q)^c$, $\|\psi_Q\|_\infty \leq C$ and $\|\nabla \psi_Q\|_\infty \leq C\,l(Q)^{-1}$. Then one is tempted to write

$$\begin{aligned}|(K_1 * \varphi_Q T)(x) - \varphi_Q(x)(K_1 * T)(x)| &\leq |\langle T, \psi_Q(\varphi_Q - \varphi_Q(x))K_1(x-\cdot)\rangle| \\ &\quad + \|\varphi_Q\|_\infty |\langle T, (1-\psi_Q)K_1(x-\cdot)\rangle|.\end{aligned}$$

The problem is that the first term on the right hand side above does not make any sense because T is acting on a function of y which is not necessarily differentiable at the point x. To overcome this difficulty one needs to resort to a standard regularization process. Take $\chi \in \mathcal{C}^\infty(B(0,1))$ such that $\int \chi = 1$ and set $\chi_\varepsilon(x) = \varepsilon^{-2}\,\chi(x/\varepsilon)$. It is enough to prove that $\chi_\varepsilon * K_1 * \varphi_Q T$ is uniformly bounded, since $\chi_\varepsilon * K_1 * \varphi_Q T$ converges weakly to $K_1 * \varphi_Q T$ in the distributinal sense, as $\varepsilon \to 0$. We have

$$\begin{aligned}|(\chi_\varepsilon * K_1 * \varphi_Q T)(x) - \varphi_Q(x)(\chi_\varepsilon * K_1 * T)(x)| &\\ \leq |\langle T, \psi_Q(\varphi_Q - \varphi_Q(x))(\chi_\varepsilon * K_1)(x-\cdot)\rangle| &\\ + \|\varphi_Q\|_\infty |\langle T, (1-\psi_Q)(\chi_\varepsilon * K_1)(x-\cdot)\rangle| &\\ = A_1 + A_2.&\end{aligned}$$

To deal with term A_1 set $K_{1,\varepsilon}^x(y) = (\chi_\varepsilon * K_1)(x-y)$. We claim that, for an appropriate small constant c, the test function

$$f_Q = c\,l(Q)\psi_Q(\varphi_Q - \varphi_Q(x))K_{1,\varepsilon}^x,$$

satisfies the normalization inequalities (42) in the definition of $G(T)$, with φ_Q replaced by f_Q and Q by $4Q$. If this is the case, then

$$A_1 \leq c^{-1} l(Q)^{-1} |\langle T, f_Q\rangle| \leq C\,G(T).$$

To prove the normalization inequalities (42) for the function f_Q we have to show that for $1 \leq i \leq 2$,

$$\|\partial_i f_Q\|_{L^{q_0}(4\,Q)} \leq C l(Q)^{2/q_0 - 1}. \tag{45}$$

To prove (45) we first notice that the regularized kernel $\chi_\varepsilon * K_1$ satisfies the inequality

$$|(\chi_\varepsilon *\ K_1)(x)| \leq \frac{C}{|x|}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{46}$$

where C is a positive constant, which, in particular, is independent of ϵ. This can be proved by standard estimates which we omit. Moreover, by (44), for $1 \leq i \leq 2$, we have

$$(\chi_\varepsilon * \partial_i\,K_1)(x) = c\,\chi_\varepsilon(x) + (\chi_\varepsilon * S)(x),$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator. As such, its kernel H satisfies the usual growth condition $|H(x)| \leq C/|x|^2$. From this is not difficult to show that for some positive constant C,

$$|(\chi_\varepsilon * S)(x)| \leq \frac{C}{|x|^2}, \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{47}$$

We have, for $1 \leq i \leq 2$,

$$\partial_i \left(\psi_Q(\varphi_Q - \varphi_Q(x))k_\varepsilon^{1,x}\right) = \psi_Q \left(\varphi_Q - \varphi_Q(x)\right)\partial_i\, k_\varepsilon^{1,x} + \partial_i(\psi_Q(\varphi_Q - \varphi_Q(x)))\, k_\varepsilon^{1,x}.$$

Therefore

$$\begin{aligned}
\|\partial_i f_Q\|_{L^{q_0}(4Q)} &\leq Cl(Q) \left(\int_{4Q} |\psi_Q(y)\left(\varphi_Q(y) - \varphi_Q(x)\right)\partial_i k_\varepsilon^{1,x}(y)|^{q_0}\, dy\right)^{\frac{1}{q_0}} \\
&\quad + Cl(Q) \left(\int_{4Q} |\partial_i\left(\psi_Q(\varphi_Q - \varphi_Q(x))\ k_\varepsilon^{1,x}(y)\right|^{q_0}\, dy\right)^{\frac{1}{q_0}} \\
&= A_{11} + A_{12}.
\end{aligned}$$

Using (46) one obtains

$$A_{12} \leq Cl(Q)\frac{1}{l(Q)} \left(\int_{4Q} |(k_\varepsilon^{1,x})(y)|^{q_0}\, dy\right)^{\frac{1}{q_0}} \leq Cl(Q)^{\frac{2}{q_0}-1}.$$

To estimate A_{11} we resort to (47) and the fact that $q_0 < 2$, which yields

$$\begin{aligned}
A_{11} &= Cl(Q) \left(\int_{4Q} |\psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))\partial_i k_\varepsilon^{1,x}(y)|^{q_0}\, dy\right)^{\frac{1}{q_0}} \\
&\leq Cl(Q)\|\nabla\varphi_Q\|_\infty \left(\int_{4Q} \frac{dy}{|y-x|^{q_0}}\, dy\right)^{\frac{1}{q_0}} \leq Cl(Q)^{\frac{2}{q_0}-1}.
\end{aligned}$$

We now turn to A_2. By Lemma 15, there exists a Lebesgue point of $K_1 * \varphi_Q T$, $x_0 \in Q$, such that $|(K_1 * \psi_Q T)(x_0)| \leq C\, G(T)$. Then

$$|(K_1 * (1 - \psi_Q)T)(x_0)| \leq C\left(\|K_1 * T\|_\infty + G(T)\right).$$

The analogous inequality holds as well for the regularized potentials appearing in A_2, for ϵ small enough and with constants independent of ϵ. Therefore

$$A_2 \leq C\,|\langle T, (1 - \psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle| + C\left(\|K_1 * T\|_\infty + G(T)\right).$$

To estimate $|\langle T, (1 - \psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle|$, we decompose $\mathbb{R}^2 \setminus \{x\}$ into a union of rings

$$N_j = \{z \in \mathbb{R}^2 : 2^j\, l(Q) \leq |z - x| \leq 2^{j+1}\, l(Q)\}, \quad j \in \mathbb{Z},$$

and consider functions φ_j in $\mathcal{C}_0^\infty(\mathbb{R}^2)$, with support contained in

$$N_j^* = \{z \in \mathbb{R}^2 : 2^{j-1}\, l(Q) \leq |z - x| \leq 2^{j+2}\, l(Q)\}, \quad j \in \mathbb{Z},$$

such that $\|\varphi_j\|_\infty \leq C$ and $\|\nabla \varphi_j\|_\infty \leq C\,(2^j\,l(Q))^{-1}$, and $\sum_j \varphi_j = 1$ on $\mathbb{R}^2 \setminus \{x\}$. Since $x \in \frac{3}{2}Q$ the smallest ring N_j^* that intersects $(2Q)^c$ is N_{-3}^*. Therefore we have

$$\begin{aligned}|\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle| &= \left|\left\langle T, \sum_{j\geq -3} \varphi_j (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\right\rangle\right| \\ &\leq \left|\left\langle T, \sum_{j\in I} \varphi_j (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\right\rangle\right| \\ &\quad + \sum_{j\in J} |\langle T, \varphi_j (k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle|,\end{aligned}$$

where I denotes the set of indices $j \geq -3$ such that the support of φ_j intersects $4Q$ and J the remaining indices, namely those $j \geq -3$ such that φ_j vanishes on $4Q$. Notice that the cardinality of I is bounded by a positive constant.

Set

$$g = C\,l(Q) \sum_{j\in I} \varphi_j (1-\psi_Q)\,(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}),$$

and for $j \in J$

$$g_j = C\,2^{2j}\,l(Q)\,\varphi_j\,(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0}).$$

We now show that the test functions g and g_j, $j \in J$, satisfy the normalization inequalities (42) in the definition of $G(T)$ for an appropriate choice of the (small) constant C . Once this is available, using the linear growth condition of T we obtain

$$\begin{aligned}|\langle T, (1-\psi_Q)(k_\varepsilon^{1,x} - k_\varepsilon^{1,x_0})\rangle| &\leq Cl(Q)^{-1}|\langle T, g\rangle| \\ &\quad + C\sum_{j\in J} (2^{2j} l(Q))^{-1} |\langle T, g_j\rangle| \\ &\leq C\,G(T) + C\sum_{j\geq -3} 2^{-j}\,G(T) \leq C\,G(T),\end{aligned}$$

which completes the proof of Lemma 14.

Checking the normalization inequalities for g and g_j is easy. First notice that the support of g is contained in a square $\lambda\,Q$ for some universal constant λ. On the other hand the support of g_j is contained in $2^{j+2}\,Q$. By Lemma 13, we have to show that for $1 \leq i \leq 2$, and some $1 < q_0 < \infty$,

$$\|\partial_i g\|_{L^{q_0}(\lambda\,Q)} \leq Cl(Q)^{2/q_0 - 1}, \tag{48}$$

and for $j \in J$,

$$\|\partial_i g_j\|_{L^{q_0}(2^{j+2}\,Q)} \leq C(2^j l(Q))^{2/q_0 - 1}. \tag{49}$$

To show (48) we take ∂_i in the definition of g, apply Leibnitz's formula and estimate in the supremum norm each term in the resulting sum. We get

$$\|\partial_i g\|_\infty \leq C\,l(Q) \sum_{k=0}^{1} \frac{1}{l(Q)^k}\,\frac{1}{l(Q)^{2-k}} = C\,\frac{1}{l(Q)},$$

which yields (48) immediately.

For (49), applying a gradient estimate, we get

$$\|\partial_i g_j\|_\infty \leq C\, 2^{2j}\, l(Q) \sum_{k=0}^{1} \frac{1}{(2^j\, l(Q))^k}\ \frac{l(Q)}{(2^j\, l(Q))^{2+1-k}} = C\, \frac{1}{2^j\, l(Q)},$$

which yields (49) readily. □

4. Outer regularity

In what follows, we will show that the capacities γ_n are outer regular.

Lemma 16. *Let $\{E_k\}_k$ be a decreasing sequence of compact sets in $\mathbb{R}^2$, with intersection the compact set $E \subset \mathbb{R}^2$. Then $\gamma_n(E) = \lim_{k\to\infty} \gamma_n(E_k)$.*

Proof. The limit $\lim_{k\to\infty} \gamma_n^1(E_k)$ clearly exists and $\lim_{k\to\infty} \gamma_n(E_k) \geq \gamma_n(E)$. To prove the converse inequality, let T_k be a distribution supported on E_k such that, for $1 \leq i \leq 2$, $f_k^i = K_i * T_k$ is in the unit ball of $L^\infty(\mathbb{R}^2)$ and

$$\gamma_n(E_k) - \frac{1}{k} < |\langle T_k, 1\rangle| \leq \gamma_n(E_k).$$

By taking a subsequence if necessary, we may assume that, for $1 \leq i \leq 2$, f_k^i converges weakly $*$ in $L^\infty(\mathbb{R}^2)$ to some function f^i such that $\|f^i\|_\infty \leq 1$.

We will show that T_k converges to some distribution T such that $T * K_1$ and $T * K_2$ are also in the unit ball of $L^\infty(\mathbb{R}^2)$. Then

$$\gamma_n(E) \geq \langle T,\, 1\rangle = \lim_{k\to\infty} \langle T_k,\, 1\rangle = \lim_{k\to\infty} \gamma_n(E_k),$$

and we will be done.

Let us first check that the limit of $\{T_k\}_k$ exists in the topology of distributions. This is equivalent to saying that, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, the limit $\lim_{k\to\infty}\langle T_k, \varphi\rangle$ exists. Using the reproducing formula (29), we deduce that

$$\langle T_k, \varphi\rangle = \langle T_k,\, S_1(\partial_1\varphi) * K_1 + S_2(\partial_2\varphi) * K_2\rangle = \langle T_k * K_1,\, S_1(\partial_1\varphi)\rangle + \langle T_k * K_2,\, S_2(\partial_2\varphi)\rangle,$$

which is convergent, since by Lemma 13, $S_i(\partial_i\varphi) \in L^1(\mathbb{R}^2)$, $1 \leq i \leq 2$, and $f_k^i = T_k * K_1$, $1 \leq i \leq 2$, is weak $*$ convergent in $L^\infty(\mathbb{R}^2)$.

To see that, for $1 \leq i \leq 2$, $T * K_i$ is in the unit ball of $L^\infty(\mathbb{R}^2)$, we take a radial function $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$, $\int \chi = 1$, supported in the unit ball and, as usual, we denote $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(\varepsilon^{-1}x)$. Then it is enough to prove that $\chi_\varepsilon * T * K_i$ is in the unit ball of $L^\infty(\mathbb{R}^2)$ for all $\varepsilon > 0$. This follows easily: denoting $K_i^\varepsilon = \chi_\varepsilon * K_i$, for each $x \in \mathbb{R}^2$, we have

$$T_k * K_i^\varepsilon(x) = \langle T_k,\, K_i^\varepsilon(x - \cdot)\rangle.$$

Notice moreover that $\|T_k * K_i^\varepsilon\|_\infty = \|\chi_\varepsilon * (T_k * K_i)\|_\infty \leq 1$. Now, let $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be such that it equals 1 in the 1-neighborhood of E, so that $T_k = \psi_0\, T_k$ for k big enough. Then

$$T_k * K_i^\varepsilon(x) = \langle \psi_0\, T_k,\, K_i^\varepsilon(x - \cdot)\rangle = \langle T_k,\, \psi_0\, K_i^\varepsilon(x - \cdot)\rangle,$$

which converges to $\langle T,\, \psi_0\, K_i^\varepsilon(x - \cdot)\rangle = \langle T,\, K_i^\varepsilon(x - \cdot)\rangle = T * K_i^\varepsilon(x)$ as $k \to \infty$. Since $|T_k * K_i^\varepsilon(x)| \leq 1$ for all k, we deduce that $|T * K_i^\varepsilon(x)| \leq 1$ as wished, too. □

5. Proof of Theorem 2

For the proof of Theorem 2, recall the following result from [MPrVe]:

Theorem 17. *([MPrVe]) For a compact set $E \subset \mathbb{R}^d$,*

$$\Gamma(E) \approx \sup \mu(E), \tag{50}$$

the supremum taken over those positive measures μ supported on E with linear growth such that for $1 \leq i \leq d$, $i \neq k$, the potentials $\frac{x_i}{|x|^2} * \mu$ are in $L^\infty(\mu)$ with $\left\| \frac{x_i}{|x|^2} * \mu \right\|_\infty \leq 1$.

Notice that the only difference between (50) and Theorem 2 is the extra linear growth condition required on the positive measure μ. Hence, to prove Theorem 2, we have to get rid of this growth condition and still mantain the comparability between the capacities. Below, in Lemma 20, we show that if we are given a positive measure supported on E with $\|\frac{x_i}{|x|^2} * \mu\|_\infty \leq 1$ for $i \neq k$, $1 \leq i \leq d$, then this measure grows linearly in a big piece of its support E. Thus Theorem 2 holds.

For a Borel measure μ, the curvature of μ, which was introduced in [Me], is the non-negative number $c^2(\mu)$ defined by

$$c^2(\mu) = \iiint c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z),$$

where $c(x,y,z)$ is the inverse of the radius of the circumcircle of the triangle (x,y,z), that is the Menger curvature of the triple (x,y,z) (see Section 2).

The following result, that will be needed in what follows, is a version of [T1, Lemma 5.2] for $\mathbb{R}^d$. Its proof uses the curvature theorem of G. David and Léger [Lé, Proposition 1.2].

Lemma 18. *Let μ be some Radon measure supported on $B(x_0, R)$, with*

$$\Theta^*_\mu(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r} \leq 1 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

If $c^2(\mu) \leq C_2\, \mu(B(x_0,R))$, then $\mu(B(x_0,R)) \leq MR$, where M is some constant depending only on C_2.

From the preceding lemma we get the following.

Lemma 19. *Let μ be a finite Borel measure supported on a ball $B(x_0,R)$. Suppose that*

$$\Theta^1_\mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{r} = 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

Then,

$$\left(\frac{\mu(B(x_0,R))}{R} \right)^2 \leq c_1 \frac{c^2(\mu)}{\mu(B(x_0,R))}, \tag{51}$$

for some absolute constant c_1.

Proof. Consider the measure $\widetilde{\mu} = \left(\frac{\|\mu\|}{c^2(\mu)} \right)^{1/2} \mu$. Notice that

$$c^2(\widetilde{\mu}) = \left(\frac{\|\mu\|}{c^2(\mu)} \right)^{3/2} c^2(\mu) = \|\widetilde{\mu}\|.$$

Applying Lemma 18 to $\widetilde{\mu}$ with $C_2 = 1$, we infer that there exists an absolute constant M such that $\widetilde{\mu}(B(x_0, R)) \leq MR$, and thus

$$\mu(B(x_0, R)) \leq M \left(\frac{c^2(\mu)}{\mu(B(x_0, R))} \right)^{1/2} R,$$

which is equivalent to (51), with $c_1 = M^2$. □

Remark. For $x, y, z \in \mathbb{R}^d$ set $K_i(x) = x_i/|x|^2$, $1 \leq i \leq d$, and let

$$p_i(x, y, z) = K_i(x - y)\, K_i(x - z) + K_i(y - x)\, K_i(y - z) + K_i(z - x)\, K_i(z - y).$$

Given any subset of $d - 1$ elements of $\{1, 2, \cdots, d\}$, $\mathcal{S}_{d-1}$, we define, for a positive measure μ (without atoms, say),

$$p(\mu) = \sum_{i \in \mathcal{S}_{d-1}} \iiint p_i(x, y, z)\, d\mu(x)\, d\mu(y)\, d\mu(z).$$

Due [MPrVe, Corollary 2 and Theorem 4], Lemma 18 also holds in $\mathbb{R}^d$ when replacing the Menger curvature by the permutations associated with any set of $d - 1$ components of the vectorial kernel $x/|x|^2$ in $\mathbb{R}^d$. Therefore we recover Lemma 18 and Lemma 19 with $c^2(\mu)$ replaced by $p(\mu)$.

Given $M > 0$, we say that a ball $B = B(x, r)$ is non M-Ahlfors (or simply, a non Ahlfors ball) if

$$\Theta_\mu(B) := \frac{\mu(B)}{r} > M.$$

Lemma 20. *Let μ be a positive measure on $\mathbb{R}^d$ such that $\left\| \frac{x_i}{|x|^2} * \mu \right\|_\infty \leq 1$, for $i \in \mathcal{S}_{d-1}$. Let $A_M^\mu \subset \mathbb{R}^d$ be the union of all non M-Ahlfors balls. If M is big enough, then*

$$\mu(A_M^\mu) \leq \frac{1}{2}\, \mu(\mathbb{R}^d).$$

Proof. Let φ be a non negative radial $\mathcal{C}^\infty$ function supported on $B(0, 1)$ with L^1 norm equal to 1, and denote $\varphi_t(x) = t^{-n}\varphi(x/t)$, for $t > 0$. Observe that for $i \in \mathcal{S}_{d-1}$, the measure $\mu_t = \varphi_t * \mu$ satisfies

$$\left\| \frac{x_i}{|x|^2} * \mu_t \right\|_\infty = \left\| \varphi_t * (\frac{x_i}{|x|^2} * \mu) \right\|_\infty \leq 1.$$

Moreover, $\Theta^1_{\mu_t}(x) = 0$ for every $x \in \mathbb{R}^d$ and μ_t has linear growth with some constant depending on t (since the density of μ_t is a $\mathcal{C}^\infty$ function with compact support), and thus,

$$p(\mu_t) = 3 \sum_{i \in \mathcal{S}_{d-1}} \left\| \frac{x_i}{|x|^2} * \mu_t \right\|^2_{L^2(\mu_t)} \leq 3 \sum_{i \in \mathcal{S}_{d-1}} \left\| \frac{x_i}{|x|^2} * \mu_t \right\|^2_\infty \mu_t(\mathbb{R}^d) \leq 3(d-1)\, \|\mu\|. \tag{52}$$

For $t > 0$, denote

$$A^\mu_{M,t} = \bigcup_{\substack{B \text{ ball: } \Theta_\mu(B) \geq M \\ r(B) \geq t}} B.$$

Notice that if $r(B) \geq t$, then $\mu_t(2B) \geq \mu(B)$ and thus $\Theta_{\mu_t}(2B) \geq \Theta_\mu(B)/2$. Then by the preceding remark, if B is one of the balls appearing in the union that defines $A^\mu_{M,t}$,

$$p(\mu_t \lfloor 2B) \geq c_1^{-1}\, \frac{M^2}{4}\, \mu_t(2B) \geq c_1^{-1}\, \frac{M^2}{4}\, \mu(B). \tag{53}$$

By the $5r$-covering lemma, there exists a family of non M-Ahlfors balls (for μ), B_j, $j \in I$, such that the balls $2B_j$ are disjoint, and

$$A^\mu_{M,t} \subset \bigcup_{j\in I} 10B_j.$$

Moreover, the balls B_j can be taken so that aB_j is an M-Ahlfors ball for each $a \geq 2$ (just by considering maximal balls in the union that defines $A^\mu_{M,r}$). So we have

$$\mu(10B_j) \leq 10M\, r(B_j) \leq 10\,\mu(B_j).$$

Then, by (53) and (52),

$$\begin{aligned}\mu(A^\mu_{M,t}) &\leq \sum_{j\in I} \mu(10B_j) \leq 10\sum_{j\in I}\mu(B_j) \leq \frac{40\,c_1}{M^2}\sum_{j\in I} p(\mu_t\lfloor 2B_j)\\ &\leq \frac{40\,c_1}{M^2} p(\mu_t) \leq \frac{120(d-1)\,c_1}{M^2}\,\|\mu\|.\end{aligned}$$

So if M is chosen big enough, $\mu(A^\mu_{M,t}) \leq \mu(\mathbb{R}^d)/2$, and letting $t \to 0$, the lemma follows. □

Remark. Lemma 19 and Lemma 20 also hold in $\mathbb{R}^2$ replacing Menger curvature by the permutations of the kernel $x_i^{2n-1}/|x|^{2n}$, $1 \leq i \leq 2$, $n \geq 1$, and the kernel $x_i/|x|^2$ by the kernel $x_i^{2n-1}/|x|^{2n}$, respectively, because in [ChMPT] we proved David-Léger's theorem with these permutations instead of the usual curvature.

6. Some remarks on related capacities

6.1. **Extensions of Theorem 1 to other capacities.** For $n \geq 1$ and $1 \leq j \leq 2$, we set $K^n_j(x) = x_j^{2n-1}/|x|^{2n}$. For $n,\ m \geq 1$ and each compact set $E \subset \mathbb{R}^2$, we define the following capacity:

$$\gamma_{n,m}(E) = \sup\{|\langle T, 1\rangle|\},$$

the supremum taken over all distributions T supported on E with potentials $T * K^n_1$ and $T * K^m_2$ in the unit ball of $L^\infty(\mathbb{R}^2)$.

Using the same arguments as in Lemma 10, one could show that each function $f(x)$ with continuous derivatives up to order one is representable in the form

$$f(x) = (\varphi_1 * K^n_1)(x) + (\varphi_2 * K^m_2)(x),\ x \in \mathbb{R}^2,$$

where the functions φ_i, $i = 1, 2$, are defined by the formula $\varphi_i(x) = S_i(\partial_i f)(x)$, with S_i, $1 \leq i \leq 2$, being Calderón-Zygmund operators. Moreover the localization result of Lemma 14 and the outer regularity property of Lemma 16 also apply in this setting. Therefore, using the same techniques, one obtains the comparability between analytic capacity and $\gamma_{n,m}$, namely that there exists some positive constant C such that for all compact sets E of the plane

$$C^{-1}\gamma_{n,m}(E) \leq \gamma(E) \leq C\gamma_{n,m}(E).$$

In fact, following the proofs in [T2] and [T3] (see also [MPrVe]), one can show that for compact sets $E \subset \mathbb{R}^2$, a given capacity (associated twith some Calderón-Zygmund kernel K with homogeneity -1) defined as

$$\gamma_K(E) = \sup\{|\langle T, 1\rangle| :\ T \ \text{ distribution },\ \operatorname{spt} T \subset E,\ \|T * K\|_\infty \leq 1\},$$

is comparable to the analytic capacity $\gamma(E)$ provided the following properties hold:

- The symmetrization method: one has to ensure that when symmetrizing the kernel K (as in (16)) the quantity obtained is non-negative and comparable to Menger curvature.
- The localization property: we need that our kernel K localizes in the uniform norm. By this we mean that if T is a compactly supported distribution such that $T * K$ is a bounded function then $\varphi T * K$ is also bounded for each compactly supported $\mathcal{C}^1$ function φ and we have the corresponding estimate.
- The outer regularity property (see Section 4).

6.2. **Outer regularity and finiteness of the capacities γ_n^1 and γ_n^2.** Motivated by [MPrVe] and [ChMPT], we introduce now capacities related to only one kernel, $K_i = x_i^{2n-1}/|x|^{2n}$, $n \in \mathbb{N}$. For $1 \leq i \leq 2$, we set

$$\gamma_n^i(E) = \sup |\langle T, 1\rangle|,$$

the supremum taken over those real distributions T supported on E such that the potential $K_i * T$ is in the unit ball of $L^\infty(\mathbb{R}^2)$.
It is clear from the definition that for each compact set E, $1 \leq i \leq 2$,

$$\gamma_n(E) \leq \gamma_n^i(E).$$

Little is known about these capacities $\gamma_n^i(E)$, because a growth condition like (5) (see also (40)) cannot be deduced from the $L^\infty-$boundedness of only one potential (see Section 5 of [MPrVe] for some examples on this fact for the case $n = 1$). We show that these capacities are finite and satisfy the outer regularity property. For this, we need the reproduction formula stated below.

Lemma 21. *If a function $f(x_1, x_2)$ has continuous derivatives up to order 2, then, for $1 \leq i \leq 2$, it is representable in the form*

$$f(x) = (\varphi_i * K_i)(x), \tag{54}$$

where

$$\partial_i \varphi_i = S_i(\Delta f) = c\Delta f + \widetilde{S}_i(\Delta f), \tag{55}$$

for some constant c and the operators S_i, $\widetilde{S}_i$ as in Lemma 10.

Proof. Without loss of generality fix $i = 1$. By Lemma 12, we know that

$$\widehat{K_1}(\xi) = c\frac{\xi_1}{|\xi|^{2n}} p(\xi_1, \xi_2), \tag{56}$$

where $p(\xi_1, \xi_2)$ is a homogeneous polynomial of degree $2n-2$ with no non-vanishing zeros. Let S_1 be the operator with kernel

$$\widehat{s_1}(\xi) = \frac{|\xi|^{2n-2}}{p(\xi_1, \xi_2)}.$$

By [Du, Theorem 4.15, p.82] (see also the proof of Lemma 10), since the polynomial p has no non-vanishing zeros, the operators S_i, $1 \leq i \leq 2$, can be writen as $S_i = c\,\mathrm{id} + \widetilde{S}_i$, were $\widetilde{S}_1$ and $\widetilde{S}_2$ are Calderón-Zygmund operators.

Now taking Fourier transforms on (55) with $i = 1$, we obtain

$$\xi_1 \widehat{\varphi}_1(\xi) = |\xi|^2 \widehat{f}(\xi) \frac{|\xi|^{2n-2}}{p(\xi_1, \xi_2)},$$

which together with (56) gives

$$\widehat{f}(\xi) = c\widehat{\varphi}_1(\xi)\frac{\xi_1}{|\xi|^{2n}}p(\xi_1,\xi_2) = \widehat{\varphi}_1(\xi)\widehat{K}_1(\xi).$$

Therefore the lemma is proven. □

In [MPrVe] it was shown that for a square $Q \subset \mathbb{R}^2$, the capacities γ_1^i, $1 \le i \le 2$, satisfy $\gamma_1^i(Q) \le Cl(Q)$. We will now extend this result to the capacities γ_n^i, $1 \le i \le 2$ and $n \ge 1$.

Lemma 22. *For any square $Q \subset \mathbb{R}^2$ and $1 \le i \le 2$, we have $\gamma_n^i(Q) \le Cl(Q)$.*

Proof. Without loss of generality assume $i = 1$. Let T be a distribution supported on Q such that the potential $K_1 * T \in L^\infty(\mathbb{R}^2)$. Write $Q = I_1 \times I_2$, with I_j, $1 \le j \le 2$, being intervals in $\mathbb{R}$, and let $\varphi_Q \in \mathcal{C}_0^\infty(2Q)$ be such that $\|\varphi_Q\|_\infty \le C$, $\|\nabla\varphi_Q\|_\infty \le C\ l(Q)^{-1}$, $\|\nabla^2\varphi_Q\|_\infty \le C\ l(Q)^{-2}$ and

$$\varphi_Q(x) = \varphi_1(x_1)\varphi_2(x_2),$$

with $\varphi_1(x_1) = 1$ on I_1, $\varphi_1(x_1) = 0$ on $(2I_1)^c$, $\int_{-\infty}^{\infty}\varphi_1 = 0$, $\varphi_2 \ge 0$, $\varphi_2 \equiv 1$ on I_2 and $\varphi_2 \equiv 0$ on $(2I_2)^c$.

Since our distribution T is supported on Q, using (54) with f and φ_1 replaced by φ_Q and ψ respectively,

$$|\langle T, 1\rangle| = |\langle T, \varphi_Q\rangle| = |\langle K_1 * T, \psi\rangle| \le \|K_1 * T\|_\infty \|\psi\|_1,$$

where $\psi(x_1,x_2) = \int_{-\infty}^{x_1}\Delta\varphi_Q(t,x_2)dt + \int_{-\infty}^{x_1}\widetilde{S}_i(\Delta\varphi_Q)(t,x_2)dt$. Therefore, the lemma will be proven once we show that $\|\psi\|_1 \le Cl(Q)$.

Set $\psi_1(x_1,x_2) = \int_{-\infty}^{x_1}\Delta\varphi_Q(t,x_2)dt$. Notice that since the support of φ_Q is $2Q$ and $\int_{-\infty}^{\infty}\varphi_1 = 0$, then the support of ψ_1 is also $2Q$ and writing $2I_1 = [a,b]$, we get

$$\|\psi_1\|_1 \le \|\partial_1\varphi_Q\|_1 + \int_{2Q}|\partial_2^2\varphi_2(x_2)|\Big|\int_a^{x_1}\varphi_1(t)dt\Big|dx_1dx_2 \le Cl(Q).$$

Set $\psi_2(x_1,x_2) = \int_{-\infty}^{x_1}\widetilde{S}_1(\Delta\varphi_Q)(t,x_2)dt$ and let $K(x) = K(x_1,x_2)$ be the kernel of $\widetilde{S}_1$. Then,

$$\begin{aligned}\|\psi_2\|_1 &= \int_{3Q}|\psi_2(x)|dx + \int_{(3Q)^c}|\psi_2(x)|dx\\ &\le \int_{3Q}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx + \int_{(3Q)^c}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx\\ &= A + B.\end{aligned}$$

Recall that $Q = I_1 \times I_2$ and write $3I_1 = [z_1, z_2]$. Then

$$B = \int_{(3Q)^c} \left| \int_{-\infty}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx$$

$$\leq \int_{\substack{(3Q)^c \\ x_1 < z_1}} \left| \int_{-\infty}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx + \int_{\substack{(3Q)^c \\ x_1 \in [z_1, z_2]}} \left| \int_{-\infty}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx$$

$$+ \int_{\substack{(3Q)^c \\ x_1 > z_2}} \left| \int_{-\infty}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx = B_1 + B_2 + B_3.$$

We deal now with B_1. By Fubini and standard estimates for the kernel of a Calderón-Zygmund operator we get

$$B_1 = \int_{\substack{(3Q)^c \\ x_1 < z_1}} \left| \int_{-\infty}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx$$

$$\leq C \int_{\substack{(3Q)^c \\ x_1 < z_1}} \int_{2Q} |\varphi_Q(w)| \int_{-\infty}^{x_1} \frac{dt}{|w - (t, x_2)|^4} dw \, dx$$

$$\leq C l(Q)^2 \int_{\substack{(3Q)^c \\ x_1 < z_1}} \int_{-\infty}^{x_1} \frac{dt}{|(t, x_2)|^4} dw \, dx.$$

Using that

$$\int_{-\infty}^{x_1} \frac{dt}{|(t, x_2)|^4} \leq \frac{C}{|x|^3},$$

we get

$$B_1 \leq C l(Q)^2 \int_{(3Q)^c} \frac{dx}{|x|^3} \leq C l(Q)^2 l(Q)^{-1} = C l(Q).$$

Now we split B_2 in two terms:

$$B_2 = \int_{(3Q)^c} \left| \int_{-\infty}^{z_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx + \int_{\substack{(3Q)^c \\ x_1 \in [z_1, z_2]}} \left| \int_{z_1}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx.$$

The first term above is B_1 with x_1 replaced by z_1. For the second term in B_2, say B_{22}, we use Tonelli and estimates for the kernel of a Calderón-Zygmund operator. Then we obtain

$$B_{22} = \int_{\substack{(3Q)^c \\ x_1\in[z_1,z_2]}} \left|\int_{z_1}^{x_1} (K*\Delta\varphi_Q)(t,x_2)dt\right| dx$$

$$\leq \int_{\substack{(3Q)^c \\ x_1\in[z_1,z_2]}} \int_{z_1}^{x_1}\int_{2Q} |\varphi_Q(w)||\Delta K(w-(t,x_2))|\,dw\,dt\,dx$$

$$\leq C\int_{\substack{(3Q)^c \\ x_1\in[z_1,z_2]}} \int_{2Q} |\varphi_Q(w)| \int_{z_1}^{x_1} \frac{dt}{|w-(t,x_2)|^4}\,dw\,dx$$

$$\leq Cl(Q)l(Q)^2 l(Q)^{-2} = Cl(Q).$$

To deal with B_3, notice that since $\displaystyle\int_{-\infty}^{\infty} \widetilde{S}_1(\Delta\varphi_Q)(t,x_2)dt = 0$, one has

$$\int_{-\infty}^{x_1} (K*\Delta\varphi_Q)(t,x_2)dt = -\int_{x_1}^{\infty} (K*\Delta\varphi_Q)(t,x_2)dt,$$

so one argues as above.

We are now left with the term A. Recall that $3I_1 = [z_1,z_2]$ and write

$$A = \int_{3Q}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx$$

$$\leq \int_{\substack{3Q \\ x_1<z_1}}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx + \int_{\substack{3Q \\ x_1\in[z_1,z_2]}}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx$$

$$+ \int_{\substack{3Q \\ x_1>z_2}}\left|\int_{-\infty}^{x_1}(K*\Delta\varphi_Q)(t,x_2)dt\right|dx = A_1 + A_2 + A_3.$$

By Fubini and standard estimates for the kernel of a Calderón-Zygmund operator, we obtain

$$A_1 = \int_{\substack{3Q \\ x_1<z_1}}\left|\int_{-\infty}^{x_1}\int_{2Q}\Delta\varphi_Q(w)K(w-(t,x_2))dwdt\right|dx$$

$$\leq C\int_{\substack{3Q \\ x_1<z_1}}\int_{2Q}|\Delta\varphi_Q(w)|\int_{-\infty}^{x_1}\frac{dt}{|w-(t,x_2)|^2}dwdx$$

$$\leq C\int_{3Q}\frac{dx}{|x|} \leq Cl(Q).$$

Now we split A_2 in two terms

$$A_2 = \int_{3Q} \left| \int_{-\infty}^{z_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx + \int_{\substack{3Q \\ x_1 \in [z_1, z_2]}} \left| \int_{z_1}^{x_1} (K * \Delta\varphi_Q)(t, x_2) dt \right| dx$$

$$= A_{21} + A_{22}.$$

The term A_{21} is treated as A_1 with x_1 replaced by z_1. For A_{22}, we use Tonelli, the Cauchy-Schwarz inequality and the fact that Calderón-Zygmund operators are bounded in L^2. Then we get,

$$A_{22} \leq \int_{3I_2} \int_{3I_1} \int_{3I_1} \left| \widetilde{S}_1(\Delta\varphi_Q)(t, x_2) \right| dt \, dx_1 \, dx_2$$

$$= \int_{3I_1} \int_{3I_2} \int_{3I_1} \left| \widetilde{S}_1(\Delta\varphi_Q)(t, x_2) \right| dt \, dx_2 \, dx_1$$

$$\leq C \, l(Q) \, \|\widetilde{S}_1(\Delta\varphi_Q)\|_{L^1(3Q)} \leq C \, l(Q)^2 \, \|\Delta\varphi_Q\|_2 \leq C l(Q).$$

The estimate of A_3 is obtained similarly to B_3. □

As a consequence of the above result we have

Corollary 23. *For any compact set $E \subset \mathbb{R}^2$ and $1 \leq i \leq 2$, $\gamma_n^i(E) \leq C\mathrm{diam}(E)$.*

We show now that the capacities γ_n^i, $1 \leq i \leq 2$, satisfy the exterior regularity property, like the γ_n (see Lemma 16).

Lemma 24. *Let $\{E_k\}_k$ be a decreasing sequence of compact sets in $\mathbb{R}^2$, with intersection the compact set $E \subset \mathbb{R}^2$. Then, for $1 \leq i \leq 2$, $\gamma_n^i(E) = \lim_{k\to\infty} \gamma_n^i(E_k)$.*

Proof. Without loss of generality set $i = 1$. Let us see that $\lim_{k\to\infty} \gamma_n^1(E_k) = \gamma_n^1(E)$. Clearly, the limit exists and $\lim_{k\to\infty} \gamma_n^1(E_k) \geq \gamma_n^1(E)$. To prove the converse inequality, let T_k be a distribution supported on E_k such that $f_k = K_1 * T_k$ is in the unit ball of $L^\infty(\mathbb{R}^2)$ and

$$\gamma_n^1(E_k) - \frac{1}{k} < |\langle T_k, 1\rangle| \leq \gamma_n^1(E_k).$$

By taking a subsequence if necessary, we may assume that f_k converges weakly $*$ in $L^\infty(\mathbb{R}^2)$ to some function f such that $\|f\|_\infty \leq 1$.

We will show that T_k converges to some distribution T such that $T * K_1$ is also in the unit ball of $L^\infty(\mathbb{R}^2)$. Then

$$\gamma_n^1(E) \geq \langle T, 1\rangle = \lim_{k\to\infty} \langle T_k, 1\rangle = \lim_{k\to\infty} \gamma_n^1(E_k),$$

and we will be done.

Let us first check that the limit of $\{T_k\}_k$ exists in the topology of distributions. This is equivalent to saying that, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, the limit $\lim_{k\to\infty} \langle T_k, \varphi\rangle$ exists. To this end, let u be a vector of the form $u = (u_1, 0)$ such that

$$\mathrm{supp}(\varphi(\cdot - u)) \cap U_1(E) = \varnothing,$$

where $U_1(E)$ denotes the 1-neighborhood of E. In this way, for k big enough,

$$\langle T_k, \varphi\rangle = \langle T_k, \varphi - \varphi(\cdot - u)\rangle.$$

It is easy to check that there exists a function $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ such that $\partial_1 \psi = \varphi - \varphi(\cdot - u)$. Then, using the reproducing formula (54), we deduce that

$$\langle T_k, \varphi\rangle = \langle T_k, \partial_1\psi\rangle = \langle T_k,\, S_1(\Delta\psi) * K_1\rangle = \langle T_k * K_1,\, S_1(\Delta\psi)\rangle,$$

which is convergent, since $S_1(\Delta\psi) \in L^1(\mathbb{R}^2)$ arguing as in Lemma 13, and $f_k = T_k * K_1$ is weak * convergent in $L^\infty(\mathbb{R}^2)$.

To see that $T * K_1$ is in the unit ball of $L^\infty(\mathbb{R}^2)$, we take a radial function $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$, $\int \chi = 1$, supported in the unit ball and, as usual, we denote $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(\varepsilon^{-1}x)$. Then it is enough to prove that $\chi_\varepsilon * T * K_1$ is in the unit ball of $L^\infty(\mathbb{R}^2)$ for all $\varepsilon > 0$. This follows easily: denoting $K_1^\varepsilon = \chi_\varepsilon * K_1$, for each $x \in \mathbb{R}^2$, we have

$$T_k * K_1^\varepsilon(x) = \langle T_k,\, K_1^\varepsilon(x - \cdot)\rangle.$$

Notice moreover that $\|T_k * K_1^\varepsilon\|_\infty = \|\chi_\varepsilon * (T_k * K_1)\|_\infty \leq 1$. Now, let $\psi_0 \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ be such that it equals 1 in $U_1(E)$, so that $T_k = \psi_0\, T_k$ for k big enough. Then

$$T_k * K_1^\varepsilon(x) = \langle \psi_0\, T_k,\, K_1^\varepsilon(x - \cdot)\rangle = \langle T_k,\, \psi_0\, K_1^\varepsilon(x - \cdot)\rangle,$$

which converges to $\langle T,\, \psi_0\, K_1^\varepsilon(x - \cdot)\rangle = \langle T,\, K_1^\varepsilon(x - \cdot)\rangle = T * K_1^\varepsilon(x)$ as $k \to \infty$. Since $|T_k * K_1^\varepsilon(x)| \leq 1$ for all k, we deduce that $|T * K_1^\varepsilon(x)| \leq 1$ as wished, too. □

Remark. With little additional effort one can show that $T * K_1 = f$ in the above proof.

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