BMO, H^1, AND CALDERÓN-ZYGMUND OPERATORS FOR NON DOUBLING MEASURES

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Abstract. Given a Radon measure μ on $\mathbb{R}^d$, which may be non doubling, we introduce a space of type *BMO* with respect to this measure. It is shown that many properties which hold for the classical space $BMO(\mu)$ when μ is a doubling measure remain valid for the space of type *BMO* introduced in this paper, without assuming μ doubling. For instance, Calderón-Zygmund operators which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into the new *BMO* space. Moreover, this space also satisfies a John-Nirenberg inequality, and its predual is an atomic space H^1. Using a sharp maximal operator it is shown that operators which are bounded from $L^\infty(\mu)$ into the new *BMO* space and from its predual H^1 into $L^1(\mu)$ must be bounded on $L^p(\mu)$, $1 < p < \infty$. From this result one can obtain a new proof of the $T(1)$ theorem for the Cauchy transform for non doubling measures. Finally, it is proved that commutators of Calderón-Zygmund operators bounded on $L^2(\mu)$ with functions of the new *BMO* are bounded on $L^p(\mu)$, $1 < p < \infty$.

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1991 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B30.

Key words and phrases. BMO, atomic spaces, Hardy spaces, Calderón-Zygmund operators, non doubling measures, $T(1)$ theorem, commutators.

Partially supported by grants DGICYT PB96-1183 and CIRIT 1998-SGR00052 (Spain).

1. Introduction

In this paper, given a Radon measure μ on $\mathbb{R}^d$ which may be non doubling, we introduce a *BMO* space and an atomic space (the predual of the *BMO* space), with respect to the measure μ. It is shown that, in some ways, these spaces play the role of the classical spaces *BMO* and H^1_{at} in case μ is a doubling measure.

Recently it has been proved that many results of the Caderón-Zygmund theory of singular integrals remain valid for non doubling measures on $\mathbb{R}^d$. A version of the $T(1)$ theorem for the Cauchy transform was obtained in [15], and another for more general Calderón-Zygmund operators in [10]. Cotlar's inequality and weak $(1,1)$ estimates were studied in [10] (for the particular case of the Cauchy transform, the weak $(1,1)$ estimate was studied also in [15]). G. David [2] obtained a theorem of $T(b)$ type for non doubling measures that solved Vitushkin's conjecture for sets with positive finite 1-dimensional Hausdorff measure. Another $T(b)$ theorem suitable for solving this conjecture was proved later by Nazarov, Treil and Volberg [12]. In [16], it is shown that if the Cauchy transform is bounded on $L^2(\mu)$, then the principal values of the Cauchy transform exist μ-almost everywhere in $\mathbb{C}$. In [18], it is given another proof for the $T(1)$ theorem for the Cauchy transform, and in [17] a $T(1)$ theorem suitable for non doubling measures *with atoms* is proved. Also, in [13], another $T(b)$ theorem for non doubling measures (closer to the classical one than the ones stated above) is obtained.

However, for the moment, the attempts to find good substitutes for the space *BMO* and its predual H^1_{at} for non doubling measures have not been completely succesful. Mateu, Mattila, Nicolau and Orobitg [7] have studied the spaces $BMO(\mu)$ and $H^1_{at}(\mu)$ (with definitions similar to the classical ones) for a non doubling measure μ. They have shown that some of the properties that these spaces satisfy when μ is a doubling measure are satisfied also if μ is non doubling. For example, the John-Nirenberg inequality holds, $BMO(\mu)$ is the dual of $H^1_{at}(\mu)$ and the operators which are bounded from $H^1_{at}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $BMO(\mu)$ are bounded on $L^p(\mu)$, $1<p<\infty$. Nevertheless, unlike in the case of doubling measures, Calderón-Zygmund operators may be bounded on $L^2(\mu)$ but not from $L^\infty(\mu)$ into $BMO(\mu)$ or from $H^1_{at}(\mu)$ into $L^1(\mu)$, as it is shown by Verdera [18]. This is the main drawback of the spaces $BMO(\mu)$ and $H^1_{at}(\mu)$ considered in [7].

On the other hand, Nazarov, Treil and Volberg [13] have introduced another space of *BMO* type. Calderón-Zygmund operators which are bounded on $L^2(\mu)$ are bounded from $L^\infty(\mu)$ into their *BMO* space. However, the *BMO* space considered in [13] does not satisfy John-Nirenberg inequality, it is not known which is its predual, and (by now) there is no any interpolation result such as the one given in [7].

Let us introduce some notation and definitions. Let d,n be some fixed integers with $1\leq n\leq d$. A kernel $k(\cdot,\cdot)\in L^1_{loc}(\mathbb{R}^d\times\mathbb{R}^d\setminus\{(x,y):x=y\})$ is called a Calderón-Zygmund kernel if

(1) $|k(x,y)| \leq \dfrac{C}{|x-y|^n}$,

(2) there exists $0 < \delta \leq 1$ such that

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq C\,\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}$$

if $|x-x'| \leq |x-y|/2$.

Throughout all the paper we will assume that μ is a Radon measure on $\mathbb{R}^d$ satisfying the following growth condition:

$$\mu(B(x,r)) \leq C_0\, r^n \quad \text{for all } x \in \mathbb{R}^d,\ r > 0. \tag{1.1}$$

The Calderón-Zygmund operator (CZO) associated to the kernel $k(\cdot,\cdot)$ and the measure μ is defined (at least, formally) as

$$Tf(x) = \int k(x,y)\, f(y)\, d\mu(y).$$

The above integral may be not convergent for many functions f because the kernel k may have a singularity for $x = y$. For this reason, one introduces the truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x,y)\, f(y)\, d\mu(y),$$

and then one says that T is bounded on $L^p(\mu)$ if the operators T_ε are bounded on $L^p(\mu)$ uniformly on $\varepsilon > 0$.

Recall that a function $f \in L^1_{loc}(\mu)$ is said to belong to $BMO(\mu)$ if there exists some constant C_1 such that

$$\sup_Q \frac{1}{\mu(Q)} \int_Q |f - m_Q(f)|\, d\mu \leq C_1, \tag{1.2}$$

where the supremum is taken over all the cubes $Q \subset \mathbb{R}^d$ centered at some point of $\operatorname{supp}(\mu)$ (in the paper by a cube we mean a closed cube with sides parallel to the axes, and if $\|\mu\| < \infty$, we allow $Q = \mathbb{R}^d$ too) and $m_Q(f)$ stands for the mean of f over Q with respect to μ, i.e. $m_Q(f) = \int_Q f\, d\mu/\mu(Q)$. The optimal constant C_1 is the *BMO* norm of f.

Let us remark that there is a slight difference between the space $BMO(\mu)$ that we have just defined and the one considered in [7]: We have taken the supremum in (1.2) over cubes which are centered at some point in $\operatorname{supp}(\mu)$, while in [7] that supremum is taken over all the cubes in $\mathbb{R}^d$.

It is well known that if μ is a doubling measure, i.e. $\mu(2Q) \leq C\,\mu(Q)$ for all the cubes Q centered at some point of $\operatorname{supp}(\mu)$, and T is bounded on $L^2(\mu)$, then T is also bounded from $L^\infty(\mu)$ into $BMO(\mu)$. As stated above, this may fail if μ is non doubling. Hence if one wants to work with a *BMO* space which fulfils some of the usual and fundamental properties related with CZO's, then one must introduce a new space *BMO*. So, for a fixed

$\rho > 1$, one says that a function $f \in L^1_{loc}(\mu)$ belongs to $BMO_\rho(\mu)$ if for some constant C_2

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)| \, d\mu \leq C_2, \tag{1.3}$$

again with the supremum taken over all the cubes Q centered at some point of $\mathrm{supp}(\mu)$. This is *almost* the definition taken by Nazarov, Treil and Volberg in [13].

In fact, in [13], the supremum in the definition (1.3) of BMO_ρ is taken not only over all cubes Q centered at some point of $\mathrm{supp}(\mu)$, but over *all* the cubes $Q \subset \mathbb{R}^d$. We have prefered to take the supremum only over cubes centered in points of $\mathrm{supp}(\mu)$, to be coherent with our definitions below.

It is straigthforward to check that if T is bounded on $L^2(\mu)$, then T is bounded from $L^\infty(\mu)$ into $BMO_\rho(\mu)$. This is the main advantage of $BMO_\rho(\mu)$ over $BMO(\mu)$. Nevertheless, the new space $BMO_\rho(\mu)$ does not have all the nice properties that one may expect. First of all, it happens that the definition of $BMO_\rho(\mu)$ depends on the constant $\rho > 1$ that we choose. Obviously, the BMO_ρ norm of f (i.e. the optimal constant C_2 in (1.3)) depends on ρ. Moreover, it is shown in [13] that there exist measures μ and functions f which for some $\rho > 1$ are in $BMO_\rho(\mu)$, but not for other $\rho > 1$.

Given $p \in [1, \infty)$, one says that $f \in BMO^p_\rho(\mu)$ if

$$\sup_Q \int_Q |f - m_Q(f)|^p \, d\mu \leq C \, \mu(\rho Q). \tag{1.4}$$

In case μ is doubling measure, by John-Nirenberg inequality, all the spaces $BMO^p(\mu) \equiv BMO^p_{\rho=1}(\mu)$ coincide. This is not the case if μ is non doubling. In [13] it is shown that there are measures μ and functions f such that f is in $BMO^p_\rho(\mu)$ only for a proper subset of $p \in [1, \infty)$.

In this paper we will introduce a new variant of the space BMO suitable for non doubling measures, which will satisfy some of the properties of the usual BMO, such as for example the John-Nirenberg inequality. This space will be a (proper, in general) subspace of the spaces $BMO^p_\rho(\mu)$. It will be small enough to fulfil the properties that we have mentioned and big enough in order that CZO's which are bounded on $L^2(\mu)$ be also bounded from $L^\infty(\mu)$ into our new space of BMO type.

We will show that if T is bounded on $L^2(\mu)$ and $g \in L^\infty(\mu)$, then the oscillations of $f = T(g)$ satisfy not only the condition given by (1.3), but other regularity conditions. Then, the functions of our new space will be the functions satisfying (1.3) and, also, these additional regularity conditions about their oscillations. We will denote it as $RBMO(\mu)$ (this stands for '*regular* bounded mean oscillations'). Notice that we have not written $RBMO_\rho(\mu)$. This is because, as we will see, the definition will not depend on ρ, for $\rho > 1$.

If one says that f is in $BMO_\rho(\mu)$ when it satisfies (1.3), it seems that we have to consider the atomic space $H^{1,\infty}_{at,\rho}(\mu)$ made up with functions of the form

$$f = \sum_i \lambda_i \, a_i, \tag{1.5}$$

where $\lambda_i \in \mathbb{R}$, $\sum_i |\lambda_i| < \infty$ and, for each i, a_i is a function supported in a cube Q_i, with $\|a_i\|_{L^\infty(\mu)} \leq \mu(\rho Q_i)^{-1}$, and $\int a_i \, d\mu = 0$ (that is a_i is an *atom*). Obviously, $H^{1,\infty}_{at,\rho}(\mu)$ is the usual atomic space $H^{1,\infty}_{at}(\mu) \equiv H^{1,\infty}_{at,\rho=1}(\mu)$ when μ is a doubling measure. With this definition, a CZO which is bounded in $L^2(\mu)$, is also bounded from $H^{1,\infty}_{at,\rho}(\mu)$ into $L^1(\mu)$ (taking $\rho > 1$).

In this paper we will introduce another space of atomic type: $H^{1,\infty}_{atb}(\mu)$ (the subindex 'atb' stands for 'atomic block'). This space will be made up of functions of the form

$$f = \sum_i b_i, \tag{1.6}$$

where the functions b_i will be some elementary functions, which we will call *atomic blocks* (in particular, any atom a_i such as the one of (1.5) will be an atomic block). So we will have $H^{1,\infty}_{at,\rho}(\mu) \subset H^{1,\infty}_{atb}(\mu)$ but, in general, $H^{1,\infty}_{atb}(\mu)$ will be strictly bigger than $H^{1,\infty}_{at,\rho}(\mu)$.

We will see that this new atomic space enjoys some very interesting properties. First of all, the definition of the space will be independent of the chosen constant $\rho > 1$. Also, CZO's which are bounded on $L^2(\mu)$ will be also bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$. Moreover, we will show that $H^{1,\infty}_{atb}(\mu)$ is the predual of $RBMO(\mu)$, and that, as in the doubling case, there is a collection of spaces $H^{1,p}_{atb}(\mu)$, $p > 1$, that coincide with $H^{1,\infty}_{atb}(\mu)$.

We will show two applications of all the results obtained about $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$. In our first application we will obtain an interpolation theorem: We will prove that if a linear operator is bounded from L^∞ into $RBMO(\mu)$ and from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$, then it is bounded on $L^p(\mu)$, $1 < p < \infty$. As a consequence we will obtain a new proof of the $T(1)$ theorem for the Cauchy transform for non doubling measures.

We have already mentioned that in [7] it is also proved a theorem of interpolation between $(H^1_{at}(\mu), L^1(\mu))$ and $(L^\infty(\mu), BMO(\mu))$, with μ non doubling. However, from this result it is not possible to get the $T(1)$ theorem for the Cauchy transform, as it is explained in [7].

Finally, in our second application we will show that if a CZO is bounded on $L^2(\mu)$, then the commutator of this operator with a function of $RBMO(\mu)$ is bounded on $L^p(\mu)$, $1 < p < \infty$.

2. The space $RBMO(\mu)$

2.1. **Introduction.** If μ is a doubling measure and f is a function belonging to $BMO(\mu)$, it is easily checked that if Q, R are two cubes of comparable size with $Q \subset R$, then

$$|m_Q(f) - m_R(f)| \leq C\,\|f\|_{BMO(\mu)} \tag{2.1}$$

In case μ is not doubling and $f \in BMO_\rho(\mu)$, it is easily seen that

$$|m_Q(f) - m_R(f)| \leq \frac{\mu(\rho R)}{\mu(Q)}\,\|f\|_{BMO_\rho(\mu)}, \tag{2.2}$$

and that's all one can obtain. So if $\mu(Q)$ is much smaller than $\mu(R)$, then $m_Q(f)$ may be very different from $m_R(f)$, and one does not have any useful information. However, to prove most results dealing with functions in *BMO*, some kind of control in the changes of the mean values of f, such as the one in (2.1), appears to be essential.

We will see that if T is a CZO that is bounded on $L^2(\mu)$ and $g \in L^\infty(\mu)$, then the oscillations of $T(g)$ satisfy some properties which will be stronger than (2.2). Some of these properties will be stated in terms of some coefficients $K_{Q,R}$, for $Q \subset R$ cubes in $\mathbb{R}^d$, which now we proceed to describe.

2.2. **The coefficients $K_{Q,R}$.** Throughout the rest of the paper, unless otherwise stated, any cube will be a cube in $\mathbb{R}^d$ *with sides parallel to the axes and centered at some point of* $\text{supp}(\mu)$.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, we set

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n}, \tag{2.3}$$

where $N_{Q,R}$ is the first integer k such that $l(2^k Q) \geq l(R)$ (in case $R = \mathbb{R}^d \neq Q$, we set $N_{Q,R} = \infty$). The coefficient $K_{Q,R}$ measures how close Q is to R, in some sense. For example, if Q and R have comparable sizes, then $K_{Q,R}$ is bounded above by some constant which depends on the ratio $l(R)/l(Q)$ (and on the constant C_0 of (1.1)).

Given $\alpha > 1$ and $\beta > \alpha^n$, we say that some cube $Q \subset \mathbb{R}^d$ is (α, β)-doubling if $\mu(\alpha Q) \leq \beta\,\mu(Q)$. Due to the fact that μ satisfies the growth condition (1.1), there are a lot of "big" doubling cubes. To be precise, given any point $x \in \text{supp}(\mu)$ and $d > 0$, there exists some (α, β)-doubling cube Q centered at x with $l(Q) \geq d$. This is easily seen by the growth condition (1.1) for μ and the fact that $\beta > \alpha^n$.

In the following lemma we show some of the properties of the coefficients $K_{Q,R}$.

Lemma 2.1. *We have:*

1. *If $Q \subset R \subset S$ are cubes in $\mathbb{R}^d$, then $K_{Q,R} \leq K_{Q,S}$, $K_{R,S} \leq C\,K_{Q,S}$ and $K_{Q,S} \leq C\,(K_{Q,R} + K_{R,S})$.*
2. *If $Q \subset R$ have comparable sizes, $K_{Q,R} \leq C$.*
3. *If N is some positive integer and the cubes $2Q$, $2^2Q, \ldots 2^{N-1}$ are non $(2, \beta)$-doubling (with $\beta > 2^n$), then $K_{Q,2^N Q} \leq C$, with C depending on β and C_0.*

(4) *If N is some positive integer and for some $\beta < 2^n$,*

$$\mu(2^N Q) \leq \beta\mu(2^{N-1}Q) \leq \beta^2\mu(2^{N-2}Q) \leq \ldots \leq \beta^N\mu(Q),$$

then $K_{Q,2^NQ} \leq C$, with C depending on β and C_0.

Proof. The properties 1 and 2 are immediate. Let us see 3. For $\beta > 2^n$, we have $\mu(2^{k+1}Q) > \beta\,\mu(2^kQ)$ for $k = 1, \ldots, N-1$. Thus

$$\mu(2^kQ) < \frac{\mu(2^NQ)}{\beta^{N-k}}$$

for $k = 1, \ldots, N-1$. Therefore,

$$\begin{aligned} K_{Q,2^NQ} &\leq 1 + \sum_{k=1}^{N-1} \frac{\mu(2^NQ)}{\beta^{N-k}\, l(2^kQ)^n} + \frac{\mu(2^NQ)}{l(2^NQ)^n} \\ &\leq 1 + C_0 + \frac{\mu(2^NQ)}{l(2^NQ)^n} \sum_{k=1}^{N-1} \frac{1}{\beta^{N-k}\, 2^{(k-N)n}} \\ &\leq 1 + C_0 + C_0 \sum_{k=1}^{\infty} (2^n/\beta)^k \leq C. \end{aligned}$$

Let us check the fourth property. For $\beta < 2^n$, we have

$$\begin{aligned} K_{Q,2^NQ} &\leq 1 + \sum_{k=1}^{N} \frac{\beta^k\mu(Q)}{l(2^kQ)^n} \\ &\leq 1 + \frac{\mu(Q)}{l(Q)^n} \sum_{k=1}^{N} \frac{\beta^k}{2^{kn}} \\ &\leq 1 + C_0 \sum_{k=1}^{\infty} \left(\frac{\beta}{2^n}\right)^k \leq C. \end{aligned}$$

□

Notice that, in some sense, the property 3 of Lemma 2.1 says that if the density of the measure μ in the concentric cubes grows much faster than the size of cubes, then the coefficients $K_{Q,2^NQ}$ remain bounded, while the fourth property says that if the measure grows too slowly, then they also remain bounded.

Remark 2.2. If we substitute the numbers 2^k in the definition (2.3) by α^k, for some $\alpha > 1$, we will obtain a coefficient $K^{\alpha}_{Q,R}$. It is easy to check that $K_{Q,R} \approx K^{\alpha}_{Q,R}$ (with constants that may depend on α and C_0).

Also, if we set

$$K'_{Q,R} = 1 + \int_{l(Q)}^{l(R)} \frac{\mu(B(x_Q, r))}{r^{n-1}}\, dr,$$

or

$$K''_{Q,R} = 1 + \int_{l(Q)\leq|y-x_Q|\leq l(R)} \frac{1}{|y-x_Q|^n}\,d\mu(y),$$

where x_Q is the center of Q, then it is easily seen that $K_{Q,R} \approx K'_{Q,R} \approx K''_{Q,R}$. The definitions of $K'_{Q,R}$ and $K''_{Q,R}$ have the advantage of not depending on the grid of cubes, unlike the one of $K_{Q,R}$.

We have stated above that there a lot of "big" (α,β)-doubling cubes. In the next remark we show that, for β big enough, there are also many "small" (α,β)-doubling cubes.

Remark 2.3. Given $\alpha > 1$, if μ is any Radon measure on $\mathbb{R}^d$, it is known that for β big enough (depending on α and d), for μ-almost all $x \in \mathbb{R}^d$ there is a sequence of (α,β)-*doubling* cubes $\{Q_n\}_n$ centered at x with $l(Q_n)$ tending to 0 as $n \to \infty$.

For $\alpha = 2$, we denote by β_d one of these big constants β. For definiteness, one can assume that β_d is twice the infimum of these β's.

If α and β are not specified, by a doubling cube we will mean a $(2,\beta_d)$-doubling cube.

Let $f \in L^1_{loc}(\mu)$ be given. Observe that, by the Lebesgue differentiation theorem, for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of $(2,\beta_d)$-doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f\,d\mu = f(x).$$

Thus, for any fixed $\lambda > 0$, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists a sequence of $(2,\beta_d)$-doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ such that

$$\limsup_{k\to\infty} \frac{1}{\mu(2Q_k)} \int_{Q_k} |f|\,d\mu > \frac{\lambda}{\beta_d}.$$

2.3. **The definition of** *RBMO*(μ)**.** Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer ≥ 0 such that $2^N Q$ is doubling. We denote this cube by $\widetilde{Q}$ (recall that this cube $\widetilde{Q}$ exists because otherwise the growth condition (1.1) on μ would fail).

Let $\rho > 1$ be some fixed constant. We say that $f \in L^1_{loc}(\mu)$ is in *RBMO*(μ) if there exists some constant C_3 such that for any cube Q (centered at some point of $\operatorname{supp}(\mu)$),

$$\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\widetilde{Q}} f|\,d\mu \leq C_3 \tag{2.4}$$

and

$$|m_Q f - m_R f| \leq C_3\, K_{Q,R} \quad \text{for any two doubling cubes } Q \subset R. \tag{2.5}$$

The minimal constant C_3 is the *RBMO*(μ) norm of f (in fact, it is a norm in the space of functions modulo additive constants), and it will be denoted by $\|\cdot\|_*$.

Let us remark that the space *RBMO*(μ) depends on the integer n because of the definition of the coefficients $K_{Q,R}$.

Notice that if (2.4) is satisfied, then (1.3) also holds. Indeed, for any cube Q and any $a \in \mathbb{R}$ one has

$$\int_Q |f - m_Q f| \, d\mu \leq 2 \int_Q |f - a| \, d\mu.$$

In particular this holds for $a = m_{\widetilde{Q}} f$. So, the condition (2.4) is stronger than (1.3). Moreover, in the definition of *RBMO*(μ) we ask also the regularity condition (2.5).

Observe also that, as a consequence of (2.5), if $Q \subset R$ are doubling cubes with comparable size, then

$$|m_Q f - m_R f| \leq C \, \|f\|_*, \tag{2.6}$$

taking into account the property 2 of Lemma 2.1.

Remark 2.4. In fact, (2.6) also holds for any two doubling cubes with comparable sizes such that $\operatorname{dist}(Q,R) \lessapprox l(Q)$. To see this, let Q_0 be a cube concentric with Q, containing Q and R, and such that $l(Q_0) \approx l(Q)$. Then $K(Q_0, \widetilde{Q_0}) \leq C$, and thus we have $K(Q, \widetilde{Q_0}) \leq C$ and $K(R, \widetilde{Q_0}) \leq C$ (we have used the properties 1, 2 and 3 of Lemma 2.1). Then $|m_Q f - m_{\widetilde{Q_0}} f| \leq C \, \|f\|_*$ and $|m_R f - m_{\widetilde{Q_0}} f| \leq C \, \|f\|_*$. So (2.6) holds.

Let us remark that the condition (2.6) is not satisfied, in general, by functions of the bigger space $BMO_\rho(\mu)$ and cubes Q, R as above.

We have the following properties:

Proposition 2.5.
1. *RBMO*(μ) *is a Banach space of functions (modulo additive constants).*
2. $L^\infty(\mu) \subset$ *RBMO*(μ), *with* $\|f\|_* \leq 2\|f\|_{L^\infty(\mu)}$.
3. *If* $f \in$ *RBMO*(μ), *then* $|f| \in$ *RBMO*(μ) *and* $\| \, |f| \, \|_* \leq C \, \|f\|_*$.
4. *If* $f, g \in$ *RBMO*(μ), *then* $\min(f,g)$, $\max(f,g) \in$ *RBMO*(μ) *and*

$$\|\min(f,g)\|_*, \, \|\max(f,g)\|_* \leq C \, (\|f\|_* + \|g\|_*).$$

Proof. The properties 1 and 2 are easy to check. The third property is also easy to prove with the aid of Lemma 2.8 below. The fourth property follows from the third. □

Before showing that CZO's which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into *RBMO*(μ), we will see other equivalent norms for *RBMO*(μ). Suppose that for a given a function $f \in L^1_{loc}(\mu)$ there exist some constant C_4 and a collection of numbers $\{f_Q\}_Q$ (i.e. for each cube Q, there exists $f_Q \in \mathbb{R}$) such that

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f(x) - f_Q| \, d\mu(x) \leq C_4, \tag{2.7}$$

and,

$$|f_Q - f_R| \leq C_4\, K_{Q,R} \quad \text{for any two cubes } Q \subset R. \tag{2.8}$$

Then, we write $\|f\|_{**} = \inf C_4$, where the infimum is taken over all the constants C_4 and all the numbers $\{f_Q\}$ safisfying (2.7) and (2.8). It is esily checked that $\|\cdot\|_{**}$ is a norm in the space of functions modulo constants.

The definition of the norm $\|\cdot\|_{**}$ depends on the constant ρ chosen in (2.7) (the same occurs for $\|\cdot\|_*$). However, if we write $\|\cdot\|_{**,(\rho)}$ instead of $\|\cdot\|_{**}$, we have

Lemma 2.6. *The norms $\|\cdot\|_{**,(\rho)}$, $\rho > 1$, are equivalent.*

Proof. Let $\rho > \eta > 1$ be some fixed constants. Obviously, $\|f\|_{**,(\rho)} \leq \|f\|_{**,(\eta)}$. So we only have to show $\|f\|_{**,(\eta)} \leq C\,\|f\|_{**,(\rho)}$. It is enough to prove that for a fixed collection of numbers $\{f_Q\}_Q$ satisfying

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f(x) - f_Q|\, d\mu(x) \leq 2\,\|f\|_{**,(\rho)}$$

and

$$|f_Q - f_R| \leq 2\, K_{Q,R}\, \|f\|_{**,(\rho)} \quad \text{for any two cubes } Q \subset R,$$

we have

$$\frac{1}{\mu(\eta Q_0)} \int_{Q_0} |f - f_{Q_0}|\, d\mu \leq C\,\|f\|_{**,(\rho)} \quad \text{for any fixed cube } Q_0. \tag{2.9}$$

For any $x \in Q_0 \cap \operatorname{supp}(\mu)$, let Q_x be a cube centered at x with side length $\frac{\eta-1}{10\rho}\, l(Q_0)$. Then $l(\rho Q_x) = \frac{\eta-1}{10}\, l(Q_0)$, and so $\rho Q_x \subset \eta Q_0$. By Besicovich's covering theorem, there exists a family of points $\{x_i\}_i \subset Q_0 \cap \operatorname{supp}(\mu)$ such that the cubes $\{Q_{x_i}\}_i$ form an almost disjoint covering of $Q_0 \cap \operatorname{supp}(\mu)$. Since Q_{x_i} and Q_0 have comparable sizes,

$$|f_{Q_{x_i}} - f_{Q_0}| \leq C\,\|f\|_{**,(\rho)},$$

with C depending on η and ρ. Therefore,

$$\begin{aligned} \int_{Q_{x_i}} |f - f_{Q_0}|\, d\mu &\leq \int_{Q_{x_i}} |f - f_{Q_{x_i}}|\, d\mu + |f_{Q_0} - f_{Q_{x_i}}|\, \mu(Q_{x_i}) \\ &\leq C\,\|f\|_{**,(\rho)}\, \mu(\rho Q_{x_i}). \end{aligned}$$

Then we get

$$\int_{Q_0} |f - f_{Q_0}|\, d\mu \leq \sum_i \int_{Q_{x_i}} |f - f_{Q_0}|\, d\mu \leq C\,\|f\|_{**,(\rho)} \sum_i \mu(\rho Q_{x_i}).$$

Since $\rho Q_{x_i} \subset \eta Q_0$ for all i, we obtain

$$\int_{Q_0} |f - f_{Q_0}|\, d\mu \leq C\,\|f\|_{**,(\rho)}\, \mu(\eta Q_0)\, N,$$

where N is the number of cubes of the Besicovich covering. Now it is easy to check that N is bounded some constant depending only on η, ρ and d: If

$\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and B_d is the Besicovich constant in $\mathbb{R}^d$, we have

$$N\,\mathcal{L}^d(Q_{x_i}) = \sum_i \mathcal{L}^d(Q_{x_i}) \leq B_d\,\mathcal{L}^d(\eta Q_0).$$

Thus

$$N \leq \frac{B_d\,\mathcal{L}^d(\eta Q_0)}{\mathcal{L}^d(Q_{x_i})} = B_d \left(\frac{10\eta\rho}{\eta - 1}\right)^d,$$

and (2.9) holds. □

Remark 2.7. In fact, in the preceding lemma we have seen that if C_f is some constant and $\{f_Q\}_Q$ is some fixed collection of numbers satisfying

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f(x) - f_Q|\,d\mu(x) \leq C_f$$

and

$$|f_Q - f_R| \leq K_{Q,R}\,C_f \quad \text{for any two cubes } Q \subset R,$$

then for the same numbers $\{f_Q\}_Q$, for any $\eta > 1$ we have

$$\sup_Q \frac{1}{\mu(\eta Q)} \int_Q |f(x) - f_Q|\,d\mu(x) \leq C\,C_f,$$

with C depending on η.

We also have:

Lemma 2.8. *For a fixed $\rho > 1$, the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are equivalent.*

Proof. Let $f \in L^1_{loc}(\mu)$. To see that $\|f\|_{**} \leq C\,\|f\|_*$ we set $f_Q = m_{\widetilde{Q}} f$ for all cubes Q. Then (2.7) holds with $C_4 = \|f\|_*$. Let us check that the second condition (2.8) is also satisfied. We have to prove that for any two cubes $Q \subset R$,

$$|m_{\widetilde{Q}} f - m_{\widetilde{R}} f| \leq C\,K_{Q,R}\,\|f\|_*. \tag{2.10}$$

Notice that if $\widetilde{Q} \subset \widetilde{R}$, then

$$|m_{\widetilde{Q}} f - m_{\widetilde{R}} f| \leq K_{\widetilde{Q},\widetilde{R}}\,\|f\|_*,$$

because $\widetilde{Q}, \widetilde{R}$ are doubling. So (2.10) follows if $K_{\widetilde{Q},\widetilde{R}} \leq C\,K_{Q,R}$. However, in general, $Q \subset R$ does not imply $\widetilde{Q} \subset \widetilde{R}$, and so we have to modify the argument.

Suppose first that $l(\widetilde{R}) \geq l(\widetilde{Q})$. Then $\widetilde{Q} \subset 4\widetilde{R}$. We denote $R_0 = \widetilde{4\widetilde{R}}$. Then we have

$$|m_{\widetilde{Q}} f - m_{\widetilde{R}} f| \leq |m_{\widetilde{Q}} f - m_{R_0} f| + |m_{R_0} f - m_{\widetilde{R}} f|. \tag{2.11}$$

Using the properties of Lemma 2.1 repeatedly, we get

$$\begin{aligned} K_{\widetilde{Q},R_0} &\leq C\,K_{Q,R_0} \leq C\,(K_{Q,R} + K_{R,R_0}) \\ &\leq C\,(K_{Q,R} + K_{R,\widetilde{R}} + K_{\widetilde{R},4\widetilde{R}} + K_{4\widetilde{R},R_0}) \leq C\,K_{Q,R}. \end{aligned}$$

Since $\widetilde{Q} \subset R_0$ and they are doubling cubes, we have

$$|m_{\widetilde{Q}}f - m_{R_0}f| \leq K_{\widetilde{Q},R_0}\, \|f\|_* \leq C\, K_{Q,R}\, \|f\|_*.$$

Now we are left with the second term on the right hand side of (2.11). We have

$$K_{\widetilde{R},R_0} \leq C(K_{\widetilde{R},4\widetilde{R}} + K_{4\widetilde{R},R_0}) \leq C \leq C\, K_{Q,R}.$$

Due to the fact that $\widetilde{R} \subset R_0$ are doubling cubes,

$$|m_{R_0}f - m_{\widetilde{R}}f| \leq K_{\widetilde{R},R_0}\, \|f\|_* \leq C\, K_{Q,R}\, \|f\|_*,$$

and, by (2.11), we get that (2.10) holds in this case.

Assume now $l(\widetilde{R}) < l(\widetilde{Q})$. Then $\widetilde{R} \subset 4\widetilde{Q}$. There exists some $m \geq 1$ such that $l(\widetilde{R}) \geq l(2^mQ)/10$ and $\widetilde{R} \subset 2^mQ \subset 4\widetilde{Q}$. Since $\widetilde{R}$ and 2^mQ have comparable sizes, we have $K_{\widetilde{R},2^mQ} \leq C$. Then, if we denote $Q_0 = \widetilde{4\widetilde{Q}}$, we get

$$K_{\widetilde{R},Q_0} \leq C\,(K_{\widetilde{R},2^mQ} + K_{2^mQ,4\widetilde{Q}} + K_{4\widetilde{Q},Q_0}) \leq C.$$

Also,

$$K_{\widetilde{Q},Q_0} \leq C\,(K_{\widetilde{Q},4\widetilde{Q}} + K_{4\widetilde{Q},Q_0}) \leq C.$$

Therefore,

$$\begin{aligned} |m_{\widetilde{Q}}f - m_{\widetilde{R}}f| &\leq |m_{\widetilde{Q}}f - m_{Q_0}f| + |m_{Q_0}f - m_{\widetilde{R}}f| \\ &\leq K_{\widetilde{Q},Q_0}\, \|f\|_* + K_{\widetilde{R},Q_0}\, \|f\|_* \leq C\, \|f\|_* \leq C\, K_{Q,R}\, \|f\|_*. \end{aligned}$$

Now we have to check that $\|f\|_* \leq C\, \|f\|_{**}$. If Q is a doubling cube, since (2.7) holds with $\rho = 2$ (by Lemma 2.6), we have

$$|f_Q - m_Q f| = \left| \frac{1}{\mu(Q)} \int_Q (f - f_Q)\, d\mu \right| \leq \|f\|_{**}\, \frac{\mu(2Q)}{\mu(Q)} \leq C\, \|f\|_{**}.$$

Therefore, for any cube Q (non doubling, in general), using $K_{Q,\widetilde{Q}} \leq C$ we get

$$|f_Q - m_{\widetilde{Q}}f| \leq |f_Q - f_{\widetilde{Q}}| + |f_{\widetilde{Q}} - m_{\widetilde{Q}}f| \leq C\, \|f\|_{**}.$$

Thus

$$\begin{aligned} \frac{1}{\mu(\rho Q)} \int_Q |f(x) - m_{\widetilde{Q}}f|\, d\mu(x) &\leq \frac{1}{\mu(\rho Q)} \int_Q |f(x) - f_Q|\, d\mu(x) \\ &\quad + \frac{1}{\mu(\rho Q)} \int_Q |f_Q - m_{\widetilde{Q}}f|\, d\mu(x) \\ &\leq C\, \|f\|_{**}. \end{aligned}$$

It only remains to show that (2.5) also holds with $C\, \|f\|_{**}$ instead of C_3. This follows easily. Indeed, if $Q \subset R$ are doubling cubes, we have

$$\begin{aligned} |m_Q f - m_R f| &\leq |m_Q f - f_Q| + |f_Q - f_R| + |f_R - m_R f| \\ &\leq C\, \|f\|_{**} + K_{Q,R}\, \|f\|_{**} \leq C\, K_{Q,R}\, \|f\|_{**}. \end{aligned}$$

□

Remark 2.9. By the preceding lemma, it is easily seen that we obtain equivalent norms and the same space $RBMO(\mu)$ if we replace $(2, \beta_d)$-doubling cubes in the definition of the space $RBMO(\mu)$ by (α, β)-doubling cubes, for any choice of $\alpha > 1$ and $\beta > \alpha^n$. We have taken $(2, \beta_d)$-doubling cubes in the definition of $\| \cdot \|_*$ (and not $(2, 2^{n+1})$-doubling, say) because to prove some of the results below it will be necessary to work with doubling cubes having the properties explained in Remark 2.3.

On the other hand, by Lemmas 2.6 and 2.8, the definition of $RBMO(\mu)$ does not depend on the number $\rho > 1$ chosen in (2.4). So, throughout the rest of the paper we will assume that the constant ρ in the definition of $RBMO(\mu)$ is 2.

Also, it can be seen that we also obtain equivalent definitions for the space $RBMO(\mu)$ if instead of cubes centered at points in $\mathrm{supp}(\mu)$, we consider all the cubes in $\mathbb{R}^d$ (with sides parallel to the axes). Furthermore, it does not matter if we take balls instead of cubes.

Notice that in Lemma 2.8 we have shown that if we choose $f_Q = m_{\widetilde{Q}} f$ for all cubes Q, then (2.7) and (2.8) are satisfied with $C_4 = C\,\|f\|_*$.

Other possible ways of defining $RBMO(\mu)$ are shown in the following lemma.

Lemma 2.10. *Let $\rho > 1$ be fixed. For a function $f \in L^1_{loc}(\mu)$, the following are equivalent:*

a) $f \in RBMO(\mu)$.

b) *There exists some constant C_b such that for any cube Q*

$$\int_Q |f - m_Q f| \, d\mu \leq C_b \, \mu(\rho Q) \tag{2.12}$$

and

$$|m_Q f - m_R f| \leq C_b \, K_{Q,R} \left(\frac{\mu(\rho Q)}{\mu(Q)} + \frac{\mu(\rho R)}{\mu(R)} \right) \quad \text{for any two cubes } Q \subset R. \tag{2.13}$$

c) *There exists some constant C_c such that for any* doubling *cube Q*

$$\int_Q |f - m_Q f| \, d\mu \leq C_c \, \mu(Q) \tag{2.14}$$

and

$$|m_Q f - m_R f| \leq C_c \, K_{Q,R} \quad \text{for any two doubling cubes } Q \subset R. \tag{2.15}$$

Moreover, the best constants C_b and C_c are comparable to the $RBMO(\mu)$ norm of f.

Proof. Assume $\rho = 2$ for simplicity. First we show a) $\Rightarrow$ b). If $f \in RBMO(\mu)$, then (2.12) holds with $C_b = 2\|f\|_*$. Moreover, for any cube Q we have

$$|m_Q f - m_{\widetilde{Q}} f| \leq m_Q(|f - m_{\widetilde{Q}} f|) \leq \|f\|_* \frac{\mu(2Q)}{\mu(Q)}. \tag{2.16}$$

Therefore,

$$|m_Q f - m_R f| \le |m_Q f - m_{\widetilde{Q}} f| + |m_{\widetilde{Q}} f - m_{\widetilde{R}} f| + |m_R f - m_{\widetilde{R}} f|.$$

The second term on the right hand side is estimated as (2.10) in the preceding lemma. For the first and third terms on the right, we apply (2.16). So we get,

$$\begin{aligned}
|m_Q f - m_R f| &\le \left(C\, K_{Q,R} + \frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)} \right) \|f\|_* \\
&\le C\, K_{Q,R} \left(\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)} \right) \|f\|_*.
\end{aligned}$$

Thus f satisfies (2.13) too.

The implication b) $\Rightarrow$ c) is easier: One only has to consider doubling cubes in b).

Let us see now c)$\Rightarrow$ a). Let Q be some cube, non doubling in general. We only have to show that (2.4) holds. We know that for μ-almost all $x \in Q$ there exists some *doubling* cube centered at x with sidelength $2^{-k}\, l(Q)$, for some $k \ge 1$. We denote by Q_x the biggest cube satisfying these properties. Observe that $K_{Q_x,\widetilde{Q}} \le C$, and then

$$|m_{Q_x} f - m_{\widetilde{Q}} f| \le C\, C_c. \tag{2.17}$$

By Besicovich's covering theorem, there are points $x_i \in Q$ such that μ-almost all Q is covered by a family of cubes $\{Q_{x_i}\}_i$ with bounded overlap. By (2.17), using that $Q_{x_i} \subset 2Q$, we get

$$\begin{aligned}
\int_Q |f - m_{\widetilde{Q}} f|\, d\mu &\le \sum_i \int_{Q_{x_i}} |f - m_{\widetilde{Q}} f|\, d\mu \\
&\le \sum_i \int_{Q_{x_i}} |f - m_{Q_{x_i}} f|\, d\mu + \sum_i |m_{\widetilde{Q}} f - m_{Q_{x_i}} f|\, \mu(Q_{x_i}) \\
&\le C\, C_c\, \mu(2Q).
\end{aligned}$$

□

2.4. **Boundedness of CZO's from $L^\infty(\mu)$ into $RBMO(\mu)$.** Now we are going to see that if a CZO is bounded on $L^2(\mu)$, then it is bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. In fact, we will replace the assumption of $L^2(\mu)$ boundedness by another weaker assumption.

Theorem 2.11. *If for any cube Q and any function a supported on Q*

$$\int_Q |T_\varepsilon a|\, d\mu \le C\, \|a\|_{L^\infty}\, \mu(\rho Q) \tag{2.18}$$

uniformly on $\varepsilon > 0$, then T is bounded from $L^\infty(\mu)$ into $RBMO(\mu)$.

Let us remark that when we say that T is bounded from $L^\infty(\mu)$ into $RBMO(\mu)$, we mean that the operators T_ε, $\varepsilon > 0$, are uniformly bounded from $L^\infty(\mu)$ into $RBMO(\mu)$.

Proof. First we will show that if $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ for some $p_0 \in [1,\infty)$, then

$$\|T_\varepsilon f\|_{RBMO(\mu)} \leq C\|f\|_{L^\infty(\mu)}. \tag{2.19}$$

We will use the characterization of *RBMO*(μ) given by (2.12) and (2.13) in Lemma 2.10.

The condition (2.12) follows by standard methods. The same proof that shows that $T_\varepsilon f \in BMO(\mu)$ when μ is a doubling measure works here. We omit the details.

Let us see how (2.13) follows. For simplicity, we assume $\rho = 2$. We have to show that if $Q \subset R$, then

$$|m_Q(T_\varepsilon f) - m_R(T_\varepsilon f)| \leq C\,K_{Q,R}\left(\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)}.$$

Recall that $N_{Q,R}$ is the first integer k such that $2^kQ \supset R$. We denote $Q_R = 2^{N_{Q,R}+1}Q$. Then, for $x \in Q$ and $y \in R$, we set

$$\begin{aligned} T_\varepsilon f(x) - T_\varepsilon f(y) &= T_\varepsilon f\,\chi_{2Q}(x) + \sum_{k=1}^{N_{Q,R}} T_\varepsilon f\,\chi_{2^{k+1}Q\setminus 2^kQ}(x) + T_\varepsilon f\,\chi_{\mathbb{R}^d\setminus Q_R}(x) \\ &\quad - \Big(T_\varepsilon f\,\chi_{Q_R}(y) + T_\varepsilon f\,\chi_{\mathbb{R}^d\setminus Q_R}(y)\Big). \end{aligned}$$

Since

$$|T_\varepsilon f\,\chi_{\mathbb{R}^d\setminus Q_R}(x) - T_\varepsilon f\,\chi_{\mathbb{R}^d\setminus Q_R}(y)| \leq C\,\|f\|_{L^\infty(\mu)},$$

we get

$$\begin{aligned} |T_\varepsilon f(x) - T_\varepsilon f(y)| &\leq |T_\varepsilon f\,\chi_{2Q}(x)| + C\sum_{k=1}^{N_{Q,R}} \frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q)^n}\|f\|_{L^\infty(\mu)} \\ &\quad + |T_\varepsilon f\,\chi_{Q_R}(y)| + C\,\|f\|_{L^\infty(\mu)}. \end{aligned} \tag{2.20}$$

Now we take the mean over Q for x, and over R for y. Using the $L^2(\mu)$ boundedness of T, we obtain

$$\begin{aligned} m_Q(|T_\varepsilon f\,\chi_{2Q}|) &\leq \left(\frac{1}{\mu(Q)}\int_Q |T_\varepsilon f\,\chi_{2Q}|^2\,d\mu\right)^{1/2} \\ &\leq C\left(\frac{\mu(2Q)}{\mu(Q)}\right)^{1/2}\|f\|_{L^\infty(\mu)} \\ &\leq C\,\frac{\mu(2Q)}{\mu(Q)}\|f\|_{L^\infty(\mu)}. \end{aligned}$$

For R we write

$$m_R(|T_\varepsilon f\,\chi_{Q_R}|) \leq m_R(|T_\varepsilon f\,\chi_{Q_R\cap 2R}|) + m_R(|T_\varepsilon f\,\chi_{Q_R\setminus 2R}|).$$

The estimate for the first term on the right hand side is similar to the previous estimate for Q:

$$\begin{aligned} m_R(|T_\varepsilon f\,\chi_{Q_R\cap 2R}|) &\leq \left(\frac{1}{\mu(R)}\int_R |T_\varepsilon f\,\chi_{Q_R\cap 2R}|^2\,d\mu\right)^{1/2} \\ &\leq C\left(\frac{\mu(Q_R\cap 2R)}{\mu(R)}\right)^{1/2}\|f\|_{L^\infty(\mu)} \\ &\leq C\,\frac{\mu(2R)}{\mu(R)}\|f\|_{L^\infty(\mu)}. \end{aligned}$$

On the other hand, since $l(Q_R)\approx l(R)$, we have $m_R(|T_\varepsilon f\,\chi_{Q_R\setminus 2R}|)\leq C\,\|f\|_{L^\infty(\mu)}$. Therefore,

$$\begin{aligned} |m_Q(T_\varepsilon f)-m_R(T_\varepsilon f)| &\leq C\sum_{k=1}^{N_{Q,R}}\frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q)^n}\,\|f\|_{L^\infty(\mu)} \\ &\quad + C\left(\frac{\mu(2Q)}{\mu(Q)}+\frac{\mu(2R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)} \\ &\leq C\,K_{Q,R}\left(\frac{\mu(2Q)}{\mu(Q)}+\frac{\mu(2R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)}. \end{aligned}$$

So we have proved that (2.19) holds for $f\in L^\infty(\mu)\cap L^{p_0}(\mu)$.

If $f\notin L^p(\mu)$ for all $p\in[1,\infty)$, then the integral $\int_{|x-y|>\varepsilon}k(x,y)\,f(y)\,d\mu(y)$ may be not convergent. The operator T_ε can be extended to the whole space $L^\infty(\mu)$ following the usual arguments: Given a cube Q_0 centered at the origin with side length $>3\varepsilon$, we write $f=f_1+f_2$, with $f_1=f\,\chi_{2Q_0}$. For $x\in Q_0$, we define

$$T_\varepsilon f(x)=T_\varepsilon f_1(x)+\int(k(x,y)-k(0,y))\,f_2(y)\,d\mu(y).$$

Now both integrals in this equation are convergent and with this definition one can check that (2.19) holds too, with arguments similar to the case $f\in L^\infty(\mu)\cap L^{p_0}(\mu)$. □

Let us remark that in Theorem 8.1 we will see that the condition (2.18) holds if and only if T is bounded from $L^\infty(\mu)$ into *RBMO*(μ).

2.5. **Examples.**

Example 2.12. Assume $d=2$ and $n=1$. So we can think that we are in the complex plane and T is the Cauchy transform. Let $E\subset\mathbb{C}$ be a 1-dimensional Ahlfors-David (AD) regular set. That is,

$$C^{-1}\,r\leq\mathcal{H}^1(E\cap B(x,r))\leq C\,r\quad\text{for all } x\in E,\ 0<r\leq\operatorname{diam}(E).$$

(Here $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.) We set $\mu=\mathcal{H}^1_{|E}$. Notice that μ is a doubling measure.

For any Q centered at some point of $\operatorname{supp}(\mu)$, one has

$$\mu(2^k Q) \approx l(2^k Q)$$

if $l(2^k Q) \leq \operatorname{diam}(E)$. Then, given $Q \subset R$, it is easy to check that if $l(R) \leq \operatorname{diam}(E)$,

$$K_{Q,R} \approx 1 + \log \frac{l(R)}{l(Q)}, \tag{2.21}$$

and if $l(R) > \operatorname{diam}(E)$,

$$K_{Q,R} \approx 1 + \log \frac{\operatorname{diam}(E)}{l(Q)}. \tag{2.22}$$

So, in this case, we have $RBMO(\mu) = BMO(\mu)$, since any function $f \in BMO(\mu)$ satisfies (2.7) and (2.8), with $f_Q = m_Q f$ for all cubes Q. Notice that (2.8) holds because of (2.21) and (2.22).

Example 2.13. We assume again $d = 2$ and $n = 1$. Let μ be the planar Lebesgue measure restricted to the unit square $[0,1] \times [0,1]$. This measure is doubling, but not AD-regular (for $n = 1$). Now one can check that the coefficients $K_{Q,R}$ are uniformly bounded. That is, for any two squares $Q \subset R$,

$$K_{Q,R} \approx 1.$$

Let us take $R_0 = [0,1]^2$ and $Q \subset R_0$. Then, if $f \in RBMO(\mu)$,

$$|m_Q(f - m_{R_0} f)| = |m_Q f - m_{R_0} f| \leq K_{Q,R_0} \|f\|_* \leq C\, \|f\|_*.$$

Since this holds for any square $Q \subset R_0$, by the Lebesgue differentiation theorem $f - m_{R_0} f$ is a bounded function, with

$$\|f - m_{R_0} f\|_{L^\infty(\mu)} \leq \|f\|_*.$$

Therefore, now $RBMO(\mu)$ coincides with $L^\infty(\mu)$ modulo constants functions, which is strictly smaller than $BMO(\mu)$.

Example 2.14. This example is borrowed in part from [13]. Suppose $d = 2$ (i.e. we are in the complex plane) and $n = 1$. Let μ be a measure on the real axis such that in the intervals $[-2,-1]$ and $[1,2]$ is the linear Lebesgue measure, on the interval $[-1/2, 1/2]$ is the linear Lebesgue measure times ε, with $\varepsilon > 0$ very small, and $\mu = 0$ elsewhere. We consider the function $f = \varepsilon^{-1}\left(\chi_{[1/4,1/2]} - \chi_{[-1/2,-1/4]}\right)$. It is easily checked that for $\rho \leq 2$,

$$\|f\|_{BMO_\rho(\mu)} \approx \varepsilon^{-1},$$

while for $\rho = 5$,

$$\|f\|_{BMO_5(\mu)} \approx 1.$$

On the other hand, the $RBMO(\mu)$ norm of f is

$$\|f\|_* \approx \varepsilon^{-1},$$

since

$$C\, \|f\|_* \geq |m_{[-2,2]} f - m_{[1/4,1/2]} f| = \varepsilon^{-1}$$

and $\|f\|_* \leq C\,\|f\|_{L^\infty(\mu)} \leq C\,\varepsilon^{-1}$.

3. The inequality of John-Nirenberg

The following result is a version of John-Nirenberg's inequality suitable for the space *RBMO*(μ). To prove it we will adapt the arguments in [6, p.31-32] to the present situation.

Theorem 3.1. *Let $f \in$ RBMO(μ) and let $\{f_Q\}_Q$ be a collection of numbers satisfying*

$$\sup_Q \frac{1}{\mu(2Q)} \int_Q |f(x) - f_Q|\, d\mu(x) \leq C\,\|f\|_* \tag{3.1}$$

and

$$|f_Q - f_R| \leq C\,K_{Q,R}\,\|f\|_* \quad \text{for any two cubes } Q \subset R. \tag{3.2}$$

Then, for any cube Q and any $\lambda > 0$ we have

$$\mu\{x \in Q : |f(x) - f_Q| > \lambda\} \leq C_5\,\mu(\rho Q)\,\exp\left(\frac{-C_6\,\lambda}{\|f\|_*}\right), \tag{3.3}$$

with C_5 and C_6 depending on the constant $\rho > 1$ (but not on λ).

To prove this theorem, we will use the following straightforward result:

Lemma 3.2. *Let $f \in$ RBMO(μ) and let $\{f_Q\}_Q$ be a collection of numbers satisfying (3.1) and (3.2). If Q and R are cubes such that $l(Q) \approx l(R)$ and* $\operatorname{dist}(Q,R) \lessapprox l(Q)$, *then*

$$|f_Q - f_R| \leq C\,\|f\|_*.$$

Proof. Let R' be the smallest cube concentric with R containing Q and R. Since $l(Q) \approx l(R') \approx l(R)$, we have $K_{Q,R'} \leq C$ and $K_{R,R'} \leq C$. Then,

$$|f_Q - f_R| \leq |f_Q - f_{R'}| + |f_R - f_{R'}| \leq C\,(K_{Q,R} + K_{R,R'})\,\|f\|_* \leq C\,\|f\|_*.$$

□

We will use the following lemma too.

Lemma 3.3. *Let $f \in$ RBMO(μ). Given $q > 0$, we set*

$$f_q(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq q, \\ q\,\dfrac{f(x)}{|f(x)|} & \text{if } |f(x)| > q. \end{cases}$$

Then $f_q \in$ RBMO(μ), with $\|f_q\|_* \leq C\,\|f\|_*$.

Proof. For any function g, we set $g = g_+ - g_-$, with $g_+ = \max(g,0)$ and $g_- = -\min(g,0)$.

By Proposition 2.5, $\|f_+\|_*, \|f_-\|_* \leq C\,\|f\|_*$. Since $f_{q,+} = \min(f_+,q)$ and $f_{q,-} = \min(f_-,q)$, we have $\|f_{q,+}\|_*, \|f_{q,-}\|_* \leq C\,\|f\|_*$. Thus $\|f_q\|_* \leq \|f_{q,+}\|_* + \|f_{q,-}\|_* \leq C\,\|f\|_*$. □

Remark 3.4. Let $f \in \mathit{RBMO}(\mu)$ and let $\{f_Q\}_Q$ be a collection of numbers satisfying (3.1) and (3.2). We set $f_{Q,+} = \max(f_Q, 0)$ and $f_{Q,-} = -\min(f_Q, 0)$ and we take

$$f_{q,Q} = \min(f_{Q,+}, q) - \min(f_{Q,-}, q).$$

It is easily seen that

$$\sup_Q \frac{1}{\mu(2Q)} \int_Q |f_q(x) - f_{q,Q}|\, d\mu(x) \leq C\, \|f\|_*$$

and

$$|f_{q,Q} - f_{q,R}| \leq C\, K_{Q,R}\, \|f\|_* \quad \text{for any two cubes } Q \subset R.$$

Proof of Theorem 3.1. We will prove (3.3) for $\rho = 2$. The proof for other values of ρ is similar. Recall that if (3.1) and (3.2) are satisfied, then (3.1) is also satisfied subsituting "$\mu(2Q)$" by "$\mu(\rho Q)$", for any $\rho > 1$ (see Remark 2.7).

Let $f \in \mathit{RBMO}(\mu)$. Assume first that f is bounded. Let Q_0 be some fixed cube in $\mathbb{R}^d$. We write $Q_0' = \frac{3}{2} Q_0$.

Let B be some positive constant which will be fixed later. By Remark 2.3, for μ-almost any $x \in Q_0$ such that $|f(x) - f_{Q_0}| > B\, \|f\|_*$, there exists some doubling cube Q_x centered at x satisfying

$$m_{Q_x}(|f - f_{Q_0}|) > B\, \|f\|_*. \tag{3.4}$$

Moreover, we may assume that Q_x is the biggest doubling cube satisfying (3.4) with side length $2^{-k}\, l(Q_0)$ for some integer $k \geq 0$, with

$$l(Q_x) \leq \frac{1}{10}\, l(Q_0).$$

By Besicovich's covering theorem, there exists an almost disjoint subfamily $\{Q_i\}_i$ of the cubes $\{Q_x\}_x$ such that

$$\{x : |f(x) - f_{Q_0}| > B\, \|f\|_*\} \subset \bigcup_i Q_i. \tag{3.5}$$

Then, since $Q_i \subset Q_0'$ and $|f_{Q_0} - f_{Q_0'}| \leq C\, \|f\|_*$, we have

$$\begin{aligned}
\sum_i \mu(Q_i) &\leq \sum_i \frac{1}{B\, \|f\|_*} \int_{Q_i} |f - f_{Q_0}|\, d\mu \\
&\leq \frac{C}{B\, \|f\|_*} \int_{Q_0'} |f - f_{Q_0}|\, d\mu \\
&\leq \frac{C}{B\, \|f\|_*} |f_{Q_0} - f_{Q_0'}|\, \mu(Q_0') + \frac{C}{B\, \|f\|_*} \int_{Q_0'} |f - f_{Q_0'}|\, d\mu. \qquad (3.6)
\end{aligned}$$

Since (3.1) is satisfied if we change "$\mu(2Q)$" by "$\mu(\frac{4}{3}Q)$", we have

$$\int_{Q_0'} |f - f_{Q_0'}|\, d\mu \leq C\, \mu(2Q_0)\, \|f\|_*,$$

and, by (3.6),

$$\sum_i \mu(Q_i) \leq \frac{C\,\mu(2Q_0)}{B}.$$

So if we choose B big enough,

$$\sum_i \mu(Q_i) \leq \frac{\mu(2Q_0)}{2\beta_d}. \tag{3.7}$$

Now we want to see that for each i we have

$$|f_{Q_i} - f_{Q_0}| \leq C_7\,\|f\|_*. \tag{3.8}$$

We consider the cube $\widetilde{2Q_i}$. If $l(\widetilde{2Q_i}) > 10l(Q_0)$, then there exists some cube $2^m Q_i$, $m \geq 1$, containing Q_0 and such that $l(Q_0) \approx l(2^m Q_i) \leq l(\widetilde{2Q_i})$. Thus

$$|f_{Q_i} - f_{Q_0}| \leq |f_{Q_i} - f_{2Q_i}| + |f_{2Q_i} - f_{2^m Q_i}| + |f_{2^m Q_i} - f_{Q_0}|.$$

The first and third sums on the right hand side are bounded by $C\,\|f\|_*$ because Q_i and $2Q_i$ on the one hand and $2^m Q_i$ and Q_0 on the other hand have comparable sizes. The second sum is also bounded by $C\,\|f\|_*$ due to the fact that there are no doubling cubes of the form $2^k Q_i$ between Q_i and $2^m Q_i$, and then $K_{Q_i,2^m Q_i} \leq C$.

Assume now $\frac{1}{10}\,l(Q_0) < l(\widetilde{2Q_i}) \leq 10l(Q_0)$. Then

$$|f_{Q_i} - f_{Q_0}| \leq |f_{Q_i} - f_{\widetilde{2Q_i}}| + |f_{\widetilde{2Q_i}} - f_{Q_0}|.$$

Since $\widetilde{2Q_i}$ and Q_0 have comparable sizes, by Lemma 3.2 we have $|f_{\widetilde{2Q_i}} - f_{Q_0}| \leq C\,\|f\|_*$. And since $K_{Q_i,\widetilde{2Q_i}} \leq C(K_{Q_i,2Q_i} + K_{2Q_i,\widetilde{2Q_i}})$, we also have $|f_{Q_i} - f_{\widetilde{2Q_i}}| \leq C\,\|f\|_*$. So (3.8) holds in this case too.

If $l(\widetilde{2Q_i}) \leq \frac{1}{10}\,l(Q_0)$, then, by the choice of Q_i, we have $m_{\widetilde{2Q_i}}(|f - f_{Q_0}|) \leq B\,\|f\|_*$, which implies

$$|m_{\widetilde{2Q_i}}(f - f_{Q_0})| \leq B\,\|f\|_*. \tag{3.9}$$

Thus

$$|f_{Q_i} - f_{Q_0}| \leq |f_{Q_i} - f_{\widetilde{2Q_i}}| + |f_{\widetilde{2Q_i}} - m_{\widetilde{2Q_i}} f| + |m_{\widetilde{2Q_i}} f - f_{Q_0}|.$$

As above, the term $|f_{Q_i} - f_{\widetilde{2Q_i}}|$ is bounded by $C\,\|f\|_*$. The last one equals $|m_{\widetilde{2Q_i}}(f - f_{Q_0})|$, which is estimated in (3.9). For the second one, since $\widetilde{2Q_i}$ is doubling, we have

$$|f_{\widetilde{2Q_i}} - m_{\widetilde{2Q_i}} f| \leq \frac{1}{\mu(\widetilde{2Q_i})} \int_{\widetilde{2Q_i}} |f - f_{\widetilde{2Q_i}}|\,d\mu \leq C\,\frac{\mu(2\cdot\widetilde{2Q_i})}{\mu(\widetilde{2Q_i})}\,\|f\|_* \leq C\,\|f\|_*.$$

So (3.8) holds in any case.

Now we consider the function

$$X(t) = \sup_Q \frac{1}{\mu(2Q)} \int_Q \exp\left(|f - f_Q|\,\frac{t}{\|f\|_*}\right)\,d\mu.$$

Since we are assuming that f is bounded, $X(t) < \infty$. By (3.5) and (3.8) we have

$$\begin{aligned}
&\frac{1}{\mu(2Q_0)} \int_{Q_0} \exp\left(|f - f_{Q_0}| \frac{t}{\|f\|_*}\right) d\mu \\
&\qquad \le \frac{1}{\mu(2Q_0)} \int_{Q_0 \setminus \bigcup_i Q_i} \exp(B\,t)\, d\mu \\
&\qquad + \frac{1}{\mu(2Q_0)} \sum_i \int_{Q_i} \exp\left(|f - f_{Q_i}| \frac{t}{\|f\|_*}\right) d\mu \cdot \exp(C_7\,t) \\
&\qquad \le \exp(B\,t) + \frac{1}{\mu(2Q_0)} \sum_i \mu(2Q_i)\, X(t)\, \exp(C_7\,t).
\end{aligned}$$

By (3.7) and taking into account that $\mu(2Q_i)/\mu(Q_i) \le \beta_d$, we get

$$\frac{1}{\mu(2Q_0)} \int_{Q_0} \exp\left(|f - f_{Q_0}| \frac{t}{\|f\|_*}\right) d\mu \le \exp(B\,t) + \frac{1}{2} X(t)\, \exp(C_7\,t).$$

Thus

$$X(t)\left(1 - \frac{1}{2}\exp(C_7\,t)\right) \le \exp(B\,t).$$

Then, for t_0 small enough,

$$X(t_0) \le C_8,$$

with C_8 depending on t_0, B and C_7.

Now the theorem is almost proved for f bounded. We have

$$\begin{aligned}
&\mu\{x \in Q : |f(x) - f_Q| > \lambda\, \|f\|_*/t_0\} \\
&\qquad \le \int_Q \exp\left(\frac{t_0\,|f(x) - f_Q|}{\|f\|_*}\right) \exp(-\lambda)\, d\mu(x) \\
&\qquad \le C_8\, \mu(2Q)\, \exp(-\lambda),
\end{aligned}$$

which is equivalent to (3.3).

When f is not bounded, we consider the function f_q of Lemma 3.3. By this Lemma and the subsequent remark we know that

$$\mu\{x \in Q : |f_q(x) - f_{Q,q}| > \lambda\} \le C_5\, \mu(\rho Q)\, \exp\left(\frac{-C_6\,\lambda}{\|f\|_*}\right).$$

Since $\mu\{x \in Q : |f_q(x) - f_{Q,q}| > \lambda\} \to \mu\{x \in Q : |f(x) - f_Q| > \lambda\}$ as $q \to \infty$, (3.3) holds in this case too. □

From Theorem 3.1 we can get easily that the following spaces $RBMO^p(\mu)$ coincide for all $p \in [1, \infty)$. Given $\rho > 1$ and $p \in [1, \infty)$, $RBMO^p(\mu)$ is defined as follows. We say that $f \in L^1_{loc}(\mu)$ is in $RBMO^p(\mu)$ if there exists some constant C_9 such that for any cube Q (centered at some point of $\mathrm{supp}(\mu)$)

$$\left(\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\widetilde{Q}} f|^p\, d\mu\right)^{1/p} \le C_9 \tag{3.10}$$

and

$$(3.11)\qquad |m_Q f - m_R f| \leq C_9\, K_{Q,R} \quad \text{for any two doubling cubes } Q \subset R.$$

The minimal constant C_9 is the $RBMO^p(\mu)$ norm of f, denoted by $\|\cdot\|_{*,p}$. Arguing as for $p = 1$, one can show that another equivalent definition for $RBMO^p(\mu)$ can be given in terms of the numbers $\{f_Q\}_Q$ (as in (2.7) and (2.8)), and that the definition of the space does not depend on the constant $\rho > 1$.

We have the following corollary of John-Nirenberg inequality:

Corollary 3.5. *For $p \in [1, \infty)$, the spaces $RBMO^p(\mu)$, coincide, and the norms $\|\cdot\|_{*,p}$ are equivalent.*

Proof. The conditions (2.5) and (3.11) coincide. So we only have to compare (2.4) and (3.10).

For any $f \in L^1_{loc}(\mu)$, the inequality $\|f\|_* \leq \|f\|_{*,p}$ follows from Hölder's inequality. To obtain the converse inequality we will apply John-Nirenberg. If $f \in RBMO(\mu)$, then

$$\begin{aligned}\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\widetilde{Q}} f|^p \, d\mu &= \frac{1}{\mu(\rho Q)} \int_0^\infty p\, \lambda^{p-1}\, \mu\{x : |f(x) - m_{\widetilde{Q}} f| > \lambda\}\, d\lambda \\ &\leq C_5\, p \int_0^\infty \lambda^{p-1} \exp\left(\frac{-C_6\, \lambda}{\|f\|_*}\right) d\lambda \leq C\, \|f\|_*^p,\end{aligned}$$

and so $\|f\|_{*,p} \leq C\|f\|_*$. □

4. The space $H^{1,\infty}_{atb}(\mu)$

For a fixed $\rho > 1$, a function $b \in L^1_{loc}(\mu)$ is called an *atomic block* if

1. there exists some cube R such that $\operatorname{supp}(b) \subset R$,
2. $\displaystyle\int b\, d\mu = 0$,
3. there are functions a_j supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^\infty \lambda_j a_j$, and

$$\|a_j\|_{L^\infty(\mu)} \leq \big(\mu(\rho Q_j)\, K_{Q_j,R}\big)^{-1}.$$

Then we denote

$$|b|_{H^{1,\infty}_{atb}(\mu)} = \sum_j |\lambda_j|$$

(to be rigorous, we should think that b is not only a function, but a structure formed by the function b, the cubes R and Q_j, the functions a_j, etc.)

Then, we say that $f \in H^{1,\infty}_{atb}(\mu)$ if there are atomic blocks b_i such that

$$f = \sum_{i=1}^\infty b_i,$$

with $\sum_i |b_i|_{H^{1,\infty}_{atb}(\mu)} < \infty$. The $H^{1,\infty}_{atb}(\mu)$ norm of f is

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} = \inf \sum_i |b_i|_{H^{1,\infty}_{atb}(\mu)},$$

where the infimum is taken over all the possible decompositions of f in atomic blocks.

Observe the difference with the atomic space $H^{1,\infty}_{at,\rho}(\mu)$. The size condition on the functions a_j is similar (we should forget the coefficient $K_{Q,R}$), but the cancellation property $\int a_j \, d\mu = 0$ is substituted by something which offers more possibilities: We can gather some terms $\lambda_j a_j$ in an atomic block b, and then we must have $\int b \, d\mu = 0$.

Notice also that if we take atomic blocks b_i made up of a unique function a_i, we derive $H^{1,\infty}_{at,\rho}(\mu) \subset H^{1,\infty}_{atb}(\mu)$.

We have the following properties:

Proposition 4.1. 1. *$H^{1,\infty}_{atb}(\mu)$ is a Banach space.*

2. *$H^{1,\infty}_{atb}(\mu) \subset L^1(\mu)$, with $\|f\|_{L^1(\mu)} \leq \|f\|_{H^{1,\infty}_{atb}(\mu)}$.*
3. *The definition of $H^{1,\infty}_{atb}(\mu)$ does not depend on the constant $\rho > 1$.*

Proof. The proofs of properties 1 and 2 are similar to the usual proofs for $H^{1,\infty}_{at}(\mu)$.

Let us sketch the proof of the third property, we can follow an argument similar to the one of Lemma 2.6. Given $\rho > \eta > 1$, it is obvious that $H^{1,\infty}_{atb,\rho}(\mu) \subset H^{1,\infty}_{atb,\eta}(\mu)$ and $\|f\|_{H^{1,\infty}_{atb,\eta}(\mu)} \leq \|f\|_{H^{1,\infty}_{atb,\rho}(\mu)}$. For the converse inequality, given an atomic block $b = \sum_j \lambda_j a_j$ with $\text{supp}(a_j) \subset Q_j \subset R$, it is not difficult to see that each function a_j can be decomposed in a finite fixed number of functions $a_{j,k}$ such that $\|a_{j,k}\|_{L^\infty(\mu)} \leq \|a_j\|_{L^\infty(\mu)}$ for all k, with $\text{supp}(a_{j,k}) \subset Q_{j,k}$, where $Q_{j,k}$ are cubes such that $l(Q_{j,k}) \approx l(Q_j)$ and $\rho Q_{j,k} \subset \eta Q_j$, etc.

Then, we will have $|b|_{H^{1,\infty}_{atb,\rho}(\mu)} \leq C \, |b|_{H^{1,\infty}_{atb,\eta}(\mu)}$, which yields $\|f\|_{H^{1,\infty}_{atb,\rho}(\mu)} \leq C \, \|f\|_{H^{1,\infty}_{atb,\eta}(\mu)}$. □

Unless otherwise stated, we will assume that the constant ρ in the definition $H^{1,\infty}_{atb}(\mu)$ is equal to 2.

Now we are going to see that if a CZO is bounded on $L^2(\mu)$, then it is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$. In fact, we will replace the assumption of $L^2(\mu)$ boundedness by another weaker assumption (as in Theorem 2.11).

Theorem 4.2. *If for any cube Q and any function a supported on Q*

$$\int_Q |T_\varepsilon a| \, d\mu \leq C \, \|a\|_{L^\infty(\mu)} \, \mu(\rho Q) \tag{4.1}$$

uniformly on $\varepsilon > 0$, then T is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$.

Proof. By standard arguments, it is enough to show that

$$\|T_\varepsilon b\|_{L^1(\mu)} \leq C\,|b|_{H^{1,\infty}_{atb}(\mu)}$$

for any atomic block b with $\mathrm{supp}(b) \subset R$, $b = \sum_j \lambda_j\, a_j$, where the a_j's are functions satisfying the properties 3 and 4 of the definition of atomic block.

We write

$$\int |T_\varepsilon b|\,d\mu = \int_{\mathbb{R}^d\setminus 2R} |T_\varepsilon b|\,d\mu + \int_{2R} |T_\varepsilon b|\,d\mu. \tag{4.2}$$

To estimate the first integral on the right hand side, we take into account that $\int b\,d\mu = 0$, and by usual arguments we get

$$\int_{\mathbb{R}^d\setminus 2R} |T_\varepsilon b|\,d\mu \leq C\,\|b\|_{L^1(\mu)} \leq C\,|b|_{H^{1,\infty}_{atb}(\mu)}. \tag{4.3}$$

On the other hand, for the last integral in (4.2), we have

$$\begin{aligned}\int_{2R} |T_\varepsilon b|\,d\mu &\leq \sum_j |\lambda_j| \int_{2R} |T_\varepsilon a_j|\,d\mu \\ &= \sum_j |\lambda_j| \int_{2Q_j} |T_\varepsilon a_j|\,d\mu + \sum_j |\lambda_j| \int_{2R\setminus 2Q_j} |T_\varepsilon a_j|\,d\mu.\end{aligned}$$

By (4.1), for each j we have

$$\int_{2Q_j} |T_\varepsilon a_j|\,d\mu \leq C\,\|a_j\|_{L^\infty(\mu)}\,\mu(2\rho Q_j).$$

Also,

$$\begin{aligned}\int_{2R\setminus 2Q_j} |T_\varepsilon a_j|\,d\mu &\leq \sum_{k=1}^{N_{Q_j,R}} \int_{2^{k+1}Q_j\setminus 2^k Q_j} |T_\varepsilon a_j|\,d\mu \\ &\leq C \sum_{k=1}^{N_{Q_j,R}} \frac{\mu(2^{k+1}Q_j)}{l(2^{k+1}Q_j)^n}\,\|a_j\|_{L^1(\mu)} \\ &\leq C\,K_{Q_j,R}\,\|a_j\|_{L^\infty(\mu)}\,\mu(Q_j).\end{aligned}$$

Thus

$$\int_{2R} |T_\varepsilon a_j|\,d\mu \leq C\,K_{Q_j,R}\,\|a_j\|_{L^\infty(\mu)}\,\mu(2\rho Q_j),$$

and then, taking into account the property 3 in Proposition 4.1,

$$\int_{2R} |T_\varepsilon b|\,d\mu \leq C \sum_j |\lambda_j|\,K_{Q_j,R}\,\|a_j\|_{L^\infty(\mu)}\,\mu(2\rho Q_j) \leq C\,|b|_{H^{1,\infty}_{atb}(\mu)}. \tag{4.4}$$

By (4.3) and (4.4), we are done. □

We will see below, in Theorem 8.1, that condition (4.1) holds if and only if T is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$.

The spaces $H^{1,\infty}_{atb}(\mu)$ and $RBMO(\mu)$ are closely related. In the next section we will prove that the dual of $H^{1,\infty}_{atb}(\mu)$ is $RBMO(\mu)$. Let us see one of the inclusions (the easiest one).

Lemma 4.3.

$$RBMO(\mu) \subset H^{1,\infty}_{atb}(\mu)^*.$$

That is, for $g \in RBMO(\mu)$, the linear functional

$$L_g(f) = \int f\, g\, d\mu$$

defined over bounded functions f with compact support extends to a continuous linear functional L_g over $H^{1,\infty}_{atb}(\mu)$ with

$$\|L_g\|_{H^{1,\infty}_{atb}(\mu)^*} \leq C\, \|g\|_*.$$

Proof. Following some standard arguments (see [3, p.294-296], for example), we only need to show that if b is an atomic block and $g \in RBMO(\mu)$, then

$$\left| \int b\, g\, d\mu \right| \leq C\, |b|_{H^{1,\infty}_{atb}(\mu)}\, \|g\|_*.$$

Suppose $\operatorname{supp}(b) \subset R$, $b = \sum_j \lambda_j\, a_j$, where the a_j's are functions satisfying the properties 3 and 4 of the definition of atomic blocks. Then, using $\int b\, d\mu = 0$,

$$\left| \int b\, g\, d\mu \right| = \left| \int_R b\,(g - g_R)\, d\mu \right| \leq \sum_j |\lambda_j|\, \|a_j\|_{L^\infty(\mu)} \int_{Q_j} |g - g_R|\, d\mu. \tag{4.5}$$

Since $g \in RBMO(\mu)$, we have

$$\begin{aligned}
\int_{Q_j} |g - g_R|\, d\mu \;&\leq\; \int_{Q_j} |g - g_{Q_j}|\, d\mu + |g_R - g_{Q_j}|\, \mu(Q_j) \\
&\leq\; \|g\|_*\, \mu(2Q_j) + K_{Q_j,R}\, \|g\|_*\, \mu(Q_j) \\
&\leq\; C\, K_{Q_j,R}\, \|g\|_*\, \mu(2Q_j).
\end{aligned}$$

From (4.5) we get

$$\left| \int b\, g\, d\mu \right| \leq C \sum_j |\lambda_j|\, \|g\|_* = C\, |b|_{H^{1,\infty}_{atb}(\mu)}\, \|g\|_*.$$

□

In the following lemma we prove the converse inequality to the one stated in Lemma 4.3.

Lemma 4.4. *If $g \in RBMO(\mu)$, we have*

$$\|L_g\|_{H^{1,\infty}_{atb}(\mu)^*} \approx \|g\|_*.$$

To obtain this result we need to define another equivalent norm for $RBMO(\mu)$. First we introduce some notation. Given a cube $Q \subset \mathbb{R}^d$ and $f \in L^1_{loc}(\mu)$, let $\alpha_Q(f)$ be the constant for which $\inf_{\alpha\in\mathbb{R}} m_Q(|f-\alpha|)$ is attained. It is known that the constant $\alpha_Q(f)$ (which may be not unique) satisfies

$$\mu\{x \in Q:\ f(x) > \alpha_Q(f)\} \leq \frac{1}{2}\mu(Q)$$

and

$$\mu\{x \in Q:\ f(x) < \alpha_Q(f)\} \leq \frac{1}{2}\mu(Q)$$

(see [6, p.30], for example).

Given $f \in L^1_{loc}(\mu)$, we denote by $\|f\|_\circ$ the minimal constant C_{10} such that

$$\frac{1}{\mu(2Q)} \int_Q |f - \alpha_{\widetilde{Q}}(f)|\, d\mu \leq C_{10} \tag{4.6}$$

and, for any two doubling cubes $Q \subset R$,

$$|\alpha_Q(f) - \alpha_R(f)| \leq C_{10}\, K_{Q,R}. \tag{4.7}$$

Then,

Lemma 4.5. $\|\cdot\|_\circ$ *is a norm which equivalent with* $\|\cdot\|_*$.

To prove this result, one can argue as in Lemma 2.8 and show that the norm $\|\cdot\|_\circ$ is equivalent with the norm $\|\cdot\|_{**}$. The details are left for the reader

Proof of Lemma 4.4. We have to prove that $\|L_g\|_{H^{1,\infty}_{atb}(\mu)^*} \geq C^{-1}\,\|g\|_*$. We will show that there exists some function $f \in H^{1,\infty}_{atb}(\mu)$ such that

$$|L_g(f)| \geq C^{-1}\,\|g\|_\circ\,\|f\|_{H^{1,\infty}_{atb}(\mu)}.$$

Let $\varepsilon > 0$ be some small constant which will be fixed later. There are two possibilities:

1. There exists some doubling cube $Q \subset \mathbb{R}^d$ such that

$$\int_Q |g - \alpha_Q(g)|\, d\mu \geq \varepsilon\, \|g\|_\circ\, \mu(Q). \tag{4.8}$$

2. For any doubling cube $Q \subset \mathbb{R}^d$, (4.8) does not hold.

If case 1 holds and Q is doubling and satisfies (4.8), then we take f such that $f(x) = 1$ if $g(x) > \alpha_Q(g)$, $f(x) = -1$ if $g(x) < \alpha_Q(g)$, and $f(x) = \pm 1$ if $g(x) = \alpha_Q(g)$, so that $\int f\, d\mu = 0$ (this is possible because of (4.6) and (4.7)). Then,

$$\left|\int g\, f\, d\mu\right| = \left|\int (g - \alpha_Q(g))\, f\, d\mu\right| = \int |g - \alpha_Q(g)|\, d\mu \geq \varepsilon\, \|g\|_\circ\, \mu(Q).$$

Since f is an atomic block and Q is doubling, $\|f\|_{H^{1,\infty}_{atb}(\mu)} \leq |f|_{H^{1,\infty}_{atb}(\mu)} \leq C\,\mu(Q)$. Therefore

$$|L_g(f)| = \left|\int g\,f\,d\mu\right| \geq C^{-1}\,\varepsilon\,\|g\|_\circ\,\|f\|_{H^{1,\infty}_{atb}(\mu)}.$$

In the case 2 there are again two possibilities:

a) For any two doubling cubes $Q \subset R$.

$$|\alpha_Q(g) - \alpha_R(g)| \leq \frac{1}{2}\,K_{Q,R}\,\|g\|_\circ.$$

b) There are doubling cubes $Q \subset R$ such that

$$|\alpha_Q(g) - \alpha_R(g)| > \frac{1}{2}\,K_{Q,R}\,\|g\|_\circ.$$

Assume first that a) holds. By the definition of $\|g\|_\circ$ there exists some cube Q such that

$$\int_Q |g - \alpha_{\widetilde{Q}}(g)|\,d\mu \geq \frac{1}{2}\,\|g\|_\circ\,\mu(2Q).$$

We consider the following atomic block supported on $\widetilde{Q}$: We set $f = a_1 + a_2$, where

$$a_1 = \chi_{Q\cap\{g>\alpha_{\widetilde{Q}}(g)\}} - \chi_{Q\cap\{g\leq\alpha_{\widetilde{Q}}(g)\}},$$

and a_2 is supported on $\widetilde{Q}$, constant on this cube, and such that $\int(a_1 + a_2)\,d\mu = 0$.

Let us estimate $\|f\|_{H^{1,\infty}_{atb}(\mu)}$. We have

$$\|a_2\|_{L^\infty(\mu)}\,\mu(\widetilde{Q}) = \left|\int a_2\,d\mu\right| = \left|\int a_1\,d\mu\right| \leq \mu(Q).$$

Then, since $\widetilde{Q}$ is doubling and $K_{Q,2\widetilde{Q}} \leq C$,

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} \leq \|a_1\|_{L^\infty(\mu)}\,\mu(2Q) + C\,\|a_2\|_{L^\infty(\mu)}\,\mu(2\widetilde{Q}) \leq C\,\mu(2Q).$$

Now we have

$$\begin{aligned} L_g(f) &= \int g\,f\,d\mu = \int_{\widetilde{Q}} (g - \alpha_{\widetilde{Q}}(g))\,f\,d\mu \\ &= \int_{\widetilde{Q}} (g - \alpha_{\widetilde{Q}}(g))\,a_1\,d\mu + \int_{\widetilde{Q}} (g - \alpha_{\widetilde{Q}}(g))\,a_2\,d\mu. \end{aligned} \tag{4.9}$$

By the definition of a_1,

$$\left|\int_{\widetilde{Q}} (g - \alpha_{\widetilde{Q}}(g))\,a_1\,d\mu\right| = \int_Q |g - \alpha_{\widetilde{Q}}(g)|\,d\mu \geq \frac{1}{2}\,\|g\|_\circ\,\mu(2Q). \tag{4.10}$$

On the other hand, by the computation about $\|a_2\|_{L^\infty(\mu)}$ and since (4.8) does not hold for $\widetilde{Q}$,

$$\left|\int_{\widetilde{Q}} (g - \alpha_{\widetilde{Q}}(g))\,a_2\,d\mu\right| \leq \frac{\mu(Q)}{\mu(\widetilde{Q})}\int_{\widetilde{Q}} |g - \alpha_{\widetilde{Q}}(g)|\,d\mu \leq C\,\varepsilon\|g\|_\circ\,\mu(2Q). \tag{4.11}$$

By (4.9), (4.10) and (4.11), if ε has been chosen small enough,

$$|L_g(f)| \geq \frac{1}{4}\,\|g\|_\circ\,\mu(2Q) \geq C^{-1}\,\|g\|_\circ\,\|f\|_{H^{1,\infty}_{atb}(\mu)}.$$

Now we consider the case b). Let $Q \subset R$ be doubling cubes such that

$$|\alpha_Q(g) - \alpha_R(g)| > \frac{1}{2}\,K_{Q,R}\,\|g\|_\circ. \tag{4.12}$$

We take

$$f = \frac{1}{\mu(R)}\,\chi_R - \frac{1}{\mu(Q)}\,\chi_Q.$$

So $\int f\,d\mu = 0$, and f is an atomic block. Since Q and R are doubling, $\|f\|_{H^{1,\infty}_{atb}(\mu)} \leq C\,K_{Q,R}$. We have

$$\begin{aligned}
L_g(f) &= \int_R (g - \alpha_R(g))\,f\,d\mu\\
&= \frac{1}{\mu(R)}\int_R (g - \alpha_R(g))\,d\mu - \frac{1}{\mu(Q)}\int_Q (g - \alpha_R(g))\,d\mu\\
&= \frac{1}{\mu(R)}\int_R (g - \alpha_R(g))\,d\mu - \frac{1}{\mu(Q)}\int_Q (g - \alpha_Q(g))\,d\mu\\
&\quad + (\alpha_Q(g) - \alpha_R(g)).
\end{aligned}$$

Since we are in the case 2, the terms

$$\left|\frac{1}{\mu(R)}\int_R (g - \alpha_R(g))\,d\mu\right|, \qquad \left|\frac{1}{\mu(Q)}\int_Q (g - \alpha_Q(g))\,d\mu\right|$$

are bounded by $\varepsilon\,\|g\|_\circ$. By (4.12), if ε is chosen $\leq 1/8$, then

$$|L_g(f)| \geq \frac{1}{4}K_{Q,R}\,\|g\|_\circ \geq C^{-1}\,\|g\|_\circ\,\|f\|_{H^{1,\infty}_{atb}(\mu)}.$$

□

5. The spaces $H^{1,p}_{atb}(\mu)$ and duality

To study the duality between $H^{1,\infty}_{atb}(\mu)$ and $RBMO(\mu)$ we will follow the scheme of [6, p.34-40]. We will introduce the atomic spaces $H^{1,p}_{atb}(\mu)$, and we will prove that they coincide with $H^{1,\infty}_{atb}(\mu)$ and that the dual of $H^{1,\infty}_{atb}(\mu)$ is $RBMO(\mu)$ simultaneously.

For a fixed $\rho > 1$ and $p \in (1,\infty)$, a function $b \in L^1_{loc}(\mu)$ is called a *p-atomic block* if

1. there exists some cube R such that $\mathrm{supp}(b) \subset R$,
2. $\displaystyle\int b\,d\mu = 0$,
3. there are functions a_j supported in cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^\infty \lambda_j a_j$, and

$$\|a_j\|_{L^p(\mu)} \leq \mu(\rho Q_j)^{1/p-1}\,K^{-1}_{Q_j,R}.$$

We denote

$$|b|_{H^{1,p}_{atb}(\mu)} = \sum_j |\lambda_j|$$

(as in the case of $H^{1,\infty}_{atb}(\mu)$, to be rigorous we should think that b is not only a function, but a structure formed by the function b, the cubes R and Q_j, the functions a_j, etc.)

Then, we say that $f \in H^{1,p}_{atb}(\mu)$ if there are p-atomic blocks b_i such that

$$f = \sum_{i=1}^{\infty} b_i,$$

with $\sum_i |b_i|_{H^{1,p}_{atb}(\mu)} < \infty$. The $H^{1,p}_{atb}(\mu)$ norm of f is

$$\|f\|_{H^{1,p}_{atb}(\mu)} = \inf \sum_i |b_i|_{H^{1,p}_{atb}(\mu)},$$

where the infimum is taken over all the possible decompositions of f in p-atomic blocks.

We have properties analogous to the ones for $H^{1,\infty}_{atb}(\mu)$:

Proposition 5.1.
1. *$H^{1,p}_{atb}(\mu)$ is a Banach space.*
2. *$H^{1,p}_{atb}(\mu) \subset L^1(\mu)$, with $\|f\|_{L^1(\mu)} \leq \|f\|_{H^{1,p}_{atb}(\mu)}$.*
3. *For $1 < p_1 \leq p_2 \leq \infty$, we have the continuous inclusion $H^{1,p_2}_{atb}(\mu) \subset H^{1,p_1}_{atb}(\mu)$.*
4. *The definition of $H^{1,p}_{atb}(\mu)$ does not depend on the constant $\rho > 1$.*

The proof of these properties is similar to the proof of the properties in Proposition 4.1.

As in the case of *RBMO*(μ) and $H^{1,\infty}_{atb}(\mu)$, we will assume that the constant ρ in the definition of the $H^{1,p}_{atb}(\mu)$ is $\rho = 2$.

The proof about the duality between $H^{1,\infty}_{atb}(\mu)$ and *RBMO*(μ) and the coincidence of the spaces $H^{1,p}_{atb}(\mu)$ has been split in several lemmas. The first one is the following.

Lemma 5.2. *For $1 < p < \infty$,*

$$RBMO(\mu) \subset H^{1,p}_{atb}(\mu)^*.$$

That is, for $g \in$ RBMO(μ), the linear functional

$$L_g(f) = \int f\, g\, d\mu$$

defined over bounded functions f with compact support extends to a unique continuous linear functional L_g over $H^{1,p}_{atb}(\mu)$ with

$$\|L_g\|_{H^{1,p}_{atb}(\mu)^*} \leq C\, \|g\|_*.$$

Proof. This lemma is very similar to Lemma 4.3. We only need to show that if b is a p-atomic block and $g \in RBMO(\mu)$, then

$$\left|\int b\, g\, d\mu\right| \leq C\, |b|_{H^{1,p}_{atb}(\mu)}\, \|g\|_*.$$

Suppose $\operatorname{supp}(b) \subset R$, $b = \sum_j \lambda_j\, a_j$, where the a_j's are functions satisfying the properties 3 and 4 of the definition of p-atomic block. Since $\int b\, d\mu = 0$,

(5.1)

$$\left|\int b\, g\, d\mu\right| = \left|\int_R b\,(g - g_R)\, d\mu\right| \leq \sum_j |\lambda_j|\, \|a_j\|_{L^p(\mu)} \left(\int_{Q_j} |g - g_R|^{p'}\, d\mu\right)^{1/p'},$$

where $p' = p/(p-1)$. As $g \in RBMO(\mu) = RBMO^{p'}(\mu)$, we have

$$\begin{aligned}\left(\int_{Q_j} |g - g_R|^{p'}\, d\mu\right)^{1/p'} &\leq \left(\int_{Q_j} |g - g_{Q_j}|\, d\mu\right)^{1/p'} + |g_R - g_{Q_j}|\, \mu(Q_j)^{1/p'} \\ &\leq \|g\|_*\, \mu(2Q_j)^{1/p'} + K_{Q_j,R}\, \|g\|_*\, \mu(Q_j)^{1/p'} \\ &\leq C\, K_{Q_j,R}\, \|g\|_*\, \mu(2Q_j)^{1/p'}.\end{aligned}$$

From (5.1) we get

$$\left|\int b\, g\, d\mu\right| \leq C \sum_j |\lambda_j|\, \|g\|_* = C\, |b|_{H^{1,p}_{atb}(\mu)}\, \|g\|_*.$$

□

Lemma 5.3. *For $1 < p < \infty$,*

$$H^{1,p}_{atb}(\mu)^* \cap L^{p'}_{loc}(\mu) \subset RBMO(\mu).$$

Proof. Let $g \in L^{p'}_{loc}(\mu)$ such that such that the functional L_g belongs to $H^{1,p}_{atb}(\mu)^*$. We have to show that $g \in RBMO(\mu)$ and $\|g\|_\circ \leq C\, \|L_g\|_{H^{1,p}_{atb}(\mu)^*}$. So we will see that, for any cube Q,

$$\frac{1}{\mu(2Q)} \int_Q |g - \alpha_{\widetilde{Q}}(g)|\, d\mu \leq C\, \|L_g\|_{H^{1,p}_{atb}(\mu)^*} \tag{5.2}$$

and for any two doubling cubes $Q \subset R$,

$$|\alpha_Q(g) - \alpha_R(g)| \leq C\, \|L_g\|_{H^{1,p}_{atb}(\mu)^*}\, K_{Q,R}. \tag{5.3}$$

First we will show that (5.2) holds for any *doubling* cube $Q = \widetilde{Q}$. In this case the argument is almost the same as the one of [6, p.38-39]. We will repeat it for the sake of completeness. Without loss of generality we may assume that

$$\int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu \geq \int_{Q\cap\{g<\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu.$$

We consider an atomic block defined as follows:

$$a(x) = \begin{cases} |g(x) - \alpha_Q|^{p'-1} & \text{if } x \in Q \cap \{g > \alpha_Q(g)\}, \\ C_Q & \text{if } x \in Q \cap \{g \leq \alpha_Q(g)\}, \\ 0 & \text{if } x \notin Q, \end{cases}$$

where C_Q is a constant such that $\int a\, d\mu = 0$.

By the definition of $\alpha_Q(g)$, we have

$$\mu(Q \cap \{g > \alpha_Q(g)\}) \leq \frac{1}{2}\, \mu(Q) \leq \mu(Q \cap \{g \leq \alpha_Q(g)\}).$$

Since Q is doubling,

$$\|a\|_{H^{1,p}_{atb}(\mu)} \leq C\, \|a\|_{L^p(\mu)}\, \mu(Q)^{1-1/p} \leq C\, \mu(Q)$$
$$\times \left(\frac{1}{\mu(Q)} \int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu + \frac{1}{\mu(Q)} \int_{Q\cap\{g\leq\alpha_Q(g)\}} |C_Q|^p\, d\mu \right)^{1/p}.$$

Now we have

$$\begin{aligned}
&\frac{1}{\mu(Q)} \int_{Q\cap\{g\leq\alpha_Q(g)\}} |C_Q|^p\, d\mu \\
&\qquad \leq \frac{1}{\mu(Q\cap\{g\leq\alpha_Q(g)\})} \int_{Q\cap\{g\leq\alpha_Q(g)\}} |C_Q|^p\, d\mu \\
&\qquad = \left| \frac{1}{\mu(Q\cap\{g\leq\alpha_Q(g)\})} \int_{Q\cap\{g\leq\alpha_Q(g)\}} C_Q\, d\mu \right|^p \\
&\qquad = \left(\frac{1}{\mu(Q\cap\{g\leq\alpha_Q(g)\})} \int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'-1}\, d\mu \right)^p \\
&\qquad \leq \frac{1}{\mu(Q\cap\{g\leq\alpha_Q(g)\})} \int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu.
\end{aligned}$$

Therefore,

$$\|a\|_{H^{1,p}_{atb}(\mu)} \leq C\, \mu(Q) \left(\frac{1}{\mu(Q)} \int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu \right)^{1/p}. \tag{5.4}$$

Since $(g - \alpha_Q(g))a \geq 0$ on Q, we have

$$\begin{aligned}
\int_Q g\, a\, d\mu &= \int_Q (g - \alpha_Q(g))a\, d\mu \\
&\geq \int_{Q\cap\{g>\alpha_Q(g)\}} |g - \alpha_Q(g)|^{p'}\, d\mu \geq \frac{1}{2} \int_Q |g - \alpha_Q(g)|^{p'}\, d\mu. \qquad (5.5)
\end{aligned}$$

From (5.4) and (5.5) we get

$$\left(\frac{1}{\mu(Q)}\int_Q |g-\alpha_Q(g)|^{p'}\,d\mu\right)^{1/p'} \|a\|_{H^{1,p}_{atb}(\mu)} \leq C\int_Q |g-\alpha_Q(g)|^{p'}\,d\mu$$
$$\leq C\int_Q g\,a\,d\mu = C\,L_g(a) \leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,\|a\|_{H^{1,p}_{atb}(\mu)}.$$

So (5.2) holds in this case.

Assume now that Q is *non doubling*. We consider an atomic block $b = a_1 + a_2$, with

$$a_1 = \frac{|g-\alpha_{\widetilde{Q}}(g)|^{p'}}{g-\alpha_{\widetilde{Q}}(g)}\,\chi_{Q\cap\{g\neq\alpha_{\widetilde{Q}}(g)\}} \tag{5.6}$$

and

$$a_2 = C_{\widetilde{Q}}\,\chi_{\widetilde{Q}}, \tag{5.7}$$

where $C_{\widetilde{Q}}$ is such that $\int(a_1+a_2)\,d\mu = 0$.

Let us estimate $\|b\|_{H^{1,p}_{atb}(\mu)}$. Since $\widetilde{Q}$ is doubling and $K_{Q,\widetilde{Q}} \leq C$,

$$\|b\|_{H^{1,p}_{atb}(\mu)} \leq C\left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(2Q)^{1/p'} + C\,|C_{\widetilde{Q}}|\,\mu(\widetilde{Q}). \tag{5.8}$$

Since $\int b\,d\mu = 0$, we have

$$\begin{aligned}\mu(\widetilde{Q})\,|C_{\widetilde{Q}}| &= \left|\int a_1\,d\mu\right| \leq \int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'-1}\,d\mu\\ &\leq \left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(Q)^{1/p'}.\end{aligned} \tag{5.9}$$

Thus

$$\|b\|_{H^{1,p}_{atb}(\mu)} \leq C\left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(2Q)^{1/p'}. \tag{5.10}$$

As $\int b\,d\mu = 0$, we also have

$$\int g\,b\,d\mu = \int_{\widetilde{Q}}(g-\alpha_{\widetilde{Q}}(g))b\,d\mu = \int_Q (g-\alpha_{\widetilde{Q}}(g))a_1\,d\mu + C_{\widetilde{Q}}\int_{\widetilde{Q}}(g-\alpha_{\widetilde{Q}}(g))\,d\mu.$$

Therefore, taking into account that $\widetilde{Q}$ satisfies (5.2), and using (5.9),

$$
\begin{aligned}
(5.11)\quad \int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu &= \int_Q (g-\alpha_{\widetilde{Q}}(g))a_1\,d\mu \\
&\leq \left|\int g\,b\,d\mu\right| + |C_{\widetilde{Q}}| \int_{\widetilde{Q}} |g-\alpha_{\widetilde{Q}}(g)|\,d\mu \\
&\leq \|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,\|b\|_{H^{1,p}_{atb}(\mu)} + C\,|C_{\widetilde{Q}}|\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,\mu(\widetilde{Q}) \\
&\leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\left[\|b\|_{H^{1,p}_{atb}(\mu)} + \left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(Q)^{1/p'}\right].
\end{aligned}
$$

By (5.10) we get

$$\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu \leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(2Q)^{1/p'}.$$

That is,

$$\left(\frac{1}{\mu(2Q)}\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p'} \leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*},$$

which implies (5.2).

Finally, we have to show that (5.3) holds for doubling cubes $Q \subset R$. We consider an atomic block $b = a_1 + a_2$ similar to the one defined above. We only change $\widetilde{Q}$ by R in (5.6) and (5.7):

$$a_1 = \frac{|g-\alpha_R(g)|}{g-\alpha_R(g)}\,\chi_{Q\cap\{g\neq\alpha_R(g)\}}$$

and

$$a_2 = C_R\,\chi_R,$$

where C_R is such that $\int (a_1+a_2)\,d\mu = 0$. Arguing as in (5.8), (5.9) and (5.10) (the difference is that now Q and R are doubling, and we do *not* have $K_{Q,R} \leq C$) we will obtain

$$(5.12)\qquad \|b\|_{H^{1,p}_{atb}(\mu)} \leq C\,K_{Q,R}\left(\int_Q |g-\alpha_{\widetilde{Q}}(g)|^{p'}\,d\mu\right)^{1/p} \mu(2Q)^{1/p'}.$$

As in (5.11), we get

$$
\begin{aligned}
\int_Q |g-\alpha_R(g)|^{p'}\,d\mu &\leq \|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,\|b\|_{H^{1,p}_{atb}(\mu)} + C\,|C_R|\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,\mu(R) \\
&\leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\left[\|b\|_{H^{1,p}_{atb}(\mu)} + \left(\int_Q |g-\alpha_R(g)|^{p'}\,d\mu\right)^{1/p} \mu(Q)^{1/p'}\right].
\end{aligned}
$$

By (5.12) we have

$$\int_Q |g-\alpha_R(g)|^{p'}\,d\mu \leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,K_{Q,R}\left(\int_Q |g-\alpha_R(g)|^{p'}\,d\mu\right)^{1/p} \mu(Q)^{1/p'}.$$

Thus

$$\left(\frac{1}{\mu(Q)}\int_Q |g-\alpha_R(g)|^{p'}\,d\mu\right)^{1/p'} \leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,K_{Q,R}. \tag{5.13}$$

Since Q is doubling and satisfies (5.2), by (5.13) we get

$$\begin{aligned}|\alpha_Q(g)-\alpha_R(g)| &= \frac{1}{\mu(Q)}\int_Q |\alpha_Q(g)-\alpha_R(g)|\,d\mu \\ &\leq \frac{1}{\mu(Q)}\int_Q |g-\alpha_Q(g)|\,d\mu + \frac{1}{\mu(Q)}\int_Q |g-\alpha_R(g)|\,d\mu \\ &\leq C\,\|L_g\|_{H^{1,p}_{atb}(\mu)^*}\,K_{Q,R},\end{aligned}$$

and we are done. □

Lemma 5.4. *For $1<p<\infty$,*

$$H^{1,p}_{atb}(\mu)^* \subset L^{p'}_{loc}(\mu) \quad \textit{modulo constants.}$$

Proof. This lemma is an easy consequence of the Riesz representation theorem. The same argument as the one of [6, p.39-40] works here. □

Now we have:

Theorem 5.5. *For $1<p<\infty$, $H^{1,p}_{atb}(\mu)=H^{1,\infty}_{atb}(\mu)$. Also, $H^{1,\infty}_{atb}(\mu)^*=$ $\mathit{RBMO}(\mu)$.*

Proof. Notice that, by Lemmas 5.2, 5.3 and 5.4, $H^{1,p}_{atb}(\mu)^* = \mathit{RBMO}(\mu)$ for $1<p<\infty$.

Now we repeat the arguments in [6] again. We consider the diagram

$$\begin{array}{rcl} i: H^{1,\infty}_{atb}(\mu) & \longrightarrow & H^{1,p}_{atb}(\mu) \\ i^*: \mathit{RBMO}(\mu)=H^{1,p}_{atb}(\mu)^* & \longrightarrow & H^{1,\infty}_{atb}(\mu)^*. \end{array}$$

The map i is an inclusion and i^* is the canonical injection of $\mathit{RBMO}(\mu)$ in $H^{1,\infty}_{atb}(\mu)^*$ (with the identification $g\equiv L_g$ for $g\in \mathit{RBMO}(\mu)$). By Lemma 4.4, $i^*(\mathit{RBMO}(\mu))$ is a closed subspace of $H^{1,\infty}_{atb}(\mu)^*$. By Banach's closed range theorem (see [19, p.205]), $H^{1,\infty}_{atb}(\mu)$ is also closed in $H^{1,p}_{atb}(\mu)$. Now it is easily checked that the Hahn-Banach theorem and the fact that $H^{1,p}_{atb}(\mu)^*=$ $\mathit{RBMO}(\mu)$ imply $H^{1,\infty}_{atb}(\mu)^*=\mathit{RBMO}(\mu)$. □

Example 5.6. By the previous theorem and the fact that for an AD-regular set the space $\mathit{RBMO}(\mu)$ coincides with $BMO(\mu)$, we derive that in this case we have $H^{1,\infty}_{atb}(\mu)=H^{1,\infty}(\mu)$ too (using the same sort of uniqueness argument as above). However, this does not hold for all doubling measures μ. For instance, in the example 2.13 (μ equal to the Lebesgue measure on $[0,1]^2$, with $n=1$, $d=2$) since $\mathit{RBMO}(\mu)=L^\infty(\mu)$ modulo constants, we have $H^{1,\infty}_{atb}(\mu)=\{f\in L^1(\mu):\ \int f\,d\mu=0\}$.

6. The sharp maximal operator

The classical (centered) sharp maximal operator $M^\sharp$ is defined as

$$M^\sharp f(x) = \sup_Q \frac{1}{\mu(Q)} \int_Q |f - m_Q f| \, d\mu,$$

where the supremum is taken over the cubes Q centered at x and $f \in L^1_{loc}(\mu)$. Then, one has $f \in BMO(\mu)$ if and only if $M^\sharp f \in L^\infty(\mu)$.

Since $M^\sharp f$ is pointwise bounded above by the (centered) Hardy-Littlewood maximal operator

$$Mf(x) = \sup_Q \frac{1}{\mu(Q)} \int_Q |f| \, d\mu$$

(with the supremum again over the cubes Q centered at x), one has

$$\|M^\sharp f\|_{L^p(\mu)} \leq C \, \|f\|_{L^p(\mu)} \tag{6.1}$$

for $1 < p \leq \infty$. On the other hand, the converse inequality also holds. If $f \in L^p(\mu)$, $1 < p < \infty$, then

$$\|f\|_{L^p(\mu)} \leq C \, \|M^\sharp f\|_{L^p(\mu)} \tag{6.2}$$

(assuming $\int f \, d\mu = 0$ if $\|\mu\| < \infty$).

In [7] it is shown that the inequalities (6.1) and (6.2) are satisfied too if μ is non doubling (choosing an appropiate grid of cubes). However, the above definition of the sharp operator is not useful for our purposes because we do not have the equivalence

$$f \in RBMO(\mu) \iff M^\sharp f \in L^\infty(\mu). \tag{6.3}$$

Now we want to introduce another sharp maximal operator suitable for our space $RBMO(\mu)$ enjoying properties similar to the ones of the classical sharp operator. We define

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |f - m_{\widetilde{Q}} f| \, d\mu + \sup_{\substack{Q \subset R:\, x \in Q, \\ Q, R \text{ doubling}}} \frac{|m_Q f - m_R f|}{K_{Q,R}}. \tag{6.4}$$

Notice that the cubes that appear in these supremums may be non centered at x. It is clear that $f \in RBMO(\mu)$ if and only if $M^\sharp f \in L^\infty(\mu)$ (recall Remark 2.9).

We consider the non centered doubling maximal operator N:

$$Nf(x) = \sup_{\substack{Q \ni x, \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu.$$

Observe that $|f(x)| \leq Nf(x)$ for μ-a.e. $x \in \mathbb{R}^d$, by Remark 2.3. Moreover, the operator N is of weak type $(1,1)$ and bounded on $L^p(\mu)$, $p \in (1, \infty]$. Indeed, if Q is doubling and $x \in Q$, we can write

$$\frac{1}{\mu(Q)} \int_Q |f| \, d\mu \leq \frac{\beta_d}{\mu(2Q)} \int_Q |f| \, d\mu \leq \beta_d \, M_{(2)} f(x),$$

where, for $\rho > 1$, we denote

$$M_{(\rho)}f(x) = \sup_{Q \ni x} \frac{1}{\mu(\rho Q)} \int_Q |f| \, d\mu.$$

So $Nf(x) \leq \beta_d M_{(2)} f(x)$. The maximal operator $M_{(\rho)}$ is bounded above by the operator defined as

$$M^{(\rho)}f(x) = \sup_{\rho^{-1}Q \ni x} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu.$$

This is the version of the Hardy-Littlewood operator that one obtains taking supremums over cubes Q which may be non centered at x but such that $x \in \rho^{-1}Q$. Recall that since $0 < \rho^{-1} < 1$, one can apply Besicovich's covering theorem (see [9] or [4, p.6-7], for example) and then one gets that $M^{(\rho)}$ is of weak type $(1,1)$ and bounded on $L^p(\mu)$, $p \in (1,\infty]$. As a consequence, $M_{(\rho)}$ is also of weak type $(1,1)$ and bounded on $L^p(\mu)$, $p \in (1,\infty]$

Now we derive that $M^\sharp$ also satifies the inequality (6.1) since the first supremum in the definition of $M^\sharp f$ is bounded by $M_{(3/2)}f(x) + Nf(x)$ while the second one is bounded by $2Nf(x)$.

Remark 6.1. We have

$$M^\sharp|f|(x) \leq 5\beta_d \, M^\sharp f(x).$$

This easy to check: Assume that $x \in Q$ and Q is doubling. Then we have

$$\begin{aligned} |m_Q|f| - |m_Q f|| &= \left| \frac{1}{\mu(Q)} \int_Q (|f(x)| - |m_Q f|) \, d\mu(x) \right| \\ &\leq \frac{1}{\mu(Q)} \int_Q |f(x) - m_Q f| \, d\mu(x) \\ &\leq \beta_d \, M^\sharp f(x). \qquad (6.5) \end{aligned}$$

Therefore, if $Q \subset R$ are doubling,

$$\begin{aligned} |m_Q|f| - m_R|f|\,| &\leq |m_Q f - m_R f| + 2\beta_d \, M^\sharp f(x) \\ &\leq (K_{Q,R} + 2\,\beta_d) \, M^\sharp f(x) \leq 3\beta_d K_{Q,R} \, M^\sharp f(x). \end{aligned}$$

Thus

$$\sup_{\substack{Q \subset R:\, x \in Q, \\ Q,R \text{ doubling}}} \frac{|m_Q|f| - m_R|f|\,|}{K_{Q,R}} \leq 3\beta_d \, M^\sharp f(x).$$

For the other supremum, by (6.5) we have

$$\begin{aligned}\sup_{Q\ni x}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|f(x)| - m_{\widetilde{Q}}|f|\,\big|\,d\mu(x)\\ &\leq \sup_{Q\ni x}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|f(x)| - |m_{\widetilde{Q}}f|\,\big|\,d\mu(x) + \beta_d\,M^\sharp f(x)\\ &\leq \sup_{Q\ni x}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |f(x) - m_{\widetilde{Q}}f|\,d\mu(x) + \beta_d\,M^\sharp f(x)\\ &\leq 2\beta_d\,M^\sharp f(x).\end{aligned}$$

Finally we are going to prove that our new operator $M^\sharp$ satisfies (6.2) too. This is a consequence of the next result and Remark 2.3.

Theorem 6.2. *Let $f \in L^1_{loc}(\mu)$, with $\int f\,d\mu = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$, if $\inf(1, Nf) \in L^p(\mu)$, then we have*

$$\|Nf\|_{L^p(\mu)} \leq C\,\|M^\sharp f\|_{L^p(\mu)}. \tag{6.6}$$

Proof. We assume $\|\mu\| = \infty$. The proof for $\|\mu\| < \infty$ is similar. For some fixed $\eta < 1$ and all $\varepsilon > 0$, we will prove that there exists some $\delta > 0$ such that for any $\lambda > 0$ we have the following good λ inequality:

$$\mu\{x:\,Nf(x) > (1+\varepsilon)\lambda,\,M^\sharp f(x) \leq \delta\lambda\} \leq \eta\,\mu\{x:\,Nf(x) > \lambda\}. \tag{6.7}$$

It is well known that by this inequality one gets $\|Nf\|_{L^p(\mu)} \leq C\,\|M^\sharp f\|_{L^p(\mu)}$. if $\inf(1, Nf) \in L^p(\mu)$.

We denote $\Omega_\lambda = \{x:\,Nf(x) > \lambda\}$ and

$$E_\lambda = \{x:\,Nf(x) > (1+\varepsilon)\lambda,\,M^\sharp f(x) \leq \delta\lambda\}.$$

For the moment we assume $f \in L^p(\mu)$. For each $x \in E_\lambda$, among the doubling cubes Q that contain x and such that $m_Q|f| > (1+\varepsilon/2)\lambda$, we consider one cube Q_x which has 'almost maximal' side lenght, in the sense that if some doubling cube Q' with side lenght $\geq 2l(Q_x)$ contains x, then $m_{Q'}|f| \leq (1+\varepsilon/2)\lambda$. It is easy to check that this maximal cube Q_x exists, because $f \in L^p(\mu)$.

Let R_x be the cube centered at x with side length $3l(Q_x)$. We denote $S_x = \widetilde{R_x}$. Then, assuming δ small enough we have $m_{S_x}|f| > \lambda$, and then $S_x \subset \Omega_\lambda$. Indeed, by construction, we have $K_{Q_x,S_x} \leq C$. Then, as $Q_x \subset S_x$ are doubling cubes containing x,

$$\big|m_{Q_x}|f| - m_{S_x}|f|\,\big| \leq K_{Q_x,S_x}\,M^\sharp|f|(x) \leq C_{11}5\beta_d\delta\lambda.$$

Thus, for $\delta < C_{11}\varepsilon/10\beta_d$,

$$m_{S_x}|f| > (1+\varepsilon/2)\lambda - C_{11}5\beta_d\delta\lambda > \lambda.$$

By Besicovich's covering theorem there are n_B (depending on d) subfamilies $\mathcal{D}_k = \{S_i^k\}_i$, $k = 1,\ldots,n_B$, of cubes S_x such that they cover E_λ, they are centered at points $x_i^k \in E_\lambda$, and each subfamily $\mathcal{D}_k$ is disjoint. Therefore, at least one the subfamilies $\mathcal{D}_k$ satisfies

$$\mu\left(\bigcup_i S_i^k\right) \geq \frac{1}{n_B}\,\mu\left(\bigcup_{i,k} S_i^k\right).$$

Suppose, for example, that it is $\mathcal{D}_1$. We will prove that for each cube S_i^1,

$$\mu(S_i^1 \cap E_\lambda) \leq \mu(S_i^1)/2n_B \tag{6.8}$$

if δ is chosen small enough. From this inequality one gets

$$\mu\left(E_\lambda \cap \bigcup_i S_i^1\right) \leq \frac{1}{2n_B}\sum_i \mu(S_i^1) \leq \frac{1}{2n_B}\,\mu(\Omega_\lambda).$$

Then,

$$\begin{aligned} \mu(E_\lambda) &\leq \mu\left(\bigcup_{i,k} S_i^k \setminus \bigcup_i S_i^1\right) + \mu\left(E_\lambda \cap \bigcup_i S_i^1\right) \\ &\leq \left(1 - \frac{1}{n_B}\right)\mu\left(\bigcup_i S_i^1\right) + \frac{1}{2n_B}\,\mu(\Omega_\lambda) \\ &\leq \left(1 - \frac{1}{2n_B}\right)\mu(\Omega_\lambda). \end{aligned} \tag{6.9}$$

Let us prove (6.8). Let $y \in S_i^1 \cap E_\lambda$. If $Q \ni y$ is doubling and such that $m_Q|f| > (1+\varepsilon)\lambda$, then $l(Q) \leq l(S_i^1)/8$. Otherwise, $\widetilde{30Q} \supset S_i^1 \supset Q_{x_i^1}$, and since Q and $\widetilde{30Q}$ are doubling, we have

$$\left| m_Q|f| - m_{\widetilde{30Q}}|f| \right| \leq K_{Q,\widetilde{30Q}}\, M^\sharp |f|(y) \leq C_{12}\,\delta\lambda \leq \frac{\varepsilon}{2}\,\lambda,$$

assuming $C_{12}\delta < \varepsilon/2$, and so

$$m_{\widetilde{30Q}}|f| > (1+\varepsilon/2)\,\lambda,$$

which contradicts the choice of $Q_{x_i^1}$ because $\widetilde{30Q} \supset Q_{x_i^1}$ and $l(\widetilde{30Q}) > 2l(Q_{x_i^1})$.

So $Nf(y) > (1+\varepsilon)\lambda$, implies

$$N(\chi_{\frac54 S_i^1} f)(y) > (1+\varepsilon)\lambda.$$

On the other hand, we also have

$$m_{\widetilde{\frac54 S_i^1}}|f| \leq (1+\varepsilon/2)\,\lambda,$$

since $\widetilde{\frac54 S_i^1}$ is doubling and its side length is $> 2l(Q_{x_i^1})$. Therefore, we get

$$N(\chi_{\frac54 S_i^1}|f| - m_{\widetilde{\frac54 S_i^1}}|f|)(y) > \frac{\varepsilon}{2}\,\lambda,$$

and then, by the weak $(1,1)$ boundedness of N and the fact that S_i^1 is doubling,

$$\begin{aligned}
\mu(S_i^1 \cap E_\lambda) &\leq \mu\{y : N(\chi_{\frac{5}{4}S_i^1}(|f| - m_{\widetilde{\frac{5}{4}S_i^1}}|f|))(y) > \frac{\varepsilon}{2}\lambda\} \\
&\leq \frac{C}{\varepsilon\lambda}\int_{\frac{5}{4}S_i^1}(|f| - m_{\widetilde{\frac{5}{4}S_i^1}}|f|)\,d\mu \\
&\leq \frac{C}{\varepsilon\lambda}\,\mu(2S_i^1)\,M^\sharp|f|(x_i^1) \\
&\leq \frac{C_{13}\,\delta}{\varepsilon}\,\mu(S_i^1).
\end{aligned}$$

Thus, (6.8) follows by choosing $\delta < \varepsilon/2n_BC_{13}$, which implies (6.9), and as a consequence we obtain (6.7) and (6.6) (under the assumption $f \in L^p(\mu)$).

Suppose now that $f \notin L^p(\mu)$. We consider the functions f_q, $q \geq 1$, introduced in Lemma 3.3. Since for all functions $g, h \in L^1_{loc}(\mu)$ and all x we have $M^\sharp(g+h)(x) \leq M^\sharp g(x) + M^\sharp h(x)$ and $M^\sharp|g|(x) \leq C\,M^\sharp g(x)$, operating as in Lemma 3.3 we get $M^\sharp f_q(x) \leq C\,M^\sharp f(x)$. On the other hand, $|f_q(x)| \leq q\,\inf(1,|f|)(x) \leq q\,\inf(1,Nf)(x)$ and so $f_q \in L^p(\mu)$. Therefore,

$$\|Nf_q\|_{L^p(\mu)} \leq C\,\|M^\sharp f\|_{L^p(\mu)}.$$

Taking the limit as $q \to \infty$, (6.6) follows. □

7. Interpolation results

An immediate corollary of the properties of the sharp operator is the following result.

Theorem 7.1. *Let $1 < p < \infty$ and T be a linear operator bounded on $L^p(\mu)$ and from $L^\infty(\mu)$ into RBMO(μ). Then T extends boundedly to $L^r(\mu)$, $p < r < \infty$.*

Proof. Assume $\|\mu\| = \infty$. The operator $M^\sharp \circ T$ is sublinear and it is bounded in $L^p(\mu)$ and $L^\infty(\mu)$. By the Marcinkiewitz interpolation theorem, it is bounded on $L^r(\mu)$, $p < r < \infty$. That is,

$$\|M^\sharp Tf\|_{L^r(\mu)} \leq C\,\|f\|_{L^r(\mu)}.$$

We may assume that $f \in L^r(\mu)$ has compact support. Then $f \in L^p(\mu)$ and so $Tf \in L^p(\mu)$. Thus $Nf \in L^p(\mu)$, and so $\inf(1,Nf) \in L^r(\mu)$. By Theorem 6.2, we have

$$\|Tf\|_{L^r(\mu)} \leq C\,\|M^\sharp Tf\|_{L^r(\mu)} \leq C\,\|f\|_{L^r(\mu)}.$$

The proof for $\|\mu\| < \infty$ is similar: Given $f \in L^r(\mu)$, we write $f = \big(f - \int f\,d\mu\big) + \int f\,d\mu$. It easily seen that the same argument as for $\|\mu\| = \infty$ can be applied to the function $f - \int f\,d\mu$. On the other hand, T is bounded on $L^r(\mu)$ over constant functions. Indeed, since T is bounded from $L^\infty(\mu)$ into $RBMO(\mu)$, we get

$$\begin{aligned}\|T1\|_{L^r(\mu)} &\leq \|T1 - m_{\mathbb{R}^d}(T1)\|_{L^r(\mu)} + \|m_{\mathbb{R}^d}(T1)\|_{L^r(\mu)} \\ &\leq C\,\mu(\mathbb{R}^d)^{1/r} + \|m_{\mathbb{R}^d}(T1)\|_{L^r(\mu)}.\end{aligned}$$

We also have

$$\begin{aligned}\|m_{\mathbb{R}^d}(T1)\|_{L^r(\mu)} &= \|m_{\mathbb{R}^d}(T1)\|_{L^p(\mu)}\,\mu(\mathbb{R}^d)^{\frac{1}{r}-\frac{1}{p}} \\ &\leq \|T1\|_{L^p(\mu)}\,\mu(\mathbb{R}^d)^{\frac{1}{r}-\frac{1}{p}} \leq C\,\mu(\mathbb{R}^d)^{1/r} \equiv C\,\|1\|_{L^r(\mu)},\end{aligned}$$

and so $\|T1\|_{L^r(\mu)} \leq C\,\|1\|_{L^r(\mu)}$. □

The main theorem of this section is another interpolation result which is not as immediate as the previous one. Using this result, we will be able to prove the $T(1)$ theorem for the Cauchy integral in the next section. The statement is the following.

Theorem 7.2. *Let T be a linear operator which is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $RBMO(\mu)$. Then, T extends boundedly to $L^p(\mu)$, $1 < p < \infty$.*

The proof of this theorem will follow the scheme of [6, p.43-44]. To prove it we need a substitute for the Calderón-Zygmund decomposition of a function, suitable for non doubling measures. Nazarov, Treil and Volberg [11] showed that if a CZO is bounded on $L^2(\mu)$, then it is of weak type $(1,1)$ (with μ non doubling). They used some kind of Calderón-Zygmund decomposition to obtain this result. However, their decomposition does not work in the proof of Theorem 7.2. Mateu, Mattila, Nicolau and Orobitg [7] also used a Calderón-Zygmund type decomposition to prove an interpolation theorem between $(H^1_{at}(\mu), L^1(\mu))$ and $(L^\infty(\mu), BMO(\mu))$ with μ non doubling. Their decompositon is not suitable for our purposes either. We will use the following decomposition instead.

Lemma 7.3 (Calderón-Zygmund decomposition)**.** *For $1 \leq p < \infty$, consider $f \in L^p(\mu)$ with compact support. For any $\lambda > 0$ (with $\lambda > \beta_d \|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$), we have:*

a) *There exists a finite family of almost disjoint (i.e. with a bounded overlap) cubes $\{Q_i\}_i$ such that*

$$\frac{1}{\mu(2Q_i)}\int_{Q_i} |f|^p\,d\mu > \frac{\lambda^p}{\beta_d}, \tag{7.1}$$

$$\frac{1}{\mu(2\eta Q_i)}\int_{\eta Q_i} |f|^p\,d\mu \leq \frac{\lambda^p}{\beta_d} \quad \text{for all } \eta > 2, \tag{7.2}$$

$$|f| \leq \lambda \quad \text{a.e. } (\mu) \text{ on } \mathbb{R}^d \setminus \textstyle\bigcup_i Q_i. \tag{7.3}$$

b) *For each* i, *let* $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$ *and let* R_i *be a* $(6, 6^{n+1})$*-doubling cube concentric with* Q_i, *with* $l(R_i) > 4l(Q_i)$. *Then there exists a family of functions* φ_i *with* $\operatorname{supp}(\varphi_i) \subset R_i$ *satisfying*

$$\int \varphi_i \, d\mu = \int_{Q_i} f \, w_i \, d\mu, \tag{7.4}$$

$$\sum_i |\varphi_i| \leq B\,\lambda \tag{7.5}$$

(where B *is some constant), and if* $1 < p < \infty$,

$$\left(\int_{R_i} |\varphi_i|^p \, d\mu \right)^{1/p} \mu(R_i)^{1/p'} \leq \frac{C}{\lambda^{p-1}} \int_{Q_i} |f|^p \, d\mu. \tag{7.6}$$

c) *For* $1 < p < \infty$, *if* R_i *is the smallest* $(6, 6^{n+1})$*-doubling cube of the family* $\{6^k Q_i\}_{k \geq 1}$ *and we set* $b = \sum_i (f \, w_i - \varphi_i)$, *then*

$$\|b\|_{H^{1,p}_{atb}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|^p_{L^p(\mu)}. \tag{7.7}$$

Proof. We will assume $\|\mu\| = \infty$.

a) Taking into account Remark 2.3, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)|^p > \lambda^p$, there exists a cube Q_x satisfying

$$\frac{1}{\mu(2Q_x)} \int_{Q_x} |f|^p \, d\mu > \frac{\lambda^p}{\beta_d}$$

and such that if Q'_x is centered at x with $l(Q'_x) > 2l(Q_x)$, then

$$\frac{1}{\mu(2Q'_x)} \int_{Q'_x} |f|^p \, d\mu \leq \frac{\lambda^p}{\beta_d}.$$

Now we can apply Besicovich's covering theorem to get an almost disjoint subfamily of cubes $\{Q_i\}_i \subset \{Q_x\}_x$ satisfying (7.1), (7.2) and (7.3).

b) Assume first that the family of cubes $\{Q_i\}_i$ is finite. Then we may suppose that this family of cubes is ordered in such a way that the sizes of the cubes R_i are non decreasing (i.e. $l(R_{i+1}) \geq l(R_i)$). The functions φ_i that we will construct will be of the form $\varphi_i = \alpha_i \, \chi_{A_i}$, with $\alpha_i \in \mathbb{R}$ and $A_i \subset R_i$. We set $A_1 = R_1$ and

$$\varphi_1 = \alpha_1 \, \chi_{R_1},$$

where the constant α_1 is chosen so that $\int_{Q_1} f \, w_1 \, d\mu = \int \varphi_1 \, d\mu$.

Suppose that $\varphi_1, \ldots, \varphi_{k-1}$ have been constructed, satisfy (7.4) and $\sum_{i=1}^{k-1} |\varphi_i| \leq B\,\lambda$, where B is some constant (which will be fixed below).

Let $R_{s_1},\dots,R_{s_m}$ be the subfamily of $R_1,\dots,R_{k-1}$ such that $R_{s_j}\cap R_k\neq\varnothing$. As $l(R_{s_j})\leq l(R_k)$ (because of the non decreasing sizes of R_i), we have $R_{s_j}\subset 3R_k$. Taking into account that for $i=1,\dots,k-1$

$$\int|\varphi_i|\,d\mu\leq\int_{Q_i}|f|\,d\mu$$

by (7.4), and using that R_k is $(6,6^{n+1})$-doubling and (7.2), we get

$$\begin{aligned}
\sum_j\int|\varphi_{s_j}|\,d\mu &\leq \sum_j\int_{Q_{s_j}}|f|\,d\mu\\
&\leq C\int_{3R_k}|f|\,d\mu\\
&\leq C\left(\int_{3R_k}|f|^p\,d\mu\right)^{1/p}\mu(3R_k)^{1/p'}\\
&\leq C\lambda\mu(6R_k)^{1/p}\,\mu(3R_k)^{1/p'}\\
&\leq C_{14}\lambda\,\mu(R_k).
\end{aligned}$$

Therefore,

$$\mu\left\{\textstyle\sum_j|\varphi_{s_j}|>2C_{14}\lambda\right\}\leq\frac{\mu(R_k)}{2}.$$

So we set

$$A_k=R_k\cap\left\{\textstyle\sum_j|\varphi_{s_j}|\leq 2C_{14}\lambda\right\},$$

and then

$$\mu(A_k)\geq\mu(R_k)/2.$$

The constant α_k is chosen so that for $\varphi_k=\alpha_k\,\chi_{A_k}$ we have $\int\varphi_k\,d\mu=\int_{Q_k}f\,w_k\,d\mu$. Then we obtain

$$\begin{aligned}
|\alpha_k| &\leq \frac{1}{\mu(A_k)}\int_{Q_k}|f|\,d\mu\leq\frac{2}{\mu(R_k)}\int_{Q_k}|f|\,d\mu\\
&\leq \frac{2}{\mu(R_k)}\int_{\frac12 R_k}|f|\,d\mu\leq\left(\frac{2}{\mu(R_k)}\int_{\frac12 R_k}|f|^p\,d\mu\right)^{1/p}\leq C_{15}\lambda
\end{aligned}$$

(this calculation also applies to $k=1$). Thus,

$$|\varphi_k|+\sum_j|\varphi_{s_j}|\leq(2C_{14}+C_{15})\,\lambda.$$

If we choose $B=2C_{14}+C_{15}$, (7.5) follows.

Now it is easy to check that (7.6) also holds. Indeed we have

$$\begin{aligned}\left(\int_{R_i} |\varphi_i|^p\,d\mu\right)^{1/p} \mu(R_i)^{1/p'} &= |\alpha_i|\,\mu(A_i)^{1/p}\,\mu(R_i)^{1/p'}\\ &\leq C\,|\alpha_i|\,\mu(A_i)\\ &= C\left|\int_{Q_i} f\,w_i\,d\mu\right|\\ &\leq C\left(\int_{Q_i}|f|^p\,d\mu\right)^{1/p}\mu(Q_i)^{1/p'},\end{aligned}$$

and by (7.1),

$$\left(\int_{Q_i}|f|^p\,d\mu\right)^{1/p}\mu(2Q_i)^{1/p'} \leq \frac{C}{\lambda^{p-1}}\int_{Q_i}|f|^p\,d\mu. \tag{7.8}$$

Thus we get (7.6).

Suppose now that the collection of cubes $\{Q_i\}_i$ is not finite. For each fixed N we consider the family of cubes $\{Q_i\}_{1\leq i\leq N}$. Then, as above, we construct functions $\varphi_1^N,\ldots,\varphi_N^N$ with $\operatorname{supp}(\varphi_i^N)\subset R_i$ satisfying

$$\int \varphi_i^N\,d\mu = \int_{Q_i} f\,w_i\,d\mu,$$

$$\sum_{i=1}^N |\varphi_i^N| \leq B\,\lambda \tag{7.9}$$

and, if $1<p<\infty$,

$$\left(\int_{R_i}|\varphi_i^N|^p\,d\mu\right)^{1/p}\mu(R_i)^{1/p'} \leq \frac{C}{\lambda^{p-1}}\int_{Q_i}|f|^p\,d\mu. \tag{7.10}$$

By (7.9) and (7.10) there is a subsequence $\{\varphi_1^k\}_{k\in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mu)$ and in the weak $*$ topology of $L^p(\mu)$ to some function $\varphi_1\in L^\infty(\mu)\cap L^p(\mu)$. Now we can consider a subsequence $\{\varphi_2^k\}_{k\in I_2}$ with $I_2\subset I_1$ which is convergent also in the weak $*$ topologies of $L^\infty(\mu)$ and $L^p(\mu)$ to some function $\varphi_2\in L^\infty(\mu)\cap L^p(\mu)$. In general, for each j we consider a subsequence $\{\varphi_j^k\}_{k\in I_j}$ with $I_j\subset I_{j-1}$ that converges in the weak $*$ topologies of $L^\infty(\mu)$ and $L^p(\mu)$ to some function $\varphi_j\in L^\infty(\mu)\cap L^p(\mu)$.

We have $\operatorname{supp}(\varphi_i)\subset R_i$ and, by the weak $*$ convergence in $L^\infty(\mu)$ and $L^p(\mu)$, the functions φ_i also satisfy (7.4) and (7.6). To get (7.5), notice that for each fixed m, by the weak $*$ convergence in $L^\infty(\mu)$,

$$\sum_{i=1}^m|\varphi_i|\leq B\lambda,$$

and so (7.5) follows.

c) For each i, we consider the atomic block $b_i = f\,w_i - \varphi_i$, supported on the cube R_i. Since $K_{Q_i,R_i} \leq C$, by (7.8) and (7.6) we have

$$|b_i|_{H^{1,p}_{atb}(\mu)} \leq \frac{C}{\lambda^{p-1}} \int_{Q_i} |f|^p\,d\mu,$$

which implies (7.7).

□

Proof of Theorem 7.2. For simplicity we assume $\|\mu\| = \infty$. The proof follows the same lines as the one of [6, p.43-44].

The functions $f \in L^\infty(\mu)$ having compact support with $\int f\,d\mu = 0$ are dense in $L^p(\mu)$, $1 < p < \infty$. For such functions we will show that

$$\|M^\sharp Tf\|_{L^p(\mu)} \leq C\,\|f\|_{L^p(\mu)} \qquad 1 < p < \infty. \tag{7.11}$$

By Theorem 6.2, this implies

$$\|Tf\|_{L^p(\mu)} \leq C\,\|f\|_{L^p(\mu)}.$$

Notice that if $f \in L^\infty(\mu)$ has compact support and $\int f\,d\mu = 0$, then $f \in H^{1,\infty}_{atb}(\mu)$ and $Tf \in L^1(\mu)$. Thus $N(Tf) \in L^{1,\infty}(\mu)$, and then $\inf(1, N(Tf)) \in L^p(\mu)$. So the hypotheses of Theorem 6.2 are satisfied.

Given any function $f \in L^p(\mu)$, $1 < p < \infty$, for $\lambda > 0$ we take a family of almost disjoint cubes $\{Q_i\}_i$ as in the previous lemma, and a collection cubes $\{R_i\}_i$ as in c) in the same lemma. Then we can write

$$f = b + g = \sum_i \left(\frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}\,f - \varphi_i \right) + g.$$

By (7.3) and (7.5), we have $\|g\|_{L^\infty(\mu)} \leq C\,\lambda$, and by (7.7),

$$\|b\|_{H^{1,p}_{atb}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|^p_{L^p(\mu)}.$$

Due to the boundedness of T from $L^\infty(\mu)$ into $RBMO(\mu)$, we have

$$\|M^\sharp Tg\|_{L^\infty(\mu)} \leq C_{16}\,\lambda.$$

Therefore,

$$\{M^\sharp Tf > (C_{16} + 1)\,\lambda\} \subset \{M^\sharp Tb > \lambda\}.$$

Since $M^\sharp$ is of weak type $(1,1)$, we have

$$\mu\{M^\sharp Tb > \lambda\} \leq C\,\frac{\|Tb\|_{L^1(\mu)}}{\lambda}.$$

On the other hand, as T is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$,

$$\|Tb\|_{L^1(\mu)} \leq C\,\|b\|_{H^{1,\infty}_{atb}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|^p_{L^p(\mu)}.$$

Thus

$$\mu\{x\in\mathbb{R}^d:\,M^\sharp(Tf)>\lambda\}\leq C\,\frac{\|f\|^p_{L^p(\mu)}}{\lambda^p}.$$

So the sublinear operator $M^\sharp T$ is of weak type (p,p) for all $p\in(1,\infty)$. By the Marcinkiewicz interpolation theorem we get that $M^\sharp T$ is bounded on $L^p(\mu)$ for all $p\in(1,\infty)$. In particular, (7.11) holds for a bounded function f with compact support and $\int f\,d\mu=0$. □

8. The $T(1)$ theorem for the Cauchy integral

Before studying the particular case of the Cauchy integral operator, we will see a result that shows the close relation between CZO's and the spaces *RBMO*(μ), $H^{1,\infty}_{atb}(\mu)$.

Theorem 8.1. *Let T be a CZO and $\rho>1$ some fixed constant. The following conditions are equivalent:*

a) *For any cube Q and any function a supported on Q*

$$\int_Q|T_\varepsilon a|\,d\mu\leq C\,\|a\|_{L^\infty}\,\mu(\rho Q) \tag{8.1}$$

uniformly on $\varepsilon>0$.

b) *T is bounded from $L^\infty(\mu)$ into RBMO(μ).*

c) *T is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$.*

Proof. We have already seen a)$\Longrightarrow$b) in Theorem 2.11 and a)$\Longrightarrow$c) in Theorem 4.2.

Let us prove b)$\Longrightarrow$a). Suppose that $\rho=2$, for example. Let $a\in L^\infty(\mu)$ be a function supported on some cube Q. Suppose first $l(Q)\leq\operatorname{diam}(\operatorname{supp}(\mu))/20$ (this is always the case if $\|\mu\|=\infty$). We have

$$\int_Q|T_\varepsilon a-m_{\widetilde{Q}}(T_\varepsilon a)|\,d\mu\leq C\,\|a\|_{L^\infty(\mu)}\,\mu(2Q). \tag{8.2}$$

So it is enough to show that

$$|m_{\widetilde{Q}}(T_\varepsilon a)|\leq C\,\|a\|_{L^\infty(\mu)}. \tag{8.3}$$

Let $x_0\in\operatorname{supp}(\mu)$ be the point (or one of the points) in $\mathbb{R}^d\setminus(5Q)^\circ$ which is closest to Q. We denote $d_0=\operatorname{dist}(x_0,Q)$. We assume that x_0 is a point such that some cube with side length $2^{-k}d_0$, $k\geq2$, is doubling. Otherwise, we take y_0 in $\operatorname{supp}(\mu)\cap B(x_0,l(Q)/100)$ such that satisfies this condition, and we interchange x_0 with y_0.

We denote by R a cube concentric with Q with side length $\max(10d_0,l(\widetilde{Q}))$. So $K_{\widetilde{Q},R}\leq C$. Let Q_0 be the biggest doubling cube centered at x_0 with side length $2^{-k}\,d_0$, $k\geq2$. Then $Q_0\subset R$, with $K_{Q_0,R}\leq C$, and one can easily check that

$$|m_{Q_0}(T_\varepsilon a)-m_{\widetilde{Q}}(T_\varepsilon a)|\leq C\,\|T_\varepsilon a\|_{RBMO(\mu)}\leq C\,\|a\|_{L^\infty(\mu)}. \tag{8.4}$$

Moreover, $\mathrm{dist}(Q_0, Q) \approx d_0$ and so, for $y \in Q_0$,

$$|T_\varepsilon a(y)| \leq C\,\frac{\mu(Q)}{d_0^n}\,\|a\|_{L^\infty(\mu)} \leq C\,\|a\|_{L^\infty(\mu)},$$

because $l(Q) < d_0$. Then we get $|m_{Q_0}(T_\varepsilon a)| \leq C\,\|a\|_{L^\infty(\mu)}$, and from (8.4), we obtain (8.3).

Suppose now that $l(Q) > \mathrm{diam}(\mathrm{supp}(\mu))/20$. Since Q is centered at some point of $\mathrm{supp}(\mu)$, we may assume that $l(Q) \leq 4\,\mathrm{diam}(\mathrm{supp}(\mu))$. Then $Q \cap \mathrm{supp}(\mu)$ can be covered by a finite number of cubes Q_j centered at points of $\mathrm{supp}(\mu)$ with side length $l(Q)/200$. It is quite easy to check that the number of cubes Q_j is bounded above by some fixed constant N depending only on d. We set

$$a_j = \frac{\chi_{Q_j}}{\sum_k \chi_{Q_k}}\,a.$$

Since a) holds for the cubes $2Q_j$ (which support the functions a_j), we have

$$\begin{aligned}\int_Q |T_\varepsilon a|\,d\mu &\leq \sum_j \int_{Q\setminus 2Q_j} |T_\varepsilon a_j|\,d\mu + \sum_j \int_{2Q_j} |T_\varepsilon a_j|\,d\mu \\ &\leq \sum_j C\,\|a_j\|_{L^\infty(\mu)}\,\mu(Q) + \sum_j C\,\|a_j\|_{L^\infty(\mu)}\,\mu(4Q_j) \\ &\leq C\,N\,\|a\|_{L^\infty(\mu)}\,\mu(2Q).\end{aligned}$$

Now we are going to prove c)$\Longrightarrow$a). Let $a \in L^\infty(\mu)$ be supported on a cube Q. Assume $\rho = 2$ and suppose first $l(Q) \leq \mathrm{diam}(\mathrm{supp}(\mu))/20$. We consider the same construction as the one for b)$\Longrightarrow$a). The cubes Q, Q_0 and R are taken as above, and they satisfy $Q, Q_0 \subset R$, $K_{Q,R} \leq C$, $K_{Q_0,R} \leq C$ and $\mathrm{dist}(Q_0, Q) \geq l(Q)$. Recall also that Q_0 is doubling.

We take the atomic block (supported on R)

$$b = a + c_{Q_0}\,\chi_{Q_0},$$

where c_{Q_0} is a constant such that $\int b\,d\mu = 0$. For $y \in Q$ we have

$$\begin{aligned}|T_\varepsilon(c_{Q_0}\,\chi_{Q_0})(y)| &\leq C\,\frac{|c_{Q_0}|\,\mu(Q_0)}{\mathrm{dist}(Q, Q_0)^n} \leq C\,\frac{\|a\|_{L^1(\mu)}}{\mathrm{dist}(Q, Q_0)^n} \\ &\leq C\,\frac{\mu(Q)}{l(Q)^n}\|a\|_{L^\infty(\mu)} \leq C\,\|a\|_{L^\infty(\mu)}.\end{aligned}$$

Then we have

$$\begin{aligned}\int_Q |T_\varepsilon a|\,d\mu &\leq \int_Q |T_\varepsilon b|\,d\mu + C\,\|a\|_{L^\infty(\mu)}\,\mu(Q) \\ &\leq C\,\|b\|_{H^{1,\infty}_{atb}(\mu)} + C\,\|a\|_{L^\infty(\mu)}\,\mu(Q) \\ &\leq C\,K_{Q,R}\,\|a\|_{L^\infty(\mu)}\,\mu(2Q) + C\,K_{Q_0,R}\,|c_{Q_0}|\,\mu(2Q_0) \\ &\quad + C\,\|a\|_{L^\infty(\mu)}\,\mu(Q).\end{aligned}$$

Since Q_0 is doubling, we have

$$|c_{Q_0}|\,\mu(2Q_0) \leq C\,\|a\|_{L^1(\mu)} \leq C\,\|a\|_{L^\infty(\mu)}\,\mu(Q).$$

Therefore,

$$\int_Q |T_\varepsilon a|\,d\mu \leq C\,\|a\|_{L^\infty(\mu)}\,\mu(2Q).$$

If $l(Q) > \operatorname{diam}(\operatorname{supp}(\mu))/20$, operating as in the implication b)$\Longrightarrow$a), we get that a) also holds. □

Now we are going to deal with the $T(1)$ theorem for Cauchy integral operator. So we take $d = 2$ and $n = 1$. Using the relationship of the Cauchy kernel with the curvature of measures, it is not difficult to get the following result operating as Melnikov and Verdera [8]:

Lemma 8.2. *Let μ be some measure on $\mathbb{C}$ satisfying the growth condition (1.1). If $\|\mathcal{C}_\varepsilon \chi_Q\|_{L^2(\mu_{|Q})} \leq C\,\mu(2Q)^{1/2}$ (uniformly on $\varepsilon > 0$), then for any bounded function a with* $\operatorname{supp}(a) \subset Q$*,*

$$\int_Q |\mathcal{C}_\varepsilon a|^2\,d\mu \leq C\,\|a\|^2_{L^\infty(\mu)}\,\mu(2Q)$$

uniformly on $\varepsilon > 0$.

We omit the details of the proof (see [8]). This follows from the formula, for $a \in L^\infty(\mu)$ with $\operatorname{supp}(a) \subset Q$,

$$\begin{aligned} 2\int_Q |\mathcal{C}_\varepsilon a|^2\,d\mu &+ 4\mathrm{Re}\int_Q a\,\mathcal{C}_\varepsilon a \cdot \overline{\mathcal{C}_\varepsilon \chi_Q}\,d\mu \\ &= \iiint_{S_\varepsilon} c(x,y,z)^2\,a(y)\,a(z)\,d\mu(x)\,d\mu(y)\,d\mu(z) + O(\|a\|^2_{L^\infty(\mu)}\mu(2Q)), \end{aligned}$$

where we have denoted

$$S_\varepsilon = \{(x,y,z) \in Q^3 :\, |x-y| > \varepsilon,\, |y-z| > \varepsilon,\, |z-x| > \varepsilon\},$$

and $c(x,y,z)$ is the Menger curvature of the triple (x,y,z) (i.e. the inverse of the radius of the circumference passing through x,y,z).

Using the preceding lemma and the interpolation theorem between the pairs $(H^{1,\infty}_{atb}(\mu), L^1(\mu))$ and $(L^\infty(\mu), \mathit{RBMO}(\mu))$, we get the following version of the $T1$ theorem for the Cauchy transform for non doubling measures.

Theorem 8.3. *The Cauchy integral operator is bounded on $L^2(\mu)$ if and only if*

$$\int_Q |\mathcal{C}_\varepsilon \chi_Q|^2\,d\mu \leq C\,\mu(2Q) \quad \textit{for any square } Q \subset \mathbb{C} \tag{8.5}$$

uniformly on $\varepsilon > 0$.

This result is already known. The first proofs were obtained independently in [10] and [15]. Another was given later in [18]. The proof of the present paper follows the lines of the proof found by Melnikov and Verdera for the L^2 boundedness of the Cauchy integral in the (doubling) case where μ is the arc length on a Lipschitz graph [8].

Let us remark that in the previous known proofs of the $T(1)$ theorem for the Cauchy integral, instead of the hypothesis (8.5), the assumption was

$$\int_Q |\mathcal{C}_\varepsilon \chi_Q|^2 \, d\mu \leq C \, \mu(Q) \quad \text{for any square } Q \subset \mathbb{C}.$$

This is a little stronger than (8.5). However, the arguments given in [10], [15] and [18] can be modified easily to yield the same result as the one stated in Theorem 8.3.

Using the relationship between the spaces $BMO_\rho(\mu)$ and $RBMO(\mu)$ we obtain another version of the $T(1)$ theorem, which is closer to the classical way of stating the $T(1)$ theorem:

Theorem 8.4. *The Cauchy integral operator is bounded on $L^2(\mu)$ if and only if $\mathcal{C}_\varepsilon(1) \in BMO_\rho(\mu)$ (uniformly on $\varepsilon > 0$), for some $\rho > 1$.*

Proof. Suppose that $\mathcal{C}_\varepsilon 1 \in BMO_\rho(\mu)$. Let us see that this implies $\mathcal{C}_\varepsilon 1 \in RBMO(\mu)$. The estimates are similar to the ones that we used to show that CZO's bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. Assume, for example $\rho = 2$. We have to show that if $Q \subset R$, then

$$|m_Q(\mathcal{C}_\varepsilon 1) - m_R(\mathcal{C}_\varepsilon 1)| \leq C \, K_{Q,R} \left(\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)} \right).$$

We denote $Q_R = 2^{N_{Q,R}+1} Q$. Then we write

$$\begin{aligned}
&|m_Q(\mathcal{C}_\varepsilon 1) - m_R(\mathcal{C}_\varepsilon 1)| \\
&\qquad \leq |m_Q(\mathcal{C}_\varepsilon \chi_Q)| + |m_Q(\mathcal{C}_\varepsilon \chi_{2Q \setminus Q})| + |m_Q(\mathcal{C}_\varepsilon \chi_{Q_R \setminus 2Q})| \\
&\qquad\quad + |m_Q(\mathcal{C}_\varepsilon \chi_{\mathbb{C} \setminus Q_R}) - m_R(\mathcal{C}_\varepsilon \chi_{\mathbb{C} \setminus Q_R})| \\
&\qquad\quad + |m_R(\mathcal{C}_\varepsilon \chi_R)| + |m_R(\mathcal{C}_\varepsilon \chi_{Q_R \setminus 2R})| + |m_R(\mathcal{C}_\varepsilon \chi_{Q_R \cap 2R \setminus R})| \\
&\qquad = M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7.
\end{aligned}$$

Since $\mathcal{C}_\varepsilon$ is antisymmetric, we have $M_1 = M_5 = 0$. On the other hand, since the Cauchy transform is bounded from $L^2(\mu_{|\mathbb{C} \setminus Q})$ into $L^2(\mu_{|Q})$, we also have

$$\begin{aligned}
M_2 &= |m_Q(\mathcal{C}_\varepsilon \chi_{2Q \setminus Q})| \leq \left(\frac{1}{\mu(Q)} \int_Q |\mathcal{C}_\varepsilon \chi_{2Q \setminus Q})|^2 \, d\mu \right)^{1/2} \\
&\leq C \left(\frac{\mu(2Q)}{\mu(Q)} \right)^{1/2} \leq C \, \frac{\mu(2Q)}{\mu(Q)}.
\end{aligned}$$

By the same argument, we get

$$M_7 = |m_R(\mathcal{C}_\varepsilon \chi_{Q_R \cap 2R \setminus R})| \leq C \, \frac{\mu(Q_R \cap 2R)}{\mu(R)} \leq C \, \frac{\mu(2R)}{\mu(R)}.$$

Also, it is easily seen that $|\mathcal{C}_\varepsilon \chi_{Q_R\setminus 2Q}(x)| \leq C\, K_{Q,R}$ for $x \in Q$, and so

$$M_3 = |m_Q(\mathcal{C}_\varepsilon \chi_{Q_R\setminus 2Q})| \leq C\, K_{Q,R}.$$

On the other hand, if $x \in Q$ and $y \in R$, we have

$$|\mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus Q_R}(x) - \mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus Q_R}(y)| \leq C,$$

and so $M_4 \leq C$. Finally, since $l(Q_R) \approx l(R)$, $|\mathcal{C}_\varepsilon \chi_{Q_R\setminus 2R}(x)| \leq C$ for $x \in R$, and thus $M_6 \leq C$.

Therefore,

$$\begin{aligned} |m_Q(\mathcal{C}_\varepsilon 1) - m_R(\mathcal{C}_\varepsilon 1)| &\leq C\, K_{Q,R} + C\left(\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)}\right) \\ &\leq C\, K_{Q,R}\left(\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)}\right). \end{aligned}$$

So $\mathcal{C}_\varepsilon 1 \in \mathit{RBMO}(\mu)$, and thus we also have $\mathcal{C}_\varepsilon 1 \in \mathit{BMO}_\rho^2(\mu)$, for any $\rho > 1$. Now, some standard calculations show that the condition (8.5) of Theorem 8.3 is satisfied:

$$\begin{aligned} \left(\int_Q |\mathcal{C}_\varepsilon \chi_Q|^2\, d\mu\right)^{1/2} &= \left(\int_Q |\mathcal{C}_\varepsilon \chi_Q - m_Q(\mathcal{C}_\varepsilon \chi_Q)|^2\, d\mu\right)^{1/2} \\ &\leq \left(\int_Q |\mathcal{C}_\varepsilon 1 - m_Q(\mathcal{C}_\varepsilon 1)|^2\, d\mu\right)^{1/2} \\ &\quad + \left(\int_Q |\mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus 2Q} - m_Q(\mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus 2Q})|^2\, d\mu\right)^{1/2} \\ &\quad + \left(\int_Q |\mathcal{C}_\varepsilon \chi_{2Q\setminus Q} - m_Q(\mathcal{C}_\varepsilon \chi_{2Q\setminus Q})|^2\, d\mu\right)^{1/2}. \end{aligned}$$

Since $\mathcal{C}_\varepsilon 1 \in \mathit{BMO}_2^2(\mu)$, we have $\int_Q |\mathcal{C}_\varepsilon 1 - m_Q(\mathcal{C}_\varepsilon 1)|^2\, d\mu \leq C\, \mu(2Q)$. Also, as usual, we have $\int_Q |\mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus 2Q} - m_Q(\mathcal{C}_\varepsilon \chi_{\mathbb{C}\setminus 2Q})|^2\, d\mu \leq C\ \mu(Q)$. Finally, the last integral can be estimated using the boundedness of the Cauchy transform from $L^2(\mu_{|\mathbb{C}\setminus Q})$ into $L^2(\mu_{|Q})$. Thus (8.5) holds. $\square$

Let us remark that, until now, the $T1$ theorem for the Cauchy integral was known under the assumption $\mathcal{C}_\varepsilon 1 \in \mathit{BMO}_\rho^2(\mu)$, but not under the weaker assumption $\mathcal{C}_\varepsilon 1 \in \mathit{BMO}_\rho(\mu)$.

Also, for general CZO's, the assumption $T_\varepsilon 1, T_\varepsilon^* 1 \in \mathit{BMO}_\rho^2(\mu)$ in the $T1$ theorem for non doubling measures of Nazarov, Treil and Volberg can be substituted by the weaker one $T_\varepsilon 1, T_\varepsilon^* 1 \in \mathit{BMO}_\rho(\mu)$. This is due to the fact that if T_ε is weakly bounded in the sense of [13] and $T_\varepsilon 1, T_\varepsilon^* 1 \in \mathit{BMO}_\rho(\mu)$ for some $\rho > 1$, then arguing as in the proof of Theorem 8.4 it follows that $T_\varepsilon 1, T_\varepsilon^* 1 \in \mathit{RBMO}(\mu)$, and so $T_\varepsilon 1, T_\varepsilon^* 1 \in \mathit{BMO}_\rho^2(\mu)$, for any $\rho > 1$.

9. Commutators

In this section we will prove that if $b \in RBMO(\mu)$ and T is a CZO bounded on $L^2(\mu)$, then the commutator $[b,T]$ defined by

$$[b,T](f) = b\,T(f) - T(bf)$$

is bounded on $L^p(\mu)$, $1 < p < \infty$. In this formula, T stands for a weak limit as $\varepsilon \to 0$ of some subsequence of the uniformly bounded operators T_ε.

The $L^p(\mu)$ boundedness of the commutator $[b,T]$ is a result due to Coifman, Rochberg and Weiss [1] in the classical case where μ is the Lebesgue measure on $\mathbb{R}^d$. Their proof, with some minor changes, works also for doubling measures. On the other hand, for μ being the Lebesgue measure, they showed that if R_i, $i = 1, \cdots, d$, are the Riesz transforms on $\mathbb{R}^d$, then the $L^p(\mu)$ boundedness of the commutators $[b, R_i]$, $i = 1, \cdots, d$, for some $p \in (1,\infty)$ implies $b \in BMO(\mu)$.

When μ is a non doubling measure and b satisfies (1.2), i.e. it belongs to the classical space $BMO(\mu)$, then it has been shown by Orobitg and Pérez [14] that the commutator $[b,T]$ is bounded on $L^p(\mu)$, $1 < p < \infty$.

Let us state now the result that we will obtain in this section in detail.

Theorem 9.1. *If T is a CZO bounded on $L^2(\mu)$ and $b \in RBMO(\mu)$, then the commutator $[b,T]$ is bounded on $L^p(\mu)$.*

Our proof will be based on the use of the sharp maximal operator, as the one of Janson and Strömberg [5] for the doubling case. However, the result can be obtained also by means of a good λ inequality, as in [1].

To prove Theorem 9.1 we will need a couple of lemmas dealing with the coefficients $K_{Q,R}$.

Lemma 9.2. *There exists some constant P (big enough) depending on C_0 and n such that if $Q_1 \subset Q_2 \subset \cdots \subset Q_m$ are concentric cubes with $K_{Q_i,Q_{i+1}} > P$ for $i = 1, \ldots, m-1$, then*

$$\sum_{i=1}^{m-1} K_{Q_i,Q_{i+1}} \leq C_{17}\, K_{Q_1,Q_m}, \tag{9.1}$$

where C_{17} depends only on C_0 and n.

Proof. Let Q_i' be a cube concentric with Q_i such that $l(Q_i) \leq l(Q_i') < 2l(Q_i)$, with $l(Q_i') = 2^k l(Q_1)$ for some $k \geq 0$. Then

$$C_{18}^{-1}\, K_{Q_i,Q_{i+1}} \leq K_{Q_i',Q_{i+1}'} \leq C_{18}\, K_{Q_i,Q_{i+1}},$$

for all i, with C_{18} depending on C_0 and n.

Observe also that if we take P so that $C_{18}^{-1} P \geq 2$, then $K_{Q_i',Q_{i+1}'} > 2$ and so

$$K_{Q_i',Q_{i+1}'} \leq 2 \sum_{k=1}^{N_{Q_i',Q_{i+1}'}} \frac{\mu(2^k Q_i')}{l(2^k Q_i')^n}.$$

Therefore,

$$\sum_i K_{Q'_i,Q'_{i+1}} \leq 2\sum_i \sum_{k=1}^{N_{Q'_i,Q'_{i+1}}} \frac{\mu(2^k Q'_i)}{l(2^k Q'_i)^n}. \tag{9.2}$$

On the other hand, if P is big enough, then $Q'_i \neq Q'_{i+1}$. Indeed,

$$C_0\, N_{Q_i,Q_{i+1}} \geq \sum_{k=1}^{N_{Q_i,Q_{i+1}}} \frac{\mu(2^k Q_i)}{l(2^k Q_i)^n} \geq P-1,$$

and so $N_{Q_i,Q_{i+1}} \geq (P-1)/C_0 > 2$, assuming P big enough. This implies $l(Q_{i+1}) > 2l(Q_i)$, and then, by construction, $Q'_i \neq Q'_{i+1}$.

As a consequence, on the right hand side of (9.2), there is no overlapping in the terms $\frac{\mu(2^k Q'_i)}{l(2^k Q'_i)^n}$, and then

$$\sum_i K_{Q'_i,Q'_{i+1}} \leq 2K_{Q_1,Q'_m} \leq 2C_{18}\, K_{Q_1,Q_m},$$

and (9.1) follows. □

Lemma 9.3. *There exists some constant P_0 (big enough) depending on C_0, n and β_d such that if $x \in \mathbb{R}^d$ is some fixed point and $\{f_Q\}_{Q\ni x}$ is a collection of numbers such that $|f_Q - f_R| \leq C_x$ for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \leq P_0$, then*

$$|f_Q - f_R| \leq C\, K_{Q,R}\, C_x \quad \text{for all doubling cubes } Q \subset R \text{ with } x \in Q,$$

where C depends on C_0, n, P_0 and β_d.

Proof. Let $Q \subset R$ be two doubling cubes in $\mathbb{R}^d$, with $x \in Q =: Q_0$. Let Q_1 be the first cube of the form $2^k Q$, $k \geq 0$, such that $K_{Q,Q_1} > P$. Since $K_{Q,2^{-1}Q_1} \leq P$, we have $K_{Q,Q_1} \leq P + C_0$. Therefore, for the doubling cube $\widetilde{Q}_1$, we have $K_{Q,\widetilde{Q}_1} \leq C_{19}$, with C_{19} depending on P, n, β_d and C_0.

In general, given $\widetilde{Q}_i$, we denote by Q_{i+1} the first cube of the form $2^k\widetilde{Q}_i$, $k \geq 0$, such that $K_{\widetilde{Q}_i,Q_{i+1}} > P$, and we consider the cube $\widetilde{Q}_{i+1}$. Then, we have $K_{\widetilde{Q}_i,\widetilde{Q}_{i+1}} \leq C_{19}$, and also $K_{\widetilde{Q}_i,\widetilde{Q}_{i+1}} > K_{\widetilde{Q}_i,Q_{i+1}} > P$.

Then we obtain

$$|f_Q - f_R| \leq \sum_{i=1}^{N} |f_{\widetilde{Q}_{i-1}} - f_{\widetilde{Q}_i}| + |f_{\widetilde{Q}_N} - f_R|, \tag{9.3}$$

where $\widetilde{Q}_N$ is the first cube of the sequence $\{\widetilde{Q}_i\}_i$ such that $\widetilde{Q}_{N+1} \supset R$. Since $K_{\widetilde{Q}_N,\widetilde{Q}_{N+1}} \leq C_{19}$, we also have $K_{\widetilde{Q}_N,R} \leq C_{19}$. By (9.3) and Lemma 9.2, if we set $P_0 = C_{19}$, we get

$$\begin{aligned} |f_Q - f_R| &\leq \sum_{i=1}^{N} K_{\widetilde{Q}_i,\widetilde{Q}_{i+1}}\, C_x + K_{\widetilde{Q}_N,R}\, C_x \\ &\leq C\, K_{Q,\widetilde{Q}_N}\, C_x + K_{\widetilde{Q}_N,R}\, C_x \leq C\, K_{Q,R}\, C_x. \end{aligned}$$

□

Remark 9.4. By the preceding lemma, to see if some function f belongs to $RBMO(\mu)$, the regularity condition (2.5) only needs to be checked for doubling cubes $Q \subset R$ such that $K_{Q,R} \leq P_0$. In a similar way, it can be proved that if the regularity condition (2.8) holds for any pair of cubes $Q \subset R$ with $K_{Q,R}$ not too large, then it holds for any pair of cubes $Q \subset R$.

On the other hand, one can introduce an operator $\widehat{M^\sharp}$ defined as $M^\sharp$, but with the second supremum in the definition (6.4) taken only over doubling cubes $Q \subset R$ such that $x \in Q$ and $K_{Q,R} \leq P_0$. Then, by the preceding lemma it follows that $\widehat{M^\sharp}(f) \approx M^\sharp(f)$.

Proof of Theorem 9.1. For all $p \in (1,\infty)$, we will show the pointwise inequality

$$M^\sharp([b,T]f)(x) \leq C_p \,\|b\|_* \,(M_{p,(9/8)}f(x) + M_{p,(3/2)}Tf(x) + T_*f(x)), \tag{9.4}$$

where, for $\eta > 1$, $M_{p,(\eta)}$ is the non centered maximal operator

$$M_{p,(\eta)}f(x) = \sup_{Q \ni x} \left(\frac{1}{\mu(\eta Q)} \int_Q |f|^p \, d\mu \right)^{1/p},$$

and T_* is defined as

$$T_*f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.$$

The operator $M_{p,(\eta)}$ is bounded on $L^r(\mu)$ for $r > p$, and T_* is bounded on $L^r(\mu)$ for $1 < r < \infty$ because T is bounded on $L^2(\mu)$ (see [11]). Then the pointwise inequality (9.4) for $1 < p < \infty$ implies the $L^p(\mu)$ boundedness of $M^\sharp([b,T])$ for $1 < p < \infty$. If b is a bounded function we can apply Theorem 6.2 because, by the $L^p(\mu)$ boundedness of T, it follows that $[b,T] \in L^p(\mu)$. On the other hand, by Lemma 3.3 it is easily seen that we can assume that b is a bounded function. So the inequality (9.4) implies that $[b,T]$ is bounded on $L^p(\mu)$, $1 < p < \infty$.

Let $\{b_Q\}_Q$ a family of numbers satisfying

$$\int_Q |b - b_Q| \, d\mu \leq 2\mu(2Q) \, \|b\|_{**}$$

for any cube Q, and

$$|b_Q - b_R| \leq 2K_{Q,R} \, \|b\|_{**}$$

for all cubes $Q \subset R$. For any cube Q, we denote

$$h_Q := m_Q(T((b - b_Q)\, f \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q})$$

We will show that

$$\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |[b,T]f - h_Q| \, d\mu \leq C \, \|b\|_* \,(M_{p,(9/8)}f(x) + M_{p,(3/2)}Tf(x)) \tag{9.5}$$

for all x and Q with $x \in Q$, and

$$|h_Q - h_R| \leq C\,\|b\|_*\,(M_{p,(9/8)}f(x) + T_*f(x))\,K_{Q,R}^2 \tag{9.6}$$

for *all* cubes $Q \subset R$ with $x \in Q$. In the final part of the proof we will see that from the preceding two inequalities one easily gets (9.4).

To get (9.5) for some fixed cube Q and x with $x \in Q$, we write $[b,T]f$ in the following way:

$$\begin{aligned} [b,T]f &= (b-b_Q)\,Tf - T((b-b_Q)\,f) \\ &= (b-b_Q)\,Tf - T((b-b_Q)\,f_1) - T((b-b_Q)\,f_2), \end{aligned} \tag{9.7}$$

where $f_1 = f\,\chi_{\frac{4}{3}Q}$ and $f_2 = f - f_1$. Let us estimate the term $(b-b_Q)\,Tf$:

$$\begin{aligned} \frac{1}{\mu(\frac{3}{2}Q)}\int_Q |(b-b_Q)\,Tf|\,d\mu &\leq \left(\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |(b-b_Q)|^{p'}\,d\mu\right)^{1/p'} \\ &\quad\times\left(\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |Tf|^p\,d\mu\right)^{1/p} \\ &\leq C\,\|b\|_*\,M_{p,(3/2)}Tf(x). \end{aligned} \tag{9.8}$$

Now we are going to estimate the second term on the right hand side of (9.7). We take $s = \sqrt{p}$. Then we have

$$\begin{aligned} \left[\frac{1}{\mu(\frac{3}{2}Q)}\int_{\frac{4}{3}Q} |(b-b_Q)\,f_1|^s\,d\mu\right]^{1/s} &\leq \left(\frac{1}{\mu(\frac{3}{2}Q)}\int_{\frac{4}{3}Q} |b-b_Q|^{ss'}\,d\mu\right)^{1/ss'} \\ &\quad\times\left(\frac{1}{\mu(\frac{3}{2}Q)}\int_{\frac{4}{3}Q} |f|^p\,d\mu\right)^{1/p} \\ &\leq C\,\|b\|_*\,M_{p,(9/8)}f(x). \end{aligned}$$

Notice that we have used that $\int_{\frac{4}{3}Q}|b-b_Q|^{ss'}\,d\mu \leq C\,\|b\|_*^{ss'}\,\mu(\frac{3}{2}Q)$, which holds because $|b_Q - b_{\frac{4}{3}Q}| \leq C\,\|b\|_*$. Then we get

$$\begin{aligned} \frac{1}{\mu(\frac{3}{2}Q)}\int_Q |T((b-b_Q)\,f_1)|\,d\mu &\leq \frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)}\,\|T((b-b_Q)\,f_1)\|_{L^s(\mu)} \\ &\leq C\,\frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)}\,\|(b-b_Q)\,f_1\|_{L^s(\mu)} \\ &\leq C\,\frac{\mu(Q)^{1-1/s}}{\mu(\frac{3}{2}Q)^{1-1/s}}\,\|b\|_*\,M_{p,(9/8)}f(x) \\ &\leq C\,\|b\|_*\,M_{p,(9/8)}f(x). \end{aligned} \tag{9.9}$$

By (9.7), (9.8) and (9.9), to prove (9.5) we only have to estimate the difference $|T((b-b_Q)\,f_2)-h_Q|$. For $x,y\in Q$ we have

$$
\begin{aligned}
&(9.10)\quad |T((b-b_Q)\,f_2)(x)-T((b-b_Q)\,f_2)(y)|\\
&\leq C\int_{\mathbb{R}^d\setminus\frac43 Q}\frac{|y-x|^\delta}{|z-x|^{n+\delta}}\,|b(z)-b_Q|\,|f(z)|\,d\mu(z)\\
&\leq C\sum_{k=1}^\infty\int_{2^k\frac43 Q\setminus 2^{k-1}\frac43 Q}\frac{l(Q)^\delta}{|z-x|^{n+\delta}}\,(|b(z)-b_{2^k\frac43 Q}|+|b_Q-b_{2^k\frac43 Q}|)\,|f(z)|\,d\mu(z)\\
&\leq C\sum_{k=1}^\infty 2^{-k\delta}\frac{1}{l(2^kQ)^n}\int_{2^k\frac43 Q}|b(z)-b_{2^k\frac43 Q}|\,|f(z)|\,d\mu(z)\\
&+C\sum_{k=1}^\infty k2^{-k\delta}\,\|b\|_*\,\frac{1}{l(2^kQ)^n}\int_{2^k\frac43 Q}|f(z)|\,d\mu(z)\\
&\leq C\sum_{k=1}^\infty 2^{-k\delta}\,\|b\|_*\,M_{p,(9/8)}f(x)+C\sum_{k=1}^\infty k2^{-k\delta}\,\|b\|_*\,M_{(9/8)}f(x)\\
&\leq C\,\|b\|_*\,M_{p,(9/8)}f(x).
\end{aligned}
$$

Taking the mean over $y\in Q$, we get

$$
\begin{aligned}
|T((b-b_Q)\,f_2)(x)-h_Q| &= |T((b-b_Q)\,f_2)(x)-m_Q(T((b-b_Q)\,f_2))|\\
&\leq C\,\|b\|_*\,M_{p,(9/8)}f(x),
\end{aligned}
$$

and so (9.5) holds.

Now we have to check the regularity condition (9.6) for the numbers $\{h_Q\}_Q$. Consider two cubes $Q\subset R$ with $x\in Q$. We denote $N=N_{Q,R}+1$. We write the difference $|h_Q-h_R|$ in the following way:

$$
\begin{aligned}
&|m_Q(T((b-b_Q)\,f\chi_{\frac43 Q}))-m_R(T((b-b_R)\,f\chi_{\frac43 R}))|\\
&\qquad\leq |m_Q(T((b-b_Q)\,f\chi_{2Q\setminus\frac43 Q}))|\\
&\qquad\quad+|m_Q(T((b_Q-b_R)\,f\,\chi_{\mathbb{R}^d\setminus 2Q}))|\\
&\qquad\quad+|m_Q(T((b-b_R)\,f\,\chi_{2^NQ\setminus 2Q}))|\\
&\qquad\quad+|m_Q(T((b-b_R)\,f\,\chi_{\mathbb{R}^d\setminus 2^NQ}))-m_R(T((b-b_R)\,f\,\chi_{\mathbb{R}^d\setminus 2^NQ}))|\\
&\qquad\quad+|m_R(T((b-b_R)\,f\,\chi_{2^NQ\setminus 2R}))|\\
&\qquad= M_1+M_2+M_3+M_4+M_5.
\end{aligned}
$$

Let us estimate M_1. For $y \in Q$ we have

$$\begin{aligned}
|T((b-b_Q)\, f\, \chi_{2Q\setminus\frac{4}{3}Q})(y)| &\leq \frac{C}{l(Q)^n}\int_{2Q}|b-b_Q|\,|f|\,d\mu \\
&\leq \frac{C}{l(Q)^n}\left(\int_{2Q}|b-b_Q|^{p'}\,d\mu\right)^{1/p'}\left(\int_{2Q}|f|^p\,d\mu\right)^{1/p} \\
&\leq \frac{C\,\|b\|_*}{l(Q)^{n/p}}\left(\int_{2Q}|f|^p\,d\mu\right)^{1/p} \\
&\leq C\,\|b\|_*\,M_{p,(9/8)}f(x).
\end{aligned}$$

So we derive $M_1 \leq C\,\|b\|_*\,M_{p,(9/8)}f(x)$.

Let us consider the term M_2. For $x, y \in Q$, it is easily seen that

$$\begin{aligned}
|T(f\,\chi_{\mathbb{R}^d\setminus 2Q})(y)| &\leq T_*f(x) + C\sup_{Q_0\ni x}\frac{1}{l(Q_0)^n}\int_{Q_0}|f|\,d\mu \\
&\leq T_*f(x) + C\,M_{p,(9/8)}f(x).
\end{aligned}$$

Thus

$$M_2 = |(b_R-b_Q)\,T(f\,\chi_{\mathbb{R}^d\setminus 2Q})(y)| \leq C\,K_{Q,R}\,\|b\|_*\,(T_*f(x)+M_{p,(9/8)}f(x)).$$

Let us turn our attention to the term M_4. Operating as in (9.10), for any $y, z \in R$, we get

$$|T((b-b_R)\,f\,\chi_{\mathbb{R}^d\setminus 2^NQ})(y) - T((b-b_R)\,f\,\chi_{\mathbb{R}^d\setminus 2^NQ})(z)| \leq C\,\|b\|_*\,M_{p,(9/8)}f(x).$$

Taking the mean over Q for y and over R for z, we obtain

$$M_4 \leq C\,\|b\|_*\,M_{p,(9/8)}f(x).$$

The term M_5 is easy to estimate too. Some calculations very similar to the ones for M_1 yield $M_5 \leq C\,\|b\|_*\,M_{p,(9/8)}f(x)$.

Finally, we have to deal with M_3. For $y \in Q$, we have

$$\begin{aligned}
|T((b-b_R)\,f\,\chi_{2^NQ\setminus 2Q}))(y)| &\leq C\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\int_{2^{k+1}Q\setminus 2^kQ}|b-b_R|\,|f|\,d\mu \\
&\leq C\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\left(\int_{2^{k+1}Q}|b-b_R|^{p'}\,d\mu\right)^{1/p'} \\
&\qquad\times\left(\int_{2^{k+1}Q}|f|^p\,d\mu\right)^{1/p}.
\end{aligned}$$

We have

$$\begin{aligned}
\left(\int_{2^{k+1}Q}|b-b_R|^{p'}\,d\mu\right)^{1/p'} &\leq \left(\int_{2^{k+1}Q}|b-b_{2^{k+1}Q}|^{p'}\,d\mu\right)^{1/p'} \\
&\qquad + \mu(2^{k+1}Q)^{1/p'}\,|b_{2^{k+1}Q}-b_R| \\
&\leq C\,K_{Q,R}\,\|b\|_*\,\mu(2^{k+2}Q)^{1/p'}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&|T((b-b_R)\, f\, \chi_{2^N Q\setminus 2Q}))(y)| \\
&\qquad \leq C\, K_{Q,R}\, \|b\|_*\, \sum_{k=1}^{N-1} \frac{\mu(2^{k+2}Q)}{l(2^kQ)^n} \left(\frac{1}{\mu(2^{k+2}Q)} \int_{2^{k+1}Q} |f|^p\, d\mu\right)^{1/p} \\
&\qquad \leq C\, K_{Q,R}\, \|b\|_*\, M_{(9/8)} f(x) \sum_{k=1}^{N-1} \frac{\mu(2^{k+2}Q)}{l(2^kQ)^n} \\
&\qquad \leq C\, K_{Q,R}^2\, \|b\|_*\, M_{p,(9/8)} f(x).
\end{aligned}
$$

Taking the mean over Q, we get

$$M_3 \leq C\, K_{Q,R}^2\, \|b\|_*\, M_{p,(9/8)} f(x).$$

So by the estimates on M_1, M_2, M_3, M_4 and M_5, the regularity condition (9.6) follows.

Let us see how from (9.5) and (9.6) one obtains (9.4). From (9.5), if Q is a *doubling* cube and $x \in Q$, we have

$$
\begin{aligned}
|m_Q([b,T]f) - h_Q| \;&\leq\; \frac{1}{\mu(Q)} \int_Q |[b,T]f - h_Q|\, d\mu \\
&\leq\; C\, \|b\|_*\, (M_{p,(9/8)} f(x) + M_{p,(3/2)} Tf(x)).
\end{aligned}
\tag{9.11}
$$

Also, for any cube $Q \ni x$ (non doubling, in general), $K_{Q,\widetilde{Q}} \leq C$, and then by (9.5) and (9.6) we get

$$
\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |[b,T]f - m_{\widetilde{Q}}([b,T]f)|\, d\mu \\
&\quad \leq \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |[b,T]f - h_Q|\, d\mu + |h_Q - h_{\widetilde{Q}}| + |h_{\widetilde{Q}} - m_{\widetilde{Q}}([b,T]f)| \\
&\quad \leq C\, \|b\|_*\, (M_{p,(9/8)} f(x) + M_{p,(3/2)} Tf(x) + T_* f(x)).
\end{aligned}
\tag{9.12}
$$

On the other hand, for all *doubling* cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \leq P_0$, where P_0 is the constant in Lemma 9.3, by (9.6) we have

$$|h_Q - h_R| \leq C\, \|b\|_*\, (M_{p,(9/8)} f(x) + T_* f(x))\, P_0^2.$$

So by Lemma 9.3 we get

$$|h_Q - h_R| \leq C\, \|b\|_*\, (M_{p,(9/8)} f(x) + T_* f(x))\, K_{Q,R}$$

for all doubling cubes $Q \subset R$ with $x \in Q$ and, using (9.11) again, we obtain

$$
\begin{aligned}
&|m_Q([b,T]f) - m_R([b,T]f)| \\
&\qquad\qquad \leq C\, \|b\|_*\, (M_{p,(9/8)} f(x) + M_{p,(3/2)} Tf(x) + T_* f(x))\, K_{Q,R}.
\end{aligned}
$$

From this estimate and (9.12), we get (9.4). □

Acknowledgements. The author wishes to thank the hospitality of the Department of Mathematics of the University of Jyväskylä where part of this research was done. Thanks are due especially to Pertti Mattila.

The author is also grateful to the referee for carefully reading the paper and for helpful comments.

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