PAINLEVÉ'S PROBLEM, ANALYTIC CAPACITY AND CURVATURE OF MEASURES

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Abstract. In this paper we survey some recent results in connection with the so called Painlevé's problem, the semiadditivity of analytic capacity and other related results.

1. Introduction

A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. In order to study removability, in the 1940's Ahlfors [Ah] introduced the notion of analytic capacity. The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) = \sup |f'(\infty)|, \tag{1}$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

In [Ah], Ahlfors showed that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$.

Painlevé's problem consists in characterizing removable singularities for bounded analytic functions in a metric/geometric way. By Ahlfors' result this is equivalent to describing compact sets with positive analytic capacity in metric/geometric terms.

Vitushkin in the 1950's and 1960's showed that analytic capacity and the so called continuous analytic capacity play a central role in problems of uniform rational approximation on compact sets of the complex plane. The continuous analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as

$$\alpha(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all *continuous* functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ which are analytic on $\mathbb{C} \setminus E$ and uniformly bounded by 1 on $\mathbb{C}$. Many results obtained by Vitushkin in connection with uniform rational approximation are stated in terms of α and γ. See [Vi1] and [Vi2], for example. Because of its applications to this type of problems he raised the question of the semiadditivity of γ and α. Namely, does there exist an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)) \ ?$$

And analogously for α.

In [To6] it has been recently proved that analytic capacity is indeed semiadditive. Moreover, a characterization of removable sets for bounded analytic functions in terms of the so called curvature of measures is also given in [To6]. In the present

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paper we will survey these results and we will describe the main ideas and techniques involved in their proofs. We will also deal with other related results, although we don't intend to make a complete account of all recent advances in connection with analytic capacity and Painlevé's problem.

Let us make some comments about the notation used in the paper. By a cube Q we mean a closed cube with sides parallel to the axes. We denote its side length by $\ell(Q)$. As usual, in the paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2. Analytic capacity

2.1. **Basic properties of analytic capacity.** One should keep in mind that, in a sense, analytic capacity measures the size of a set as a non removable singularity for bounded analytic functions. A direct consequence of the definition is that

$$E \subset F \;\Rightarrow\; \gamma(E) \leq \gamma(F).$$

Moreover, it is also easy to check that analytic capacity is translation invariant:

$$\gamma(z+E) = \gamma(E) \qquad \text{for all } z \in \mathbb{C}.$$

Concerning dilations, we have

$$\gamma(\lambda E) = |\lambda|\gamma(E) \qquad \text{for all } \lambda \in \mathbb{C}.$$

Further, if E is connected, then

$$\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E).$$

The second inequality (which holds for any compact set E) follows from the fact that the analytic capacity of a closed disk coincides with its radius, and the first one is a consequence of Koebe's 1/4 theorem (see [Gam, Chapter VIII] or [Gar2, Chapter I] for the details, for example). Thus if E is connected and different from a point, then it is non removable. This implies that any removable compact set must be totally disconnected.

2.2. **Relationship with Hausdorff measure.** The relationship between Hausdorff measure and analytic capacity is the following:

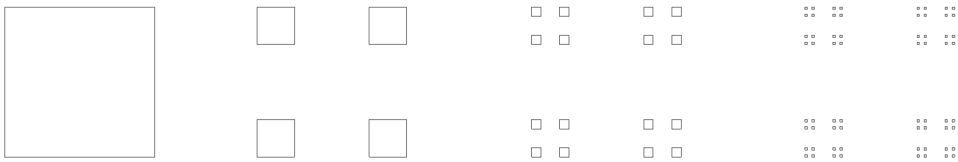
- If $\dim_H(E) > 1$ (here $\dim_H$ stands for the Hausdorff dimension), then $\gamma(E) > 0$. This result follows easily from Frostman's Lemma.
- $\gamma(E) \leq \mathcal{H}^1(E)$, where $\mathcal{H}^1$ is the one dimensional Hausdorff measure, or length. This follows from Cauchy's integral formula, and it was proved by Painlevé about one hundred years ago. Observe that, in particular we deduce that if $\dim_H(E) < 1$, then $\gamma(E) = 0$.

By the statements above, it turns out that dimension 1 is the critical dimension in connection with analytic capacity. Moreover, a natural question arises: is it true that $\gamma(E) > 0$ if and only if $\mathcal{H}^1(E) > 0$?

Vitushkin showed that the answer is *no*. He showed that there are sets with positive length and vanishing analytic capacity. A typical example of such a set is the so called corner quarters Cantor set. This set is constructed in the following way: consider a square Q^0 with side length 1. Now replace Q^0 by 4 squares Q_i^1, $i = 1, \ldots, 4$, with side length 1/4 contained in Q^0, so that each Q_i^1 contains a different vertex of Q^0. Analogously, in the next stage each Q_i^1 is replaced by 4 squares with side length 1/16 contained in Q_i^1 so that each one contains a different vertex of Q_i^1. So we will have 16 squares Q_k^2 of side length 1/16. We proceed inductively (see Figure 1), and we set $E_n = \bigcup_{i=1}^{4^n} Q_i^n$ and $E = \bigcap_{n=1}^{\infty} E_n$. This is the corner quarters Cantor set. Taking into account that

$$\sum_{i=1}^{4^n} \ell(Q_i^n) = 1$$

for each n, it is not difficult to see that $0 < \mathcal{H}^1(E) < \infty$. The proof of the fact that $\gamma(E) = 0$ is more difficult, and it is due independently to Garnett [Gar1] and Ivanov[1] [Iv].



Q^0 E_1 E_2 E_3

Fig. 1. The square Q^0 and the sets E_1, E_2 and E_3, which appear in the first stages of the construction of the corner quarters Cantor set.

Recall that a set is called rectifiable if it is $\mathcal{H}^1$-almost all contained in a countable union of rectifiable curves. On the other hand, it is called purely unrectifiable if it intersects any rectifiable curve at most in a set of zero length.

It turns out that the corner quarters Cantor set, and also Vitushkin's example, are purely unrectifiable. Motivated by this fact Vitushkin conjectured that pure unrectifiability is a necessary and sufficient condition for vanishing analytic capacity for sets with finite length.

Guy David [Dd1] showed in 1998 that Vitushkin's conjecture is true:

Theorem 1. *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then, $\gamma(E) = 0$ if and only if E is purely unrectifiable.*

To be precise, let us remark that the "if" part of the theorem is not due to David. In fact, it follows from Calderón's theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs with small Lipschitz constant. The "only if" part of the theorem, which is more difficult, is the one proved by David. See also [MMV], [DM] and [Lé] for some important preliminary contributions to the proof.

Theorem 1 is the solution of Painlevé's problem for sets with finite length. The analogous result is false for sets with infinite length (see [Ma1] and [JM]). For this type of sets there is no such a nice geometric solution of Painlevé's problem, and we have to content ourselves with a characterization such as the one in Corollary 12 below (at least, for the moment).

2.3. **The capacity γ_+ and the Cauchy transform.** Given a finite complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is defined by

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z} d\nu(\xi).$$

[1] Vitushkin constructed a different example previously.

Although the integral above is absolutely convergent a. e. with respect to Lebesgue measure, it does not make sense, in general, for $z \in \operatorname{supp}(\nu)$. This is the reason why one considers the *truncated Cauchy transform* of ν, which is defined as

$$\mathcal{C}_\varepsilon \nu(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z} d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$.

Given a positive Radon measure μ on the complex plane and a μ-measurable function f on $\mathbb{C}$, we also denote

$$\mathcal{C}_\mu f(z) := \mathcal{C}(f\, d\mu)(z)$$

for $z \notin \operatorname{supp}(f)$, and

$$\mathcal{C}_{\mu,\varepsilon} f(z) := \mathcal{C}_\varepsilon(f\, d\mu)(z)$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. We say that $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ if the operators $\mathcal{C}_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The capacity γ_+ of a compact set $E \subset \mathbb{C}$ is

$$\gamma_+(E) := \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E,\ \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\}. \tag{2}$$

That is, γ_+ is defined as γ in (1) with the additional constraint that f should coincide with $\mathcal{C}\mu$, where μ is some positive Radon measure supported on E (observe that $(\mathcal{C}\mu)'(\infty) = -\mu(\mathbb{C})$ for any Radon measure μ). To be precise, there is another slight difference: in (1) we asked $\|f\|_{L^\infty(\mathbb{C}\setminus E)} \leq 1$, while in (2), $\|f\|_{L^\infty(\mathbb{C})} \leq 1$ (for $f = \mathcal{C}\mu$). Trivially, we have $\gamma_+(E) \leq \gamma(E)$.

3. The curvature of a measure

A Radon measure μ on $\mathbb{R}^d$ has growth of degree n (or is of degree n) if there exists some constant C such that $\mu(B(x,r)) \leq Cr^n$ for all $x \in \mathbb{R}^d$, $r > 0$. When $n = 1$, we say that μ has linear growth.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x,y,z) = 0$. For a positive Radon measure μ, we set

$$c^2_\mu(x) = \iint c(x,y,z)^2\, d\mu(y) d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c^2_\mu(x)\, d\mu(x) = \iiint c(x,y,z)^2\, d\mu(x) d\mu(y) d\mu(z). \tag{3}$$

The notion of curvature of a measure was introduced by Melnikov [Me] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the big recent advances in connection with analytic capacity. The notion of curvature is connected to the Cauchy transform by the following result, proved by Melnikov and Verdera [MV].

Proposition 2. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. We have*

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6}c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{4}$$

where $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$.

In this proposition, $c^2_\varepsilon(\mu)$ stands for the ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (3), but with the triple integral over $\{x, y, z \in \mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$).

The identity (4) is remarkable because it relates an analytic notion (the Cauchy transform of a measure) with a metric-geometric one (curvature). We give a sketch of the proof.

Sketch of the proof of Proposition 2. If we don't worry about truncations and the absolute convergence of the integrals, we can write

$$\|\mathcal{C}\mu\|^2_{L^2(\mu)} = \int \left| \int \frac{1}{y-x}\, d\mu(y)\right|^2 d\mu(x) = \iiint \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x).$$

By Fubini (assuming that it can be applied correctly), permuting x, y, z, we get,

$$\|\mathcal{C}\mu\|^2_{L^2(\mu)} = \frac{1}{6}\iiint \sum_{s\in S_3} \frac{1}{(z_{s_2} - z_{s_1})\overline{(z_{s_3} - z_{s_1})}}\, d\mu(z_1)d\mu(z_2)d\mu(z_3),$$

where S_3 is the group of permutations of three elements. An elementary calculation shows that

$$\sum_{s\in S_3} \frac{1}{(z_{s_2} - z_{s_1})\overline{(z_{s_3} - z_{s_1})}} = c(z_1, z_2, z_3)^2.$$

So we get

$$\|\mathcal{C}\mu\|^2_{L^2(\mu)} = \frac{1}{6}\, c^2(\mu).$$

To argue rigorously, above we should use the truncated Cauchy transform $\mathcal{C}_\varepsilon\mu$ instead of $\mathcal{C}\mu$. Then we would obtain

$$\begin{aligned}
\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} &= \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x)\\
&= \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon\\ |y-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x) + O(\mu(\mathbb{C})).
\end{aligned} \tag{5}$$

By the linear growth of μ, it is easy to check that $|O(\mu(\mathbb{C}))| \leq \mu(\mathbb{C})$. As above, using Fubini and permuting x, y, z, one shows that the triple integral in (5) equals $c^2_\varepsilon(\mu)/6$. □

The notion of curvature is related to rectifiability, and there is a strong connection of this notion with the coefficients β which appear in the travelling salesman theorem of P. Jones [Jo]. The following nice result of Léger [Lé] is an example of this relationship.

Theorem 3. *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. If $c^2(\mathcal{H}^1_{|E}) < \infty$, then E is rectifiable.*

Observe that from the preceding result and Proposition 2 one infers that if $\mathcal{H}^1(E) < \infty$ and the Cauchy transform is bounded on $L^2(\mathcal{H}^1_{|E})$, then E must be rectifiable. A more quantitative version of this result due to Mattila, Melnikov and Verdera [MMV] asserts that if E is such that

$$\mathcal{H}^1(E \cap B(x,r)) \approx r \qquad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam}(E)$$

and the Cauchy transform is bounded on $L^2(\mathcal{H}^1_{|E})$, then E is contained in a regular curve Γ (i.e. a curve which also satisfies the preceding estimates, with Γ instead of E).

4. The $T(1)$ and $T(b)$ theorems and Calderón-Zygmund theory with non doubling measures

The study of analytic capacity has led to the extension of Calderón-Zygmund (CZ) theory to the situation where the underlying measure μ on $\mathbb{C}$ is non doubling. Recall that μ is said to be doubling if there exists some constant C such that

$$\mu(B(z,2r)) \leq C\mu(B(z,r)) \qquad \text{for } z \in \operatorname{supp}(\mu)) \text{ and } r > 0.$$

Let us remark that in the classical CZ theory this doubling assumption plays an essential role in almost all results. When one deals with analytic capacity one is forced to deal with measures which may be non doubling, and which are only assumed to have linear growth.

The use of CZ theory has been fundamental in most of the recent developments in connection with analytic capacity. For instance, the so called "$T(b)$ type theorems" are essential tools in the proofs of Vitushkin's conjecture by G. David and of the semiadditivity of analytic capacity in [To6]. In this section we will describe briefly some results of CZ theory without doubling assumptions. In particular, we will state in detail the $T(1)$ theorem and one of the $T(b)$ type theorems of Nazarov, Treil and Volberg.

Let us introduce some terminology. We say that $k(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\} \to \mathbb{C}$ is an n-dimensional Calderón-Zygmund kernel if there exist constants $C > 0$ and η, with $0 < \eta \leq 1$, such that the following inequalities hold for all $x, y \in \mathbb{R}^d$, $x \neq y$:

$$\begin{aligned} & |k(x,y)| \leq \frac{C}{|x-y|^n}, \qquad \text{and} \\ & |k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq \frac{C|x-x'|^\eta}{|x-y|^{n+\eta}} \quad \text{if } |x-x'| \leq |x-y|/2. \end{aligned} \tag{6}$$

For example, the Cauchy kernel $1/(y-x)$, with $x, y \in \mathbb{C}$, is a 1-dimensional CZ kernel.

Given a real or complex Radon measure μ on $\mathbb{R}^d$, we define

$$T\mu(x) := \int k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu). \tag{7}$$

We say that T is an n-dimensional Calderón-Zygmund operator (CZO) with kernel $k(\cdot,\cdot)$. We also consider the following ε-truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon\mu(x) := \int_{|x-y|>\varepsilon} k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

If μ is non negative and $f \in L^1_{loc}(\mu)$, we denote

$$T_\mu f(x) := T(f\,d\mu)(x) \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(f\,d\mu),$$

and

$$T_{\mu,\varepsilon} f(x) := T_\varepsilon(f\,d\mu)(x).$$

We say that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

Given $\rho > 1$, we say that $f \in L^1_{loc}(\mu)$ belongs to the space $BMO_\rho(\mu)$ if

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)|\,d\mu < \infty,$$

where the supremum is taken over all the cubes in $\mathbb{R}^d$ and $m_Q(f)$ is the μ-mean of f over Q.

Following [NTV1], a Calderón-Zygmund operator T_μ is said to be weakly bounded if

$$\left|\langle T_{\mu,\varepsilon}\chi_Q, \chi_Q\rangle\right| \leq C\mu(Q) \quad \text{ for all the cubes } Q \subset \mathbb{R}^d, \text{ uniformly on } \varepsilon > 0.$$

Notice that if T_μ is antisymmetric, then the left hand side above vanishes and so T_μ is weakly bounded.

Now we are ready to state the $T(1)$ theorem:

Theorem 4 ($T(1)$ theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. The following conditions are equivalent:*

(a) *T_μ is bounded on $L^2(\mu)$.*
(b) *T_μ is weakly bounded and, for some $\rho > 1$, we have that $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \in BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$.*
(c) *There exists some constant C such that for all $\varepsilon > 0$ and all the cubes $Q \subset \mathbb{R}^d$,*

$$\|T_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \leq C\mu(Q)^{1/2} \quad \text{and} \quad \|T^*_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \leq C\mu(Q)^{1/2}.$$

The classical way of stating the $T(1)$ theorem is the equivalence (a) $\Leftrightarrow$ (b). However, for some applications it is sometimes more practical to state the result in terms of the L^2 boundedness of T_μ and T^*_μ over characteristic functions of cubes, i.e. (a) $\Leftrightarrow$ (c).

Theorem 4 is the extension of the classical $T(1)$ theorem of David and Journé to measures of degree n which may be non doubling. The result was proved by Nazarov, Treil and Volberg in [NTV1], although not exactly in the form stated above. An independent proof for the particular case of the Cauchy transform was obtained almost simultaneously in [To1]. For the equivalence of conditions (b) and (c) above, the reader should see [To4, Remark 7.1 and Lemma 7.3]. Other (more recent) proofs of the $T(1)$ theorem for non doubling measures are in [Ve2] (for the particular case of the Cauchy transform) and in [To4].

By Proposition 2, the $T(1)$ theorem for the Cauchy transform can be rewritten in the following way:

Theorem 5. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. The Cauchy transform is bounded on $L^2(\mu)$ if and only if*

$$c^2(\mu_{|Q}) \leq C\mu(Q) \qquad \text{for all the squares } Q \subset \mathbb{C}.$$

Observe that this result is a restatement of the equivalence (a) $\Leftrightarrow$ (c) in Theorem 4, by an application of (4) to the measure $\mu_{|Q}$, for all the squares $Q \subset \mathbb{C}$.

Let us remark that the boundedness of T_μ on $L^2(\mu)$ does not imply the boundedness of T_μ from $L^\infty(\mu)$ into $BMO(\mu)$ (this is the space $BMO_\rho(\mu)$ with parameter $\rho = 1$), and in general $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \notin BMO(\mu)$ uniformly on $\varepsilon > 0$. See [Ve2] and [MMNO]. On the contrary, one can show that if T_μ is bounded on $L^2(\mu)$, then it is also bounded from $L^\infty(\mu)$ into $BMO_\rho(\mu)$, for $\rho > 1$, by arguments similar to the classical ones for homogeneous spaces. However, the space $BMO_\rho(\mu)$ has some drawbacks. For example, it depends on the parameter ρ and it does not satisfy the John-Nirenberg inequality. To solve these problems, in [To2] a new space called $RBMO(\mu)$ has been introduced. $RBMO(\mu)$ is a subspace of $BMO_\rho(\mu)$ for all $\rho > 1$, and it coincides with $BMO(\mu)$ when μ is an AD-regular measure, that is, when

$$\mu(B(x,r)) \approx r^n \quad \text{for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

Moreover, $RBMO(\mu)$ satisfies a John-Nirenberg type inequality, and all CZO's which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. For these reasons $RBMO(\mu)$ seems to be a good substitute of the classical space BMO for non doubling measures of degree n. For the precise definition of $RBMO(\mu)$ and its properties, see [To2].

$T(b)$ type theorems are other criterions for the $L^2(\mu)$ boundedness of CZO's. To state one of these theorems in detail we need to introduce the notion of weak accretivity. We say that a function $b \in L^1_{loc}(\mu)$ is weakly accretive if there exists some positive constant C such that

$$\left| \int_Q b \, d\mu \right| \geq C^{-1} \mu(Q) \quad \text{for all cubes } Q \subset \mathbb{R}^d.$$

Then we have:

Theorem 6 ($T(b)$ theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. Let b_1, b_2 be two weakly accretive functions belonging to $L^\infty(\mu)$. Then T_μ is bounded n $L^2(\mu)$ if and only if the operator $b_2 T_\mu b_1$ is weakly bounded and $T_{\mu,\varepsilon} b_1$, $T^*_{\mu,\varepsilon} b_2$ belong to $BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$, for some $\rho > 1$.*

The condition that $b_2 T_\mu b_1$ is weakly bounded means that

$$\langle b_2 T_{\mu,\varepsilon}(\chi_Q b_1), \, \chi_Q \rangle \leq C\mu(Q)$$

uniformly on $\varepsilon > 0$, for all cubes $Q \subset \mathbb{R}^d$. Notice that if T_μ is antisymmetric and $b_1 = b_2 = b$, then $b T_\mu b$ is always weakly bounded.

The preceding theorem has been proved in [NTV4], and it is a generalization of a classical theorem of David, Journé and Semmes to the case of non doubling measures (and so it requires new ideas and techniques). Other variants of this result (i.e. other $T(b)$ type theorems) can be found in [NTV3] and [NTV5].

For the particular case of the Cauchy transform, Theorem 6 yields the following result.

Theorem 7. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. Suppose that there exists a function b such that:*

(a) $b \in L^\infty(\mu)$,

(b) *b is weakly accretive,*

(c) *$\mathcal{C}_{\mu,\varepsilon} b \in BMO_\rho(\mu)$ uniformly in $\varepsilon > 0$, for some $\rho > 1$.*

Then $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$.

Many more results on Calderón-Zygmund theory with non doubling measures have been proved recently. For example, there are results concerning L^p and weak $(1,1)$ estimates [NTV2]; Hardy spaces [To3]; weights [GCM1], [MM], [OP]; commutators [CS], [HMY2], [To2]; fractional integrals [GCM2], [GCG1]; Lipschitz spaces [GCG2]; etc. See also the survey paper [Ve3].

5. Semiadditivity of γ_+ and its characterization in terms of curvature

We denote by $\Sigma(E)$ the set of Radon measures supported on E such that $\mu(B(x,r)) \leq r$ for all $x \in \mathbb{C}$, $r > 0$.

The following theorem characterizes γ_+ in terms of curvature of measures and in terms of the L^2 norm of the Cauchy transform.

Theorem 8. *For any compact set $E \subset \mathbb{C}$ we have*

$$\begin{aligned} \gamma_+(E) &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \leq \mu(E)\bigr\} \\ &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\bigr\}. \end{aligned} \tag{8}$$

In the statement above, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)}$ stands for the operator norm of $\mathcal{C}_\mu$ on $L^2(\mu)$. That is, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} = \sup_{\varepsilon>0} \|\mathcal{C}_{\mu,\varepsilon}\|_{L^2(\mu),L^2(\mu)}$.

Sketch of the proof of Theorem 8. Call S_1 and S_2 the first and second suprema on the right side of (8) respectively.

To see that $S_1 \gtrsim \gamma_+(E)$ take μ supported on E such that $\|\mathcal{C}\mu\|_\infty \leq 1$ and $\mu(E) \geq \gamma_+(E)/2$. One easily gets that $\|\mathcal{C}_\varepsilon \mu\|_\infty \lesssim 1$ on $\operatorname{supp}(\mu)$ for every $\varepsilon > 0$ and $\mu(B(x,r)) \leq Cr$ for all $r > 0$. From Proposition 2, it follows then that $c^2(\mu) \leq C\mu(E)$.

The inequality $S_2 \gtrsim S_1$ can be proved using the $T(1)$ theorem. Indeed, let μ be supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_1 \leq 2\mu(E)$. We set

$$A := \left\{x \in E : \iint c(x,y,z)^2\, d\mu(y) d\mu(z) \leq 2\right\}.$$

By Tchebychev $\mu(A) \geq \mu(E)/2$. Moreover, for any set $B \subset \mathbb{C}$,

$$c^2(\mu_{|B\cap A}) \leq \iiint_{x \in B\cap A} c(x,y,z)^2\, d\mu(x) d\mu(y) d\mu(z) \leq 2\mu(B).$$

In particular, this estimate holds when B is any square in $\mathbb{C}$, and so $\mathcal{C}_{\mu_{|A}}$ is bounded on $L^2(\mu_{|A})$, by Theorem 5. Thus $S_2 \gtrsim \mu(A) \approx S_1$.

Finally, the inequality $\gamma_+(E) \gtrsim S_2$ follows from a dualization of the weak $(1,1)$ inequality for the Cauchy transform. See [To1] for the details, for example. □

From Theorem 8, since the term

$$\sup\bigl\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\bigr\}$$

is countably semiadditive, we infer that γ_+ is also countably semiadditive.

Corollary 9. *The capacity γ_+ is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma_+\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma_+(E_i).$$

Another consequence of Theorem 8 is that the capacity γ_+ can be characterized in terms of the following potential, introduced by Verdera [Ve2]:

$$U_\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r} + c_\mu(x), \tag{9}$$

where $c_\mu(x)$ is the pointwise version of curvature defined in (3). The precise result is the following.

Corollary 10. *For any compact set $E \subset \mathbb{C}$ we have*

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, U_\mu(x) \leq 1\, \forall x \in \mathbb{C}\big\}.$$

The proof of this corollary follows easily from the fact that

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \leq \mu(E)\big\},$$

using Tchebychev.

Let us remark that the preceding characterization of γ_+ in terms of U_μ is interesting because it suggests that some techniques of potential theory can be useful to study γ_+. See [To5] and [Ve2].

6. The comparability between γ and γ_+

In [To6] the following result has been proved.

Theorem 11. *There exists an absolute constant C such that for any compact set $E \subset \mathbb{C}$ we have*

$$\gamma(E) \leq C\gamma_+(E).$$

As a consequence, $\gamma(E) \approx \gamma_+(E)$.

Let us remark that the comparability between γ and γ_+ had been previously proved by P. Jones for compact connected sets by geometric arguments, very different from the ones in [To6] (see [Pa1, Chapter 3]). Also, in [MTV] it had already been shown that $\gamma \approx \gamma_+$ holds for a big class of Cantor sets. In particular, for the corner quarters Cantor set E (see Fig. 1) it was proved in [MTV] that $\gamma(E_n) \approx \gamma_+(E_n)$. Recall that E_n is the n-th generation appearing in the construction of E. By results due to Mattila [Ma2] and Eiderman [Ei] (see also [To5]) it was already known that $\gamma_+(E_n) \approx 1/n^{1/2}$. Thus, one has $\gamma(E_n) \approx 1/n^{1/2}$.

An obvious corollary of Theorem 11 and the characterization of γ_+ in terms of curvature obtained in Theorem 8 is the following.

Corollary 12. *Let $E \subset \mathbb{C}$ be compact. Then, $\gamma(E) > 0$ if and only if E supports a non zero Radon measure with linear growth and finite curvature.*

Since we know that γ_+ is countably semiadditive, the same happens with γ:

Corollary 13. *Analytic capacity is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma(E_i).$$

In the rest of this section we will describe the main ideas involved in the proof of Theorem 11.

Notice that, by Theorem 8, to prove Theorem 11 it is enough to show that there exists some measure μ supported on E with linear growth, satisfying $\mu(E) \approx \gamma(E)$, and such that the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ with absolute constants. To implement this argument, the main tool used in [To6] is the $T(b)$ theorem of Nazarov, Treil and Volberg in [NTV3], which is similar in spirit to the $T(b)$ type theorem stated in Theorem 7 but more appropriate for the present situation. To apply this $T(b)$ theorem, one has to construct a suitable measure μ and a function $b \in L^\infty(\mu)$ fulfilling some precise conditions, analogous to the conditions (a), (b) and (c) in Theorem 7.

Because of the definition of analytic capacity, there exists some function $f(z)$ which is analytic and bounded in $\mathbb{C} \setminus E$ with $f'(\infty) = \gamma(E)$ (this is the so called Ahlfors function). By a standard approximation argument, it is not difficult to see that one can assume that E is a finite union of disjoint segments, so that in particular $\mathcal{H}^1(E) < \infty$. Then one has to construct μ and b and to prove the comparability $\gamma(E) \approx \gamma_+(E)$ with estimates independent of $\mathcal{H}^1(E)$. Since E is a finite union of disjoint segments, there exists some complex measure ν_0 (obtained from the boundary values of $f(z)$) supported on E such that $f = \mathcal{C}\nu_0$. This measure satisfies the following properties:

$$\|\mathcal{C}\nu_0\|_\infty \leq 1, \tag{10}$$

$$|\nu_0(E)| = \gamma(E), \tag{11}$$

$$d\nu_0 = b_0\, d\mathcal{H}^1|E, \quad \text{where } b_0 \text{ satisfies } \|b_0\|_\infty \leq 1. \tag{12}$$

Given this information, by a more or less direct application of a $T(b)$ type theorem we cannot expect to prove that the Cauchy transform is bounded with respect to a measure μ such us the one described above with absolute constants. Let us explain the reason in some detail. Suppose for example that there exists some function b such that $d\nu_0 = b\, d\mu$ and we use the information about ν_0 given by (10), (11) and (12) (notice the difference between b and b_0). From (10) and (11) we derive

$$\|\mathcal{C}(b\, d\mu)\|_\infty \leq 1 \tag{13}$$

and

$$\left| \int b\, d\mu \right| \approx \mu(E). \tag{14}$$

The estimate (13) is very good for our purposes. In fact, most classical $T(b)$ type theorems (like Theorem 7) require only the $BMO_\rho(\mu)$ norm of b to be bounded, which is a weaker assumption. The estimate (14) is likewise good; it is a global accretivity condition, and with some technical difficulties (which may involve some kind of stopping time argument, like in [Dd1] or [NTV3]), one can hope to be able to prove that the accretivity condition

$$\left| \int_Q b\, d\mu \right| \approx \mu(Q \cap E)$$

holds for many squares Q.

Our problems arise from (12). Notice that this implies that

$$|\nu_0|(E) \leq \mathcal{H}^1(E), \tag{15}$$

where $|\nu_0|$ stands for the variation of ν_0. This is a very bad estimate since we don't have any control on $\mathcal{H}^1(E)$ (we only know $\mathcal{H}^1(E) < \infty$ because our assumption on E). However, as far as we know, all $T(b)$ type theorems require the estimate $\|b\|_{L^\infty(\mu)} \leq C$, or variants of it, which in particular imply that

$$|\nu_0|(E) \leq C\mu(E) \approx \gamma(E). \tag{16}$$

That is to say, the estimate that we get from (12) is (15), but the one we need is (16). So by a direct application of a $T(b)$ type theorem we will obtain bad results when $\gamma(E) \ll \mathcal{H}^1(E)$.

To prove Theorem 11, we need to work with a measure "better behaved" than ν_0, which we call ν. This new measure will be a suitable modification of ν_0 with the required estimate for its total variation. To construct ν, in [To6] we consider a set F containing E made up of a finite disjoint union of squares: $F = \bigcup_{i\in I} Q_i$. One should think that the squares Q_i approximate E at some "intermediate scale". For example, if $E = E_N$ is N-th approximation of the corner quarters Cantor set, then a good choice for F would be $E_{N/2}$ (assuming N even), and the squares Q_i are the $4^{N/2}$ squares of generation $N/2$. For each square Q_i, we take a complex measure ν_i supported on Q_i such that $\nu_i(Q_i) = \nu_0(Q_i)$ and $|\nu_i|(Q_i) = |\nu_i(Q_i)|$ (that is, ν_i is a constant multiple of a positive measure). We set $\nu = \sum_i \nu_i$. So ν is some kind of approximation of ν_0, and if the squares Q_i are big enough, the variation $|\nu|$ becomes sufficiently small (because there are "cancellations" in the measure ν_0 in each Q_i). On the other hand, the squares Q_i cannot be too big, because we need

$$\gamma_+(F) \leq C\gamma_+(E). \tag{17}$$

In this way, we will have constructed a complex measure ν supported on F satisfying

$$|\nu|(F) \approx |\nu(F)| = \gamma(E). \tag{18}$$

Taking a suitable measure μ such that $\mathrm{supp}(\mu) \supset \mathrm{supp}(\nu)$ and $\mu(F) \approx \gamma(E)$, we will be ready for the application of a $T(b)$ type theorem, such as the one in [NTV3], which is a very powerful tool. Indeed, notice that (18) implies that ν satisfies a global accretivity condition and that also the variation $|\nu|$ is controlled. On the other hand, if we have been careful enough, we will have also some useful estimates on $|\mathcal{C}\nu|$, since ν is an approximation of ν_0. Then, using the $T(b)$ theorem in [NTV3], we will deduce $\gamma_+(F) \geq C^{-1}\mu(E)$, and so $\gamma_+(E) \geq C^{-1}\gamma(E)$, by (17), and we will be done. Nevertheless, in order to obtain the right estimates on the measures ν and μ it will be necessary to use an induction argument involving the sizes of the squares Q_i, which will allow to assume that $\gamma(E\cap Q_i) \approx \gamma_+(E\cap Q_i)$ for each square Q_i.

Let us remark that the choice of the right squares Q_i which approximate E at an intermediate scale is one of the key points of the argument. The potential defined in (9) plays an important role here.

7. Other results

In [To7], some results analogous to Theorems 8 and 11 have been obtained for the continuous analytic capacity α. Recall that this capacity is defined like γ in (1), with the additional requirement that the functions f considered in the sup should extend continuously to the whole complex plane. In particular, in [To7] it is shown that α is semiadditive. This result has some nice consequences for the theory of uniform rational approximation on the complex plane. For example, it implies the so called *inner boundary conjecture* (see [DØ] and [Ve1] for previous contributions).

Corollary 12 yields a characterization of removable sets for bounded analytic functions in terms of curvature of measures. Although this result has a definite geometric flavour, it is not clear if this is a really good geometric characterization. Nevertheless, in [To8] it has been shown that the characterization is invariant under bilipschitz mappings, using a corona type decomposition for non doubling measures. See also [GV] for an analogous result for some Cantor sets.

Using the corona type decomposition for measures with finite curvature and linear growth obtained in [To8], it has been proved in [To9] that if μ is a measure without atoms such that the Cauchy transform is bounded on $L^2(\mu)$, then any CZO associated to an odd kernel sufficiently smooth is also bounded in $L^2(\mu)$.

Volberg [Vo] has proved the natural generalization of Theorem 11 to higher dimensions. In this case, one should consider Lipschitz harmonic capacity instead of analytic capacity (see [MP] for the definition and properties of Lipschitz harmonic capacity). The main difficulty arises from the fact that in this case one does not have any good substitute of the notion of curvature of measures, and then one has to argue with a potential very different from the one defined in (9). See also [MT] for related results about Cantor sets in $\mathbb{R}^d$ which avoid the use of any notion similar to curvature.

However, the relationship of Lipschitz harmonic capacity with rectifiability is not well understood. That is to say, a result analogous to David's Theorem 1 is missing for this capacity. The reason is that, given a set $E \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(E) < \infty$ (where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure), it is not known if the fact that the Riesz transform, i.e. the CZO associated to the vectorial kernel $(x-y)/\|x-y\|^d$, is bounded on $L^2(\mathcal{H}^{d-1}_{|E})$ implies that E is $(d-1)$-rectifiable.

The techniques for the proof of Theorem 11 have also been used by Prat [Pr] and Mateu, Prat and Verdera [MPV] to study the capacities γ_s associated to s-dimensional signed Riesz kernels with s non integer:

$$k(x,y) = \frac{x-y}{|x-y|^{s+1}}.$$

In [Pr] it is shown that sets with finite s-dimensional Hausdorff measure have vanishing capacity γ_s when $0 < s < 1$. Moreover, for these s's it is proved in [MPV] that γ_s is comparable to the capacity $C_{\frac{2}{3}(n-s),\frac{3}{2}}$ from nonlinear potential theory. The case of non integer s with $s > 1$ seems much more difficult to study, although in the AD regular situation some results have been obtained [Pr]. The results in [Pr] and [MPV] show that the behavior of γ_s with s non integer is very different from the one with s integer.

For more information, we recommend the interested reader to look at the recent surveys [Dd2] and [Pa2], where the geometric part of the recent developments in connection with Painlevé's problem are treated in more detail than in the present paper. For open questions about the relationship between the length of projections of sets and their analytic capacity, as well as other related problems, see [Ma3].

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