PRINCIPAL VALUES FOR RIESZ TRANSFORMS AND RECTIFIABILITY

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ABSTRACT. Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. In this paper we prove that E is n-rectifiable if and only if the limit

$$\lim_{\varepsilon \to 0} \int_{y \in E : |x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, d\mathcal{H}^n(y)$$

exists $\mathcal{H}^n$-almost everywhere in E. To prove this result we obtain precise estimates from above and from below for the L^2 norm of the n-dimensional Riesz transforms on Lipschitz graphs.

1. INTRODUCTION

Given $x \in \mathbb{R}^d$, $x \neq 0$, we consider the signed Riesz kernel $K(x) = x/|x|^{n+1}$, for an integer such that $0 < n \leq d$. Observe that K is a vectorial kernel. The n-dimensional Riesz transform of a finite Borel measure μ on $\mathbb{R}^d$ is defined by

$$R^n\mu(x) = \int K(x-y)\, d\mu(y), \qquad x \notin \operatorname{supp}(\mu).$$

Notice that the integral above may fail to be absolutely convergent for $x \in \operatorname{supp}(\mu)$. For this reason one considers the ε-truncated n-dimensional Riesz transform, for $\varepsilon > 0$:

$$R^n_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} K(x-y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

The principal values are denoted by

$$\text{p.v.}R^n\mu(x) = \lim_{\varepsilon \to 0} R^n_\varepsilon\mu(x),$$

whenever the limit exists.

One says that a subset $E \subset \mathbb{R}^d$ is n-rectifiable if there exists a countable family of n-dimensional $\mathcal{C}^1$ submanifolds $\{M_i\}_{i\geq 1}$ such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_i M_i\Big) = 0,$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure.

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In this paper we are interested in the relationship between rectifiability and Riesz transforms. One of our main results is the following.

Theorem 1.1. *Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable if, and only if, the principal value* p.v.$R^n(\mathcal{H}^n_{|E})(x)$ *exists for $\mathcal{H}^n$-almost every $x \in E$.*

In fact, the "only if" part of the theorem (rectifiability implies existence of principal values) was well known (see [MPr], for example). On the other hand, under the additional assumption that

$$\liminf_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n} > 0 \qquad \mathcal{H}^n\text{-a.e. } x \in E, \tag{1.1}$$

Mattila and Preiss proved [MPr] that if the principal value p.v.$R^n(\mathcal{H}^n_{|E})(x)$ exists $\mathcal{H}^n$-almost everywhere in E, then E is rectifiable. Getting rid of the hypothesis (1.1) was an open problem raised by authors in [MPr].

Let us also remark that in the particular case $n = 1$, Theorem 1.1 was previously proved in [To1] (and in [Ma2] under the assumption (1.1)) using the relationship between the Cauchy transform and curvature of measures (for more information on this curvature, see [Me] and [MeV], for example). In higher dimensions the curvature method does not work (see [Fa]) and new techniques are required.

We do not know if Theorem 1.1 holds if one replaces the assumption on the existence of principal values for the Riesz transforms by

$$\sup_{\varepsilon>0} |R^n_\varepsilon(\mathcal{H}^n_{|E})(x)| < \infty \qquad \mathcal{H}^n\text{-a.e. } x \in E.$$

That this is the case for $n = 1$ was shown in [To1] using curvature. However, for $n > 1$ this is an open problem that looks very difficult (probably, as difficult as proving that the L^2 boundedness of Riesz transforms with respect to $\mathcal{H}^n_{|E}$ implies the n-rectifiability of E).

Given a Borel measure μ on $\mathbb{R}^d$, its upper and lower n-dimensional densities are defined, respectively, by

$$\Theta^{n,*}_\mu(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^n_{\mu,*}(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^n}.$$

So (1.1) means that the lower n-dimensional densities with respect to $\mathcal{H}^n_{|E}$ is positive $\mathcal{H}^n$-a.e. in E. We recall that if $\mathcal{H}^n(E) < \infty$, then

$$0 < \Theta^{n,*}_{\mathcal{H}^n_{|E}}(x) < \infty \qquad \mathcal{H}^n\text{-a.e. } x \in E.$$

However there are sets E with $0 < \mathcal{H}^n(E) < \infty$ such that the lower density $\Theta^n_{\mathcal{H}^n_{|E},*}(x)$ vanishes for every $x \in E$ (see [Ma1, Chapter 6], for example).

Theorem 1.1 is a particular case of the following somewhat stronger result.

Theorem 1.2. *Let μ be a finite Borel measure on $\mathbb{R}^d$. Let $E \subset \mathbb{R}^d$ be such that for all $x \in E$ we have*

$$0 < \Theta^{n,*}_\mu(x) < \infty \quad \text{and} \quad \exists\, \text{p.v.}R^n\mu(x).$$

Then E is n-rectifiable.

Our arguments to prove Theorems 1.1 and 1.2 are very different from the ones in [MPr] and [Ma2], which are based on the use of tangent measures. A fundamental step in our proof consists in obtaining precise L^2 estimates of Riesz transforms on Lipschitz graphs. In a sense, these L^2 estimates play a role analogous to curvature of measures in [To1]. Loosely speaking, the second step of the proof consists of using these L^2 estimates to construct a Lipschitz graph containing a suitable piece of E, by arguments more or less similar to the ones in [Lé2].

To describe in detail the L^2 estimates mentioned above we need to introduce some additional terminology. We denote the projection

$$(x_1, \ldots, x_n, \ldots, x_d) \mapsto (x_1, \ldots, x_n, 0, \cdots, 0)$$

by Π, and we set $\Pi^\perp = I - \Pi$. We also denote

$$R^{n,\perp}\mu(x) = \Pi^\perp(R^n\mu(x)) \quad \text{and} \quad R^{n,\perp}_\varepsilon\mu(x) = \Pi^\perp(R^n_\varepsilon\mu(x)).$$

That is to say, $R^{n,\perp}\mu(x)$ and $R^{n,\perp}_\varepsilon\mu(x)$ are made up of the components of $R^n\mu(x)$ and $R^n_\varepsilon\mu(x)$ orthogonal to $\mathbb{R}^n$, respectively (we are identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{(0, \ldots, 0)\}$).

Theorem 1.3. *Consider the n-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $d\mu(z) = g(z)\, d\mathcal{H}^n_{|\Gamma}(z)$, where $g(\cdot)$ is a function such that $C_1^{-1} \leq g(z) \leq C_1$ for all $z \in \Gamma$. Suppose that A has compact support. If $\|g-1\|_2 \leq C_2\|\nabla A\|_2$ and $\|\nabla A\|_\infty \leq \varepsilon_0$, with $0 < \varepsilon_0 < 1$ small enough (depending on C_2), then we have*

$$\|\text{p.v.}R^{n,\perp}\mu\|_{L^2(\mu)} \approx \|\text{p.v.}R^n\mu\|_{L^2(\mu)} \approx \|\nabla A\|_2.$$

Let us remark that the existence of the principal values p.v.$R^n\mu$ μ-a.e. under the assumptions of the theorem is a well know fact. If we take $g(x) \equiv 1$, we obtain:

Corollary 1.4. *Consider the n-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $\mu = \mathcal{H}^n_{|\Gamma}$. Suppose that A has compact support. If $\|\nabla A\|_\infty \leq \varepsilon_0$, with $0 < \varepsilon_0 \leq 1$ small enough, then*

$$\|\text{p.v.}R^{n,\perp}\mu\|_{L^2(\mu)} \approx \|\text{p.v.}R^n\mu\|_{L^2(\mu)} \approx \|\nabla A\|_2.$$

The upper estimate $\|\text{p.v.}R^n\mu\|_{L^2(\mu)} \lesssim \|\nabla A\|_2$ is an easy consequence of some of the results from [Do] and [To3] and also holds replacing ε_0 by any big constant (see Lemma 3.1 in Section 3 for more details). The lower estimate $\|\text{p.v.}R^{n,\perp}\mu\|_{L^2(\mu)} \gtrsim \|\nabla A\|_2$ is more difficult. To prove it we use a Fourier type estimate as well as the quasiorthogonality techniques developed in [To3]. In particular, the coefficients $\alpha(Q)$ (see Section 2 for the definition) introduced in that paper are an important tool for the proof.

We remark that we do not know if the inequalities $\|\text{p.v.}R^n\mu\|_{L^2(\mu)} \geq C_3^{-1}\|\nabla A\|_2$ or $\|\text{p.v.}R^{n,\perp}\mu\|_{L^2(\mu)} \geq C_3^{-1}\|\nabla A\|_2$ in Theorem 1.3 or Corollary

1.4 hold assuming $\|\nabla A\|_\infty \leq C_4$ instead of $\|\nabla A\|_\infty \leq \varepsilon_0$, with C_4 arbitrarily large and C_3 possibly depending on C_4.

Obtaining lower estimates for the L^2 norm of n-dimensional Riesz transforms in $\mathbb{R}^d$ is also important for other problems, such as the characterization of removable singularities for bounded analytic functions (for $n = 1$) and Lipschitz harmonic functions (for $n \geq 1$). For instance, in [MaT], in order to characterize some Cantor sets which are removable for Lipschitz harmonic functions in $\mathbb{R}^{n+1}$ first one needs to get a lower estimate of the norm $\|\text{p.v.}R^n\mu\|_{L^2(\mu)}$, where μ is the natural probability measure supported on the given Cantor set. Analogous results for bilipschitz images of Cantor sets are obtained in [GPT]. See also [ENV] for other recent results which involve lower estimates of L^2 norms of Riesz transforms, and [Da2], [To2], [Vo] for other questions on removability of singularities of bounded analytic functions and Lipschitz harmonic functions.

The plan of the paper is the following. In Section 2 we introduce some preliminary notation and state some results that will be needed in the rest of the paper. Sections 3-6 are devoted to the proof of Theorem 1.3, while Theorem 1.2 is proved in Sections 7-10 by arguments inspired in part by the corona type constructions of [Lé2] and [DS1].

2. Preliminaries

As usual, in the paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

An open ball with center x and radius x is denoted by $B(x, r)$. If we want to remark that this is an n-dimensional ball, we write $B_n(x, r)$.

Given $f \in L^1_{loc}(\mu)$, we denote $R^n_\mu(f) = R^n(f\,d\mu)$ and $R^n_{\mu,\varepsilon}(f) = R^n_\varepsilon(f\,d\mu)$. Recall also the definition of the maximal Riesz transform:

$$R^n_*\mu(x) = \sup_{\varepsilon>0} |R^n_\varepsilon\mu(x)|.$$

To simplify notation, if n is fixed, quite often we will also write $R\mu(x)$ instead of $R^n\mu(x)$, and analogously with respect to $R_\varepsilon\mu$, $R^\perp\mu$, $R_*\mu$, $R_{\mu,\varepsilon}(f)$, etc. We say that the Riesz transform operator R_μ is bounded in $L^2(\mu)$ if the truncated operators $R_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$.

Given $0 < n \leq d$, we say that a Borel measure μ on $\mathbb{R}^d$ is n-dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant C_0 such that $C_0^{-1}r^n \leq \mu(B(x, r)) \leq C_0 r^n$ for all $x \in \text{supp}(\mu)$, $0 < r \leq \text{diam}(\text{supp}(\mu))$. It is not difficult to see that such a measure μ must be of the form $d\mu = \rho\, d\mathcal{H}^n_{|\text{supp}(\mu)}$, where ρ is some positive function bounded from above and from below.

Given $E \subset \mathbb{C}$ and a cube $Q \subset \mathbb{R}^d$, we set

$$\beta_E(Q) = \inf_L \left\{ \sup_{y \in E \cap 3Q} \frac{\operatorname{dist}(y, L)}{\ell(Q)} \right\},$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$. The L^p version of β is the following,

$$\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{3Q} \left(\frac{\operatorname{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p},$$

where the infimum is taken over all n-planes in $\mathbb{R}^d$ again. In our paper we will have $E = \operatorname{supp}(\mu)$ and, to simplify notation, we will write β (or β_∞) and β_p instead of β_E and $\beta_{p,\mu}$. The definition of $\beta_p(B)$ for a ball B is analogous to the one of $\beta_p(Q)$ for a cube Q.

Remark 2.1. Consider the n-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $d\mu(z) = d\mathcal{H}^n_{|\Gamma}(z)$. Suppose that $\|\nabla A\|_\infty \leq C_5$. By [Do, Theorem 6], we have

$$\|\nabla A\|_2^2 \approx \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q) \approx \sum_{Q \in \mathcal{D}} \beta_2(Q)^2 \mu(Q),$$

with constants depending only on C_5.

Given a set $A \subset \mathbb{R}^d$ and two Borel measures σ, ν on $\mathbb{R}^d$, we set

$$\operatorname{dist}_A(\sigma, \nu) := \sup \left\{ \left| \int f \, d\sigma - \int f \, d\nu \right| : \operatorname{Lip}(f) \leq 1, \operatorname{supp}(f) \subset A \right\}.$$

Given a Borel measure μ on $\mathbb{R}^d$ and a cube Q which intersects $\operatorname{supp}(\mu)$, we consider the closed ball $B_Q := \overline{B}(z_Q, 3 \operatorname{diam}(Q))$, where z_Q and $\operatorname{diam}(Q)$ stand for the center and diameter of Q, respectively. Then we define

$$\alpha^n_\mu(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \operatorname{dist}_{B_Q}(\mu, c\mathcal{H}^n_{|L}), \tag{2.1}$$

where the infimum is taken over all the constants $c \geq 0$ and all the n-planes L. For convenience, if Q does not intersect $\operatorname{supp}(\mu)$, we set $\alpha^n_\mu(Q) = 0$. To simplify notation, sometimes we will also write $\alpha(Q)$ instead of $\alpha^n_\mu(Q)$.

We denote by c_Q and L_Q the constant and the n-plane that minimize $\operatorname{dist}_{B_Q}(\mu, \mathcal{L}_L)$ (it is easy to check that this minimum is attained). We also write $\mathcal{L}_Q := c_Q \mathcal{H}^n_{|L_Q}$, so that

$$\alpha^n_\mu(Q) = \frac{1}{\ell(Q)^{n+1}} \operatorname{dist}_{B_Q}(\mu, c_Q \mathcal{H}^n_{|L_Q}) = \frac{1}{\ell(Q)^{n+1}} \operatorname{dist}_{B_Q}(\mu, \mathcal{L}_Q).$$

Let us remark that c_Q and L_Q (and so $\mathcal{L}_Q$) may be not unique. Moreover, we may (and will) assume that $L_Q \cap B_Q \neq \varnothing$.

Recall that when μ is AD regular, one can construct some kind of dyadic lattice of cubes adapted to the measure μ. The cubes from this lattice are not true cubes, although they play the role of dyadic cubes with respect to

μ, in a sense. See [Da1, Appendix 1], for example. The definitions of $\beta_p(Q)$ and $\alpha(Q)$ are the same as above for this type of "cubes".

In [To3] it is shown that $\beta_1(Q) \leq C\alpha(Q)$ when μ is an AD regular n-dimensional measure and Q is a cube of the dyadic lattice associated to μ. The opposite inequality is false, in general.

We denote

$$\delta_\mu^n(x,r) = \frac{\mu(B(x,r))}{r^n},$$

and if $B = B(x,r)$, we set $\delta_\mu^n(B) = \delta_\mu^n(x,r)$. Sometimes, to simplify notation we will write $\delta(x,r)$ instead of $\delta_\mu^n(x,r)$.

3. Auxiliary lemmas for the proof of Theorem 1.3

3.1. **More notation and definitions.** Throughout Sections 3-6, μ stands for the measure described in the assumptions of Theorem 1.3. That is, $\mu = g\,\mathcal{H}^n_\Gamma$, where Γ is the Lipschitz graph $\{(x,y) \in \mathbb{R}^d : y = A(x)\}$. Observe that μ is AD regular.

Recall that Π is the projection $(x_1,\ldots,x_n,\ldots,x_d) \mapsto (x_1,\ldots,x_n)$. We denote $x_0 = \Pi(x) = (x_1,\ldots,x_n)$ (we identify $x_0 \in \mathbb{R}^n$ with $(x_0,0,\ldots,0) \in \mathbb{R}^d$) and, also, $x^\perp = \Pi^\perp(x) = (x_{n+1},\ldots,x_d)$.

In the particular case of a Lipschitz graph and μ as above, the construction of the dyadic lattice $\mathcal{D}$ associated to μ is very simple: let $\mathcal{D}_0$ be the lattice of the usual dyadic cubes of $\mathbb{R}^n$. A subset $Q \subset \Gamma$ is a cube from $\mathcal{D}$ if and only if it is of the form

$$Q = \Pi^{-1}(Q_0) \cap \Gamma$$

for some $Q_0 \in \mathcal{D}_0$. If $\ell(Q_0) = 2^{-j}$ (where $\ell(\cdot)$ stands for side length), we set $\ell(Q) = 2^{-j}$ and $Q \in \mathcal{D}_j$. If z_{Q_0} is the center of Q_0, then we say that $\Pi^{-1}(z_{Q_0}) \cap \Gamma$ is the center of Q. The definition of λQ, for $\lambda > 0$, is analogous.

Let ψ be a non negative radial $\mathcal{C}^\infty$ function such that $\chi_{B(0,1/8)} \leq \psi \leq \chi_{B(0,1/4)}$. For each $j \in \mathbb{Z}$, set $\psi_j(x) := \psi(2^j x)$ and $\varphi_j := \psi_j - \psi_{j+1}$, so that each function φ_j is non negative and supported on $B(0,2^{-j-2}) \setminus B(0,2^{-j-4})$, and moreover we have

$$\sum_{j\in\mathbb{Z}} \varphi_j(x) = 1 \qquad \text{for all } x \in \mathbb{R}^d \setminus \{0\}.$$

We need to consider the following vectorial kernels:

$$K_j(x) = \varphi_j(x_0)\frac{x}{|x|^{n+1}} \qquad j \in \mathbb{Z}, \tag{3.1}$$

and

$$\widetilde{K}_j(x) = \varphi_j(x_0)\frac{x}{|x_0|^{n+1}} \qquad j \in \mathbb{Z}, \tag{3.2}$$

for $x \in \mathbb{R}^d$. The operators associated to K_j and $\widetilde{K}_j$ are, respectively,

$$R_j\mu(x) = \int K_j(x-y)\,d\mu(y), \qquad \widetilde{R}_j\mu(x) = \int \widetilde{K}_j(x-y)\,d\mu(y).$$

Notice that, formally,

$$R\mu(x) = \sum_{j\in\mathbb{Z}} R_j\mu(x).$$

Moreover, abusing notation sometimes we will write $R\mu$ instead of p.v.$R\mu$. When μ is like in Theorem 1.3 this does not cause any trouble, since the μ-a.e. existence of principal values is a well known result.

Let us remark that, perhaps it would be more natural to replace $\varphi_j(x)$ by $\varphi_j(x_0)$ in the definitions of the kernels K_j and $\widetilde{K}_j$ (like in [To3]). However, for some of the calculations below the definitions above are more convenient (although the choice of $\varphi_j(x)$ instead of $\varphi_j(x_0)$ would also work with minor modifications and some additional work).

We also denote by $K_j^i(x)$ and $\widetilde{K}_j^i(x)$ the i-th component of $K_j(x)$ and $\widetilde{K}_j(x)$ respectively, and we set

$$K_j^\perp(x) = \varphi_j(x_0)\frac{x^\perp}{|x|^{n+1}}$$

and

$$\widetilde{K}_j^\perp(x) = \varphi_j(x_0)\frac{x^\perp}{|x_0|^{n+1}},$$

and we denote by $R_j^\perp$ and $\widetilde{R}_j^\perp$ the corresponding operators with kernels $K_j^\perp$ and $\widetilde{K}_j^\perp$.

3.2. The upper estimate for the L^2 norm of Riesz transforms.

Lemma 3.1. *Consider the n-dimensional Lipschitz graph $\Gamma := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$. Suppose that $\|\nabla A\|_\infty \leq C_6$ and let $d\mu(z) = g(z)\, d\mathcal{H}^n_{|\Gamma}(z)$, where $g(\cdot)$ is a function such that $C_1^{-1} \leq g(z) \leq C_1$ for all $z \in \Gamma$. Then we have*

$$\|\text{p.v.}R\mu\|_{L^2(\mu)} \lesssim \|\nabla A\|_2 + \|g-1\|_2,$$

with constants depending on C_6 and C_1.

Proof. By Theorem 1.2 in [To3] we have

$$\|\text{p.v.}R\mu\|^2_{L^2(\mu)} \lesssim \sum_{Q\in\mathcal{D}} \alpha(Q)^2\mu(Q),$$

and by Theorem 1.1 and Remark 4.1 in the same paper,

$$\sum_{Q\in\mathcal{D}} \alpha(Q)^2\mu(Q) \lesssim \sum_{Q\in\mathcal{D}} \beta_1(Q)^2\mu(Q) + \|g-1\|_2^2.$$

By [Do, Theorem 6] we have

$$\sum_{Q\in\mathcal{D}} \beta_1(Q)^2\mu(Q) \approx \|\nabla A\|_2^2,$$

and so the lemma follows. □

3.3. **Auxiliary lemmas for the lower estimate.** In the following lemma we collect a pair of trivial estimates. The easy proof is left for the reader.

Lemma 3.2. *Denote $\delta = 2^{-j}$. For all $x \in \mathbb{R}^d$ and all $1 \leq i \leq n+1$, we have*

$$|K_j^i(x)| \lesssim \frac{|x_i|}{\delta^{n+1}}\,\chi_{A(0,\delta/3,3\delta)},$$

and

$$|\nabla K_j^i(x)| \lesssim \frac{1}{\delta^{n+1}}\,\chi_{A(0,\delta/3,3\delta)}.$$

Notice that

$$\big||x| - |x_0|\big| \leq \frac{|x^\perp|^2}{|x|}.$$

From this estimate and easy calculations, one gets

Lemma 3.3. *Denote $\delta = 2^{-j}$. For $x \in \mathbb{R}^d$ such that $|x| \approx |x_0|$, and for $1 \leq i \leq d$, we have*

$$|K_j^i(x) - \widetilde{K}_j^i(x)| \lesssim \frac{|x_i||x^\perp|^2}{\delta^{n+3}}\,\chi_{A(0,\delta/16,\delta)},$$

and

$$\big|\nabla\big(K_j^i - \widetilde{K}_j^i\big)(x)\big| \lesssim \frac{|x^\perp|^2}{\delta^{n+3}}\,\chi_{A(0,\delta/16,\delta)}.$$

The proof is left for the reader again.

Lemma 3.4. *For all $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}_j$, we have*

$$\int_Q |R_j\mu|^2\,d\mu \lesssim \big[\beta_2(Q)^2 + \alpha(Q)^2\big]\mu(Q). \tag{3.3}$$

Also, if D_Q is the line that minimizes $\beta_1(Q)$ and

$$\beta_\infty(Q) \leq \varepsilon_2 \quad \text{and} \quad \sin \measuredangle(D_0, D_Q) \leq \varepsilon_2,$$

with ε_2 small enough, then

$$\int_Q |R_j\mu - \widetilde{R}_j\mu|^2\,d\mu \lesssim \varepsilon_2^4\big[\beta_2(Q)^2 + \alpha(Q)^2\big]\mu(Q). \tag{3.4}$$

Proof. The estimate (3.3) has been proved in [To3, Lemma 5.1]. The inequality (3.4) has a quite similar proof. For completeness, we show the detailed arguments. Consider the kernel $D_j = K_j - \widetilde{K}_j$, and let T_j be the operator associated to D_j.

Denote by D_Q the line that minimizes $\beta_1(Q)$ and let L_Q the one that minimizes $\alpha(Q)$. From the fact that $\beta_1(Q) \lesssim \alpha(Q)$ it easily follows that

$$\operatorname{dist}_H(L_Q \cap B_Q, D_Q \cap B_Q) \lesssim \alpha(Q)\ell(Q), \tag{3.5}$$

where dist_H stands for Hausdorff distance. Take $x \in Q \subset \Gamma$. Consider the orthogonal projection x' of x onto D_Q. Since we are assuming that $\beta_\infty(Q)$ is very small we have $|x - x'| \ll \operatorname{diam}(Q)$ and then $\operatorname{supp}(D_j(x' - \cdot)) \subset B_Q$.

First we will estimate $T_j\mu(x')$. Let U be a thin tubular neighborhood of $D_Q \cap B_Q$ of width $\leq C\varepsilon_2 \mathrm{diam}(Q)$ containing $\mathrm{supp}(\mu)\cap B_Q$ and denote $f(y) = D_j(x'-y)$. Notice that for $y \in U \cap \mathrm{supp}(D_j(x'-y)$ we have $|x'-y| \approx |x'_0 - y_0|$, and so by Lemma 3.3, for these y's,

$$|\nabla f(y)| = \big|\nabla D_j(x'-y)\big| \lesssim \frac{|x'^{\perp} - y^{\perp}|^2}{\ell(Q)^{n+3}}.$$

We have

$$|x'^{\perp} - y^{\perp}| \lesssim \ell(Q)\big(\beta_\infty(Q) + \sin \measuredangle(D_0, D_Q)\big),$$

where D_0 stands for the n-plane $D_0 = \mathbb{R}^n \times (0,\dots 0)$, and so we get

$$\big|\nabla f(y)\big| \lesssim \frac{\varepsilon_2^2}{\ell(Q)^{n+1}}. \tag{3.6}$$

We extend $f_{|U\cap B_Q}$ to a function $\widetilde{f}$ supported on B_Q with $\|\nabla \widetilde{f}\|_\infty \lesssim \varepsilon_2^2/\ell(Q)^{n+1}$. Since $K_j(\cdot)$ is odd and $x' \in D_Q$, we have $\int D_j(x'-y)\, d\mathcal{H}^n_{|D_Q}(y) = 0$, and so

$$\begin{aligned}
\left|\int D_j(x'-y)\, d\mu(y)\right| &= \left|\int D_j(x'-y)\, d\mu(y) - c_Q \int D_j(x'-y)\, d\mathcal{H}^n_{|D_Q}(y)\right| \\
&= \left|\int \widetilde{f}(y)\, d\mu(y) - c_Q \int \widetilde{f}(y)\, d\mathcal{H}^n_{|D_Q}(y)\right| \\
&\lesssim \frac{\varepsilon_2^2}{\ell(Q)^{n+1}}\, \mathrm{dist}_{B_Q}(\mu, c_Q\mathcal{H}^n_{|D_Q}).
\end{aligned}$$

In these estimates c_Q stands for the constant minimizing the definition of $\alpha(Q)$. By the definition of $\alpha(Q)$ and (3.5) one easily gets

$$\mathrm{dist}_{B_Q}(\mu, c_Q\mathcal{H}^n_{|D_Q}) \lesssim \alpha(Q)\ell(Q)^{n+1}.$$

Thus, $|T_j\mu(x')| \lesssim \varepsilon_2^2\alpha(Q)$.

Now we turn our attention to $T_j\mu(x)$. We have

$$|T_j\mu(x) - T_j\mu(x')| \lesssim |x-x'| \sup_{\xi\in[x,x']} |\nabla T_j\mu(\xi)|.$$

By an estimate analogous to (3.6) we have

$$|\nabla T_j\mu(\xi)| \leq \int \big|\nabla D_j(\xi - y)\big| d\mu(y) \lesssim \frac{\varepsilon_2^2}{\ell(Q)},$$

since $|\xi - y| \approx |\xi_0 - y_0|$ for $\xi \in [x,x']$ and $y \in \mathrm{supp}(\mu)\cap \mathrm{supp}(D_j(\xi-\cdot))$. Therefore,

$$|T_j\mu(x) - T_j\mu(x')| \lesssim \frac{\varepsilon_2^2\, \mathrm{dist}(x, D_Q)}{\ell(Q)},$$

and so

$$|T_j\mu(x)| \lesssim \frac{\varepsilon_2^2 \mathrm{dist}(x,D_Q)}{\ell(Q)} + |T_j\mu(x')| \lesssim \varepsilon_2^2\Big(\frac{\mathrm{dist}(x,D_Q)}{\ell(Q)} + \alpha(Q)\Big).$$

The lemma is a direct consequence of this estimate. □

From the preceding result we get the following.

Lemma 3.5. *For $j \in \mathbb{Z}$, let us denote*

$$\beta_{2,j}(\Gamma)^2 := \sum_{Q \in \mathcal{D}_j} \beta_2(Q)^2 \mu(Q) \quad \text{and} \quad \alpha_j(\Gamma)^2 := \sum_{Q \in \mathcal{D}_j} \alpha(Q)^2 \mu(Q).$$

Suppose that

$$\beta_\infty(Q) \leq \varepsilon_2 \quad \text{and} \quad \operatorname{sin} \measuredangle(D_0, D_Q) \leq \varepsilon_2,$$

where D_Q is the line that minimizes $\beta_1(Q)$ and ε_2 is small enough. We have

$$\big|\langle R_j^\perp \mu,\, R_k^\perp \mu\rangle - \langle \widetilde{R}_j^\perp \mu,\, \widetilde{R}_k^\perp \mu\rangle\big| \lesssim \varepsilon_2^2 \big(\beta_{2,j}(\Gamma) + \alpha_j(\Gamma)\big)\big(\beta_{2,k}(\Gamma) + \alpha_k(\Gamma)\big).$$

Proof. We set

$$\begin{aligned}
\big|\langle R_j^\perp \mu,\, R_k^\perp \mu\rangle &- \langle \widetilde{R}_j^\perp \mu,\, \widetilde{R}_k^\perp \mu\rangle\big| \\
&\leq \big|\langle R_j^\perp \mu - \widetilde{R}_j^\perp \mu,\, R_k^\perp \mu\rangle\big| + \big|\langle \widetilde{R}_j^\perp \mu,\, R_k^\perp \mu - \widetilde{R}_k^\perp \mu\rangle\big| \\
&\leq \|R_j^\perp \mu - \widetilde{R}_j^\perp \mu\|_2 \|R_k^\perp \mu\|_2 + \|\widetilde{R}_j^\perp \mu - R_j^\perp \mu\|_2 \|R_k^\perp \mu - \widetilde{R}_k^\perp \mu\|_2 \\
&\quad + \|R_j^\perp \mu\|_2 \|R_k^\perp \mu - \widetilde{R}_k^\perp \mu\|_2.
\end{aligned}$$

If we plug the estimates (3.3) and (3.4) into the preceding inequality, the lemma follows. □

4. The key Fourier estimate

Consider the image measure $\sigma := \Pi_\# \mu$ on $\mathbb{R}^n$ and set

$$H_j(x_0, y_0) = \varphi_j(x_0 - y_0) \frac{A(x_0) - A(y_0)}{|x_0 - y_0|^{n+1}}.$$

We have

$$\begin{aligned}
\langle \widetilde{R}_j^\perp \mu, \widetilde{R}_k^\perp \mu\rangle &= \iiint \widetilde{K}_j^\perp(x,y) \widetilde{K}_k^\perp(x,z) d\mu(x) d\mu(y) d\mu(z) \\
&= \iiint H_j(x_0, y_0) H_k(x_0, z_0) d\sigma(x_0) d\sigma(y_0) d\sigma(z_0) =: I_0
\end{aligned} \tag{4.1}$$

Below we will calculate I_0 using the Fourier transform in the special case in which σ coincides with the Lebesgue n-dimensional measure on $\mathbb{R}^n$. This will allow us to prove Theorem 1.3 in this particular situation. The full theorem will follow easily from this case.

Lemma 4.1. *Let us denote $\delta = 2^{-j}$, $\varepsilon = 2^{-k}$, and assume $\delta \leq \varepsilon$. We have*

(4.2)

$$\begin{aligned}
0 \leq \iiint_{(\mathbb{R}^n)^3} H_j(x,y) H_k(x,z)\, dx dy dz \approx \delta\varepsilon &\int_{|\xi| \leq 1/\varepsilon} |\widehat{A}(\xi)|^2 |\xi|^4 d\xi \\
&+ \frac{\delta}{\varepsilon} \int_{1/\varepsilon \leq |\xi| \leq 1/\delta} |\widehat{A}(\xi)|^2 |\xi|^2 d\xi \\
&+ \frac{1}{\delta\varepsilon} \int_{|\xi| \geq 1/\delta} |\widehat{A}(\xi)|^2 d\xi.
\end{aligned}$$

Proof. For $x \in \mathbb{R}^n$, we denote $\eta(x) = \varphi(x)/|x|^{n+1}$. Notice that

$$\frac{\varphi_j(x)}{|x|^{n+1}} = \frac{1}{\delta^{n+1}}\,\eta\Big(\frac{x}{\delta}\Big) =: \frac{1}{\delta}\eta_\delta(x),$$

and analogously for $\varphi_k(x)/|x|^{n+1}$. By the change of variables $y = x + s$, $z = x + t$, and by Plancherel the triple integral on the left hand side of (4.2) equals

$$\begin{aligned} I_0 &:= \iiint \left(\varphi_j(x-y)\frac{A(x)-A(y)}{|x-y|^{n+1}}\right)\left(\varphi_k(x-z)\frac{A(x)-A(z)}{|x-z|^{n+1}}\right)dxdydz \\ &= \frac{1}{\delta\varepsilon}\iiint \eta_\delta(s)\big(A(x)-A(x+s)\big)\eta_\varepsilon(t)\big(A(x)-A(x+t)\big)dxdsdt \\ &= \frac{1}{\delta\varepsilon}\iiint |\widehat{A}(\xi)|^2(1-e^{-2\pi i\xi s})\eta_\delta(s)\overline{(1-e^{-2\pi i\xi t})\eta_\varepsilon(t)}d\xi dsdt. \end{aligned}$$

By Fubini, taking Fourier transform (for the s and t variables), we get

$$I_0 = \frac{1}{\delta\varepsilon}\iiint |\widehat{A}(\xi)|^2\big(\widehat{\eta}(0)-\widehat{\eta}(\delta\xi)\big)\overline{\big(\widehat{\eta}(0)-\widehat{\eta}(\varepsilon\xi)\big)}d\xi.$$

Let

$$f_\delta(\xi) := \frac{1}{\delta}\big(\widehat{\eta}(0)-\widehat{\eta}(\delta\xi)\big).$$

It is easy to check that $f_\delta(\xi)$ is real and positive for $\xi \neq 0$[1]. Moreover, using that $\widehat{\eta}$ is radial and $\widehat{\eta} \in \mathcal{S}$, we get $f_\delta(\xi) \approx C\delta|\xi|^2$ as $\xi \to 0$, and $f_\delta(\xi) \approx C/\delta$ as $|\xi| \to \infty$. So we infer that

$$f_\delta(\xi) \approx \delta|\xi|^2 \quad \text{if } |\xi| \leq \frac{1}{\delta}, \qquad \text{and} \qquad f_\delta(\xi) \approx \frac{1}{\delta} \quad \text{if } |\xi| \geq \frac{1}{\delta}.$$

Analogous estimates hold for the corresponding function $f_\varepsilon(\xi)$. Therefore,

$$\begin{aligned} I_0 \approx \delta\varepsilon \int_{|\xi|\leq 1/\varepsilon} |\widehat{A}(\xi)|^2|\xi|^4 d\xi + \frac{\delta}{\varepsilon}\int_{1/\varepsilon\leq|\xi|\leq 1/\delta} |\widehat{A}(\xi)|^2|\xi|^2 d\xi \\ + \frac{1}{\delta\varepsilon}\int_{|\xi|\geq 1/\delta} |\widehat{A}(\xi)|^2 d\xi. \end{aligned} \tag{4.3}$$

□

5. Proof of Theorem 1.3 in the particular case $d\sigma \equiv dx$

We will need the following result from [To3] (it is not stated explicitly there, although it is proved in the paper):

[1]This follows from the fact that

$$\widehat{\eta}(0) = \int \eta(s)ds > \int \cos(2\pi\xi s)\eta(s)ds = \widehat{\eta}(\xi)$$

for all $\xi \neq 0$, since η is a non negative radial function from $\mathcal{S}$.

Theorem 5.1. *Let μ be an n-dimensional AD regular measure. For any positive integer N_0, we have*

$$\sum_{j,k:|j-k|>N_0} \left|\langle R_j^\perp \mu, R_k^\perp \mu\rangle\right| \leq C 2^{-N_0/4} \sum_{Q\in\mathcal{D}} \alpha(Q)^2 \mu(Q). \tag{5.1}$$

Moreover, under the assumptions of Theorem 1.3, if $\Pi\#\mu = \rho(x)\,dx$, we have

$$\sum_{Q\in\mathcal{D}} \alpha(Q)^2 \mu(Q) \lesssim \sum_{Q\in\mathcal{D}} \beta_1(Q)^2 \mu(Q) + \|\rho - 1\|_2^2. \tag{5.2}$$

Let us remark that in [To3] the preceding result has been proved with $\varphi_j(x)$ replacing $\varphi_j(x_0)$ in the definition of the kernel K_j in (3.1). However, it is easy to check that all the estimates of [To3] work with the slightly different definition in (3.1) when μ is supported on a Lipschitz graph.

Proof of Theorem 1.3 in the particular case $d\sigma \equiv dx$. By Lemma 3.1 we only need to prove the lower estimate $\|\text{p.v.} R^\perp\mu\|_{L^2(\mu)} \gtrsim \|\nabla A\|_2$. We set

$$\|R^\perp\mu\|_{L^2(\mu)}^2 = \sum_{j,k:|j-k|\leq N_0} \langle R_j^\perp \mu,\, R_k^\perp \mu\rangle + \sum_{j,k:|j-k|>N_0} \langle R_j^\perp \mu,\, R_k^\perp \mu\rangle =: S_1 + S_2.$$

In this identity $R^\perp\mu$ can be understood either as the principal value or as an $L^2(\mu)$ limit. We will show that if ε_0 is small enough, then

$$S_1 \approx \sum_{Q\in\mathcal{D}} \beta_2(Q)^2 \mu(Q)$$

(with constants depending on N_0), while $|S_2| \leq S_1/2$. The theorem follows from these estimates.

The inequality

$$S_1 \lesssim \sum_{Q\in\mathcal{D}} \beta_2(Q)^2 \mu(Q)$$

is a direct consequence of (3.3), (5.2), and the fact that $\rho \equiv 1$. Now we consider the converse estimate. We denote

$$T_j f(x) = \int_{\mathbb{R}^n} H_j(x,y) f(y) dy.$$

By (4.1) we have

$$\langle \widetilde{R}_j^\perp \mu,\, \widetilde{R}_k^\perp \mu\rangle = \langle T_j 1,\, T_k 1\rangle_{\mathbb{R}^n}.$$

Then we set

$$\begin{aligned}\langle R_j^\perp \mu,\, R_k^\perp \mu\rangle &= \langle T_j 1,\, T_k 1\rangle_{\mathbb{R}^n} + \big(\langle R_j^\perp \mu,\, R_k^\perp \mu\rangle - \langle \widetilde{R}_j^\perp \mu,\, \widetilde{R}_k^\perp \mu\rangle\big)\\ &=: \langle T_j 1,\, T_k 1\rangle_{\mathbb{R}^n} + E_{j,k}.\end{aligned}$$

By Lemma 4.1, since $\langle T_j\mu, T_k\mu\rangle_{\mathbb{R}^n} \geq 0$, we have

(5.3)
$$\sum_{j,k:|j-k|\leq N_0} \langle T_j 1,\, T_k 1\rangle_{\mathbb{R}^n} \geq \sum_{j\in\mathbb{Z}} \|T_j 1\|_2^2 \gtrsim \sum_{j\in\mathbb{Z}} \int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} |\widehat{A}(\xi)|^2|\xi|^2\,d\xi$$
$$\approx \|\nabla A\|_2^2 \approx \sum_{Q\in\mathcal{D}} \beta_2(Q)^2\mu(Q).$$

We consider now the terms $E_{j,k}$. Since $\|\nabla A\|_\infty \leq \varepsilon_0$, we infer that $\beta(Q) \lesssim \varepsilon_0$, and then from Lemma 3.5 if ε_0 is small enough we deduce

$$\sum_{j,k:|j-k|\leq N_0} |E_{j,k}| \lesssim \varepsilon_0^2 \sum_{j,k:|j-k|\leq N_0} \big(\beta_{2,j}(\Gamma) + \alpha_j(\Gamma)\big)\big(\beta_{2,k}(\Gamma) + \alpha_k(\Gamma)\big)$$
$$\lesssim N_0\varepsilon_0^2 \sum_{Q\in\mathcal{D}} \big(\alpha(Q)^2 + \beta_2(Q)^2\big)\mu(Q).$$

From (5.2) we obtain

$$\sum_{j,k:|j-k|\leq N_0} |E_{j,k}| \lesssim N_0\varepsilon_0^2 \sum_{Q\in\mathcal{D}} \beta_2(Q)^2\mu(Q). \tag{5.4}$$

By the estimates (5.3) and (5.4), if ε_0 is small enough (for a given N_0), we infer that

$$S_1 \gtrsim \sum_{Q\in\mathcal{D}} \beta_2(Q)^2\mu(Q). \tag{5.5}$$

Finally we turn our attention to S_2. By Theorem 5.1 we have

$$|S_2| \lesssim 2^{-N_0/4} \sum_{Q\in\mathcal{D}} \beta_1(Q)^2\mu(Q).$$

Therefore, by (5.5), $S_2 \leq C2^{-N_0/4}S_1 \leq S_1/2$ if N_0 is big enough. We are done. □

6. Proof of Theorem 1.3 in full generality

Lemma 6.1. *Consider the n-dimensional Lipschitz graph $\Gamma := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, with $\|\nabla A\|_\infty \leq C_8$, and let μ be supported on Γ such that $d\Pi_\#\mu(x) = dx$. Then $R_\mu^\perp$ is bounded in $L^2(\mu)$ with*

$$\|R_\mu^\perp\|_{L^2(\mu),L^2(\mu)} \leq C_9\|\nabla A\|_\infty,$$

with C_9 depending only on C_8.

Proof. We think that this is essentially known. However, for completeness we give some details of the proof. Consider the kernel

$$K(x,y) = \frac{A(x) - A(y)}{\big(|x-y|^2 + |A(x)-A(y)|^2\big)^{(n+1)/2}},$$

and the associated Calderón-Zygmund operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(x)\, dx,$$

for $f \in L^2(\mathbb{R}^n)$. When $n = 1 = d - 1$, we have the expansion

$$K(x,y) = \sum_{j=1}^{\infty} (-1)^j \frac{(A(x) - A(y))^{2j-1}}{|x-y|^{2j}} = \sum_{j=1}^{\infty} K_j(x,y),$$

and the corresponding associated operators are the Calderón commutators C_j. It is well known that

$$\|C_j\|_{2,2} \leq C^{2j} \|\nabla A\|_\infty^{2j-1}$$

(see [Da1, p.50], for example), and so if $\|\nabla A\|_\infty$ is small enough the lemma follows.

For other n's and d's the result also holds. For example, it can be deduced from [To3]: if A is supported on a cube Q, then we have

$$\|R^\perp \mu\|_2 \lesssim \|\nabla A\|_2 \leq \|\nabla A\|_\infty \mu(Q)^{1/2}.$$

By a localization argument, one can prove that for any cube P,

$$\|R^\perp(\chi_P \mu)\|_2 \lesssim \|\nabla A\|_\infty \mu(P)^{1/2},$$

and then by the $T1$ theorem the lemma follows (taking into account that the Calderón-Zygmund constants involved in the kernel $K(x,y)$ are bounded above by $\|\nabla A\|_\infty$ too). □

Remark 6.2. Consider the function $\widetilde{A} : \mathbb{R}^n \to \mathbb{R}^d$ given by $\widetilde{A}(x) = (x, A(x))$, where A is the Lipschitz function that defines the Lipschitz graph Γ. Notice that the density function $\rho(x)$ such that $\Pi_\# \mu = \rho(x) dx$ is given by

$$\rho(x) = g(x)\, J\widetilde{A}(x),$$

where $J\widetilde{A}(x)$ stands for the n-dimensional Jacobian of $\widetilde{A}$. Recall that

$$J\widetilde{A}(x) = \Big(\sum_B (\det B)^2\Big)^{1/2},$$

where the sum runs over all the $n \times n$ submatrices B of $D\widetilde{A}(x)$, the differential map of $\widetilde{A}$ at x (see [Mo, p. 24], for example). Then it is easy to check that

$$(J\widetilde{A}(x))^2 = 1 + e(x),$$

with

$$|e(x)| \lesssim \sup_{i,j} |\partial_i A_j(x)|^2$$

(in fact, $e(x) = \sum_{i,j} (\partial_i A_j(x))^2 + \ldots$, where "..." stands for some terms which involve higher order products of derivatives of A). So we also have

$$J\widetilde{A}(x) = 1 + e_0(x),$$

with

$$|e_0(x)| \lesssim \sup_{i,j} |\partial_i A_j(x)|^2.$$

As a consequence,

$$|\rho(x) - 1| = |g(x)(1 + e_0(x)) - 1| \leq |g(x) - 1| + C|e_0(x)|.$$

Observe that $\|e_0\|_\infty \lesssim \|\nabla A\|_\infty^2 \leq \varepsilon_0^2$ and $\|e_0\|_2 \lesssim \|\nabla A\|_\infty \|\nabla A\|_2$. Then the assumptions of Theorem 1.3 ensure that

$$\|\rho - 1\|_2 \leq \|g - 1\|_2 + C\|e_0\|_2 \leq C\|\nabla A\|_2. \tag{6.1}$$

Proof of Theorem 1.3. Recall that we only need to prove the lower estimate $\|\text{p.v.} R^\perp \mu\|_{L^2(\mu)} \gtrsim \|\nabla A\|_2$. Consider the measure μ_0 supported on Γ such that $\Pi_\# \mu_0 = dx$. Recall that

$$\|R^\perp \mu_0\|_{L^2(\mu_0)} \approx \|\nabla A\|_2. \tag{6.2}$$

Since

$$\Pi_\# \mu = g(x)\, J\widetilde{A}(x)\, dx =: \rho(x)\, dx, \qquad x \in \mathbb{R}^n,$$

it turns out that

$$\rho(\Pi(x))\, d\mu_0(x) = d\mu(x), \qquad x \in \mathbb{R}^d.$$

We denote $h(x) = \rho(\Pi(x))$, and so we have

$$d\mu(x) - d\mu_0(x) = (h(x) - 1)\, d\mu_0(x),$$

with $\|h - 1\|_{L^2(\mu_0)} \lesssim \|\nabla A\|_2$, by (6.1). So, from Lemma 6.1 we deduce

$$\begin{aligned} \big|\|R^\perp \mu\|_{L^2(\mu_0)} - \|R^\perp \mu_0\|_{L^2(\mu_0)}\big| &\leq \|R^\perp \mu - R^\perp \mu_0\|_{L^2(\mu_0)} \\ &= \|R^\perp((h-1)\, d\mu_0)\|_{L^2(\mu_0)} \\ &\lesssim \|\nabla A\|_\infty \|h - 1\|_{L^2(\mu_0)} \leq \varepsilon_0 \|\nabla A\|_2. \end{aligned}$$

If ε_0 is small enough, from (6.2) we infer that

$$\|R^\perp \mu\|_{L^2(\mu_0)} \approx \|R^\perp \mu_0\|_{L^2(\mu_0)} \approx \|\nabla A\|_2,$$

which implies that

$$\|R^\perp \mu\|_{L^2(\mu)} \approx \|\nabla A\|_2,$$

since $g(x) \approx h(x) \approx 1$ for all x. □

7. The Main Lemma for the proof of Theorem 1.2

This and the remaining sections are devoted to the proof of Theorem 1.2. For $\varepsilon > 0$ we denote

$$\widetilde{R}_\varepsilon \mu(x) \int \frac{x - y}{\big(|x - y|^2 + \varepsilon^2\big)^{(n+1)/2}}\, d\mu(y),$$

and also

$$\widehat{R}_\varepsilon \mu(x) = \int \psi(\varepsilon^{-1}(x - y)) \frac{x - y}{|x - y|^{n+1}}\, d\mu(y),$$

where ψ is a $\mathcal{C}^\infty$ radial function such that $\chi_{\mathbb{R}^d\setminus B(0,1)} \leq \psi \leq \chi_{\mathbb{R}^d\setminus B(0,1/2)}$. We also set

$$\widetilde{R}_{\varepsilon_1,\varepsilon_2}\mu(x) = \widetilde{R}_{\varepsilon_1}\mu(x) - \widetilde{R}_{\varepsilon_2}\mu(x),$$

and

$$\widehat{R}_{\varepsilon_1,\varepsilon_2}\mu(x) = \widehat{R}_{\varepsilon_1}\mu(x) - \widehat{R}_{\varepsilon_2}\mu(x).$$

It is easy to check that if p.v.$R\mu(x)$ exists for some $x \in \mathbb{R}^d$, then

$$\lim_{\varepsilon\to 0} \widetilde{R}_\varepsilon\mu(x) = \lim_{\varepsilon\to 0} \widehat{R}_\varepsilon\mu(x) = \lim_{\varepsilon\to 0} R_\varepsilon\mu(x).$$

(Hint: write $\widetilde{R}_\varepsilon\mu(x)$ and $\widehat{R}_\varepsilon\mu(x)$ as a convex combination of $R_\varepsilon\mu(x)$, $\varepsilon > 0$. We also denote $c_n = \mathcal{L}^n(B_n(0,1))$, where $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure.

Theorem 1.2 is a consequence of the following result.

Main Lemma 7.1. *Let μ be a finite Borel measure on $\mathbb{R}^d$. Let $B_0 = \overline{B}(x_0, r_0)$ be a closed ball such that there exists a compact subset $F \subset 10B_0$, with $x_0 \in F$, which satisfies*

(a) $\mu(8B_0) = c_n 8^n r_0^n$ *and* $\mu(10B_0 \setminus F) \leq \delta_1\mu(B_0)$,
(b) $\mu(B(x,r)) \leq M_1 r^n$ *for all* $x \in F, r > 0$, *and* $\mu(B(x,r)) \leq c_n(1+\delta_1)r^n$ *for all* $x \in F$ *and* $0 < r \leq 100r_0$,
(c) $\|R_\mu\|_{L^2(\mu|F),L^2(\mu|F)} \leq M_2$,
(d) $|\widetilde{R}_{\varepsilon_1,\varepsilon_2}\mu(x)| + |\widehat{R}_{\varepsilon_1,\varepsilon_2}\mu(x)| \leq \delta_2$ *for all* $x \in F$ *and* $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_2^{-2} r_0$.

If δ_1, δ_2 are small enough, with $\delta_1 = \delta_1(M_2)$ and $\delta_2 = \delta_2(M_1, M_2)$, then there exists an n-dimensional Lipschitz graph Γ such that

$$\mu(\Gamma \cap F \cap B_0) \geq \frac{9}{10}\, c_n r_0^n.$$

Let us remark that the Lipschitz constant of the graph Γ depends on the constants M_1, M_2 and δ_1, δ_2, and tends to 0 as $\delta_1 + \delta_2 \to 0$, for fixed M_1, M_2.

Proof of Theorem 1.2 using Main Lemma 7.1. Consider an arbitrary subset $\widetilde{E} \subset E$. Given $\delta > 0$, for each $i \in \mathbb{Z}$ set

$$E_i = \{x \in \widetilde{E} : (1+\delta)^i \leq \Theta_\mu^{n,*}(x) < (1+\delta)^{i+1}\},$$

so that $\mu(\widetilde{E} \setminus \bigcup_i E_i) = 0$. For $j \geq 1$, denote

$$E_{i,j} = \{x \in E_i : \delta_\mu^n(x,r) \leq (1+\delta)^{i+2} \text{ if } 0 < r \leq 1/j,\}.$$

Notice that for all $x \in E_{i,j}$ we have

$$\mu(B(x,r)) \leq M_{i,j} r^n \qquad \text{for all } r > 0 \text{ and some fixed } M_{i,j}.$$

From the fact that $R_*\mu(x) < \infty$ on E, arguing as in [To1], we can split each set $E_{i,j}$ as

$$E_{i,j} = \bigcup_{k\geq 1} E_{i,j,k},$$

so that, for each k,

$$\|R_{\mu_{|E_{i,j,k}}}\|_{L^2(\mu_{|E_{i,j,k}}),L^2(\mu_{|E_{i,j,k}})} \leq k.$$

Given any constant $\varepsilon_0 > 0$, for each $m \geq 1$ we set

$$E_{i,j,k,m} = \Big\{x \in E_{i,j,k} : \sup_{0<\varepsilon_1<\varepsilon_2\leq 1/m} \big(|\widetilde{R}_{\varepsilon_1,\varepsilon_2}\mu(x)| + |\widehat{R}_{\varepsilon_1,\varepsilon_2}\mu(x)|\big) \leq \varepsilon_0\Big\}.$$

It is clear that

$$\widetilde{E} = \bigcup_{i,j,k,m} E_{i,j,k,m}.$$

Consider $\widetilde{E}_{i,j,k,m} \subset E_{i,j,k,m}$ such that $\widetilde{E}_{i,j,k,m} \cap \widetilde{E}_{i',j',k',m'} = \varnothing$ if $(i,j,k,m) \neq (i',j',k',m')$ and we still have

$$\widetilde{E} = \bigcup_{i,j,k,m} \widetilde{E}_{i,j,k,m}.$$

For each density point x of $\widetilde{E}_{i,j,k,m}$ consider a ball $B_x = B(x, r_x)$ with radius $0 < r_x \leq \min(1/(100j),\, \varepsilon_0/m)$ such that

$$\mu(B_x \setminus \widetilde{E}_{i,j,k,m}) \leq \delta\, \mu(\widetilde{E}_{i,j,k,m})$$

and

$$(1+\delta)^{i-1} \leq \delta_\mu^n(x, r_x) \leq (1+\delta)^{i+2}.$$

If we take δ and ε_0 small enough, we set $F := \widetilde{E}_{i,j,k,m}$, and we apply Main Lemma 7.1 to the measure $\frac{c_n\, r_x^n}{\mu(B_x)}\, \mu$ and to the ball $B_0 = \frac{1}{8} B_x$, we infer the existence of a Lipschitz graph such as the one described in the Main Lemma. If we consider a Vitali type covering with a family of disjoint balls B_{x_i} we deduce that there exists a rectifiable subset $F_{i,j,k,m} \subset \widetilde{E}_{i,j,k,m}$ with $\mu(F_{i,j,k,m}) \geq \frac{9}{10}\, \mu(\widetilde{E}_{i,j,k,m})$. We set $\widetilde{F} := \bigcup_{i,j,k,m} F_{i,j,k,m}$, and then we have

$$\mu(\widetilde{F}) \geq \frac{9}{8^n 10}\, \mu(\widetilde{E}).$$

It is easy to check that this implies that E is rectifiable. □

The remaining sections of the paper are devoted to the proof of Main Lemma 7.1.

8. Flatness of μ when the Riesz transforms are small

We set

$$P(x,\varepsilon) = \int \frac{\varepsilon}{\big(|x-y|^2 + \varepsilon^2\big)^{(n+1)/2}}\, d\mu(y)$$

and

$$P_2(x,\varepsilon) = \int \frac{\varepsilon^3}{\big(|y|^2 + \varepsilon^2\big)^{(n+3)/2}} d\mu(y).$$

Lemma 8.1. *Let μ be a Borel measure on $\mathbb{R}^d$. Consider $\varepsilon > 0$ and $x \in \mathbb{R}^d$ such that $|x| \leq \varepsilon/4$. We have*

$$\widetilde{R}_\varepsilon \mu(x) - \widetilde{R}_\varepsilon \mu(0) = T(x) + E(x),$$

with

$$T(x) = \int \frac{\big(|y|^2 + \varepsilon^2\big)x - (n+1)(x\cdot y)y}{\big(|y|^2+\varepsilon^2\big)^{(n+3)/2}}\, d\mu(y), \tag{8.1}$$

and

$$|E(x)| \leq C_{10}\, \frac{|x|^2}{\varepsilon^2}\, P(0,\varepsilon).$$

Proof. The arguments are analogous to the ones of Lemma 5.1 in [To4] for the Cauchy transform. We will show the details for completeness.

The Taylor expansion of the function $1/(s+\varepsilon^2)^{(n+1)/2}$ at s_0 is

$$\begin{aligned}\frac{1}{(s+\varepsilon^2)^{(n+1)/2}} = \frac{1}{(s_0+\varepsilon^2)^{(n+1)/2}} - \frac{n+1}{2(s_0+\varepsilon^2)^{(n+3)/2}}\,(s-s_0)\\ + \frac{(n+1)(n+3)}{8(\xi+\varepsilon^2)^{(n+5)/2}}\,(s-s_0)^2,\end{aligned}$$

where $\xi \in [s_0, s]$. If we set $s_0 = |y|^2$, $s = |x-y|^2$, and we multiply by $x-y$, we obtain

$$\begin{aligned}\frac{x-y}{(|x-y|^2+\varepsilon^2)^{(n+1)/2}} =& \frac{x-y}{(|y|^2+\varepsilon^2)^{(n+1)/2}} - \frac{\frac{n+1}{2}\,(x-y)}{(|y|^2+\varepsilon^2)^{(n+3)/2}}\,(|x|^2 - 2x\cdot y)\\ &+ \frac{(n+1)(n+3)(x-y)}{8(\xi_{x,y}+\varepsilon^2)^{(n+5)/2}}\,(|x|^2-2x\cdot y)^2,\end{aligned}$$

where $\xi_{x,y} \in [|y|^2, |x-y|^2]$. If we integrate with respect to $d\mu(y)$, we get

$$\widetilde{R}_\varepsilon \mu(x) = \widetilde{R}_\varepsilon \mu(0) + T(x) + E(x),$$

with

$$\begin{aligned}E(x) =& \frac{n+1}{2}\int \frac{|x|^2(x-y) + 2(x\cdot y)x}{(|y|^2+\varepsilon^2)^{(n+3)/2}}\, d\mu(y)\\ &+ \int \frac{(n+1)(n+3)(x-y)}{8(\xi_{x,y}+\varepsilon^2)^{(n+5)/2}}\,(|x|^2-2x\cdot y)^2\, d\mu(y) =: E_1(x) + E_2(x).\end{aligned}$$

To estimate $E_1(x)$, from $|x| \leq \varepsilon/4$ and $\big||x|^2(x-y)+2(x\cdot y)x\big| \leq C|x|^2(|y|+\varepsilon)$ we deduce

$$|E_1(x)| \lesssim \int \frac{|x|^2}{(|y|^2+\varepsilon^2)^{(n+2)/2}}\, d\mu(y) \leq \frac{|x|^2}{\varepsilon^2}\, P(0,\varepsilon).$$

For $E_2(x)$ we take into account that $\xi_{x,y} + \varepsilon^2 \approx |y|^2 + \varepsilon^2$ and, again, that $|x| \leq \varepsilon/4$. Then,

$$|E_2(x)| \lesssim |x|^2 \int \frac{(|x|+|y|)^3}{\left(|y|^2+\varepsilon^2\right)^{(n+5)/2}}\, d\mu(y) \lesssim |x|^2 \int \frac{1}{(|y|^2+\varepsilon^2)^{(n+2)/2}}\, d\mu(y)$$
$$\leq \frac{|x|^2}{\varepsilon^2}\, P(0,\varepsilon).$$

□

We will need the following result. See [Lé1, Lemma 2.8] for the proof, for example.

Lemma 8.2. *Let μ be a Borel measure on $\mathbb{R}^d$. Suppose that $\mu(B(x,r)) \leq r^n$ for all $x \in \mathbb{R}^d$. Let $B(y,t)$ be a ball such that $\delta(y,t) \geq C_{11}^{-1}$. Then there are $n+1$ balls $\Delta_0, \ldots, \Delta_n$ centered at $\operatorname{supp}(\mu) \cap B(y,t)$ with radius t/C_{12} such that $\delta(B_i) \geq C_{13}^{-1}$ and for all $(x_0, \ldots, x_n) \in \Delta_0 \times \ldots \times \Delta_n$ we have*

$$\operatorname{vol}^n((x_0,\ldots,x_n)) \geq \frac{t^n}{C_{14}}, \tag{8.2}$$

where $\operatorname{vol}^n((x_0,\ldots,x_n))$ denotes the n-volume of the n-simplex with vertices $x_0, \ldots, x_n$.

The arguments for the following lemma are very similar to the ones of [To3, Lemma 7.4]. We will show again the detailed proof for the sake of completeness.

Lemma 8.3. *Let $B(y,t)$ and let $x_0, \ldots, x_n \in B(y,t)$ satisfy (8.2). Then any point $x_{n+1} \in B(y,3t)$ satisfies*

$$\operatorname{dist}(x_{n+1}, L) \lesssim \frac{\varepsilon}{P_2(x_0,\varepsilon)} \sum_{j=1}^{n+1} |\widetilde{R}_\varepsilon \mu(x_j) - \widetilde{R}_\varepsilon \mu(x_0)| + \frac{P(x_0,\varepsilon)}{P_2(x_0,\varepsilon)}\, \frac{t^2}{\varepsilon},$$

where L is the n-plane passing through $x_0, \ldots, x_n$.

Proof. We only have to consider the case $\varepsilon > t$ and moreover, without loss of generality, we assume that $x_0 = 0$. We denote by z the orthogonal projection of x_{n+1} onto L. Then by Lemma 8.1 we have

$$|T(x_j)| \lesssim |\widetilde{R}_\varepsilon \mu(x_j) - \widetilde{R}_\varepsilon \mu(x_0)| + \frac{t^2}{\varepsilon^2}\, P(0,\varepsilon). \tag{8.3}$$

for $j = 1, \ldots, n+1$. Let $e_1, \ldots, e_n$ be an orthonormal basis of L, and set $e_{n+1} = (x_{n+1} - z)/|x_{n+1} - z|$ (we suppose that $x_{n+1} \notin L$), so that e_{n+1} is a unitary vector orthogonal to L. Since the points x_j, $j = 1, \ldots, n$ are linearly independent with "good constants" (i.e. they satisfy (8.2)) we get

$$|T(e_i)| \lesssim \frac{1}{t} \sum_{j=1}^{n} |T(x_j)| \lesssim \sum_{j=1}^{n} |\widetilde{R}_\varepsilon \mu(x_j) - \widetilde{R}_\varepsilon \mu(x_0)| + \frac{t}{\varepsilon^2}\, P(0,\varepsilon).$$

for $i = 1, \ldots, n$. Also, since $z \in L$ and $|z| \lesssim t$, we have $|T(z)| \lesssim \sum_{j=1}^n |T(x_j)|$, and so by (8.3),

$$\begin{aligned}|T(e_{n+1})| &= \frac{1}{\operatorname{dist}(x_{n+1}, L)}\,|T(z - x_{n+1})| \\ &\lesssim \frac{1}{\operatorname{dist}(x_{n+1}, L)}\Big(\sum_{j=1}^{n+1} |\widetilde{R}_\varepsilon\mu(x_j) - \widetilde{R}_\varepsilon\mu(x_0)| + \frac{t^2}{\varepsilon^2}\,P(0,\varepsilon)\Big).\end{aligned}$$

Therefore,

$$\Big|\sum_{j=1}^{n+1} T(e_j)\cdot e_j\Big| \lesssim \frac{1}{\operatorname{dist}(x_{n+1}, L)}\,\Big(\sum_{j=1}^{n+1} |\widetilde{R}_\varepsilon\mu(x_j) - \widetilde{R}_\varepsilon\mu(x_0)| + \frac{t^2}{\varepsilon^2}\,P(0,\varepsilon)\Big). \tag{8.4}$$

On the other hand, from the definition of T in (8.1), if we denote $y_{(i)} = y \cdot e_i$, we get

$$\begin{aligned}\sum_{j=1}^{n+1} T(e_j)\cdot e_j &= \int \frac{(n+1)\big(|y|^2 + \varepsilon^2\big) - (n+1)\sum_{j=1}^{n+1} y_{(j)}^2}{\big(|y|^2+\varepsilon^2\big)^{(n+3)/2}}\,d\mu(y) \\ &= (n+1)\int \frac{\varepsilon^2 + \sum_{j=n+2}^{d} y_{(j)}^2}{\big(|y|^2+\varepsilon^2\big)^{(n+3)/2}}\,d\mu(y) \geq \frac{(n+1)}{\varepsilon}\,P_2(0,\varepsilon).\end{aligned} \tag{8.5}$$

The lemma follows from (8.4) and (8.5). □

Lemma 8.4. *Let μ be a Borel measure on $\mathbb{R}^d$ and $B(x,r)$ such that*

$$\mu(B(x,r)) \geq C_{15}^{-1}\, r^n, \qquad \mu(B(x,t)) \leq M t^n \text{ for all } t \geq r.$$

Then there exists r_1 with $r \leq r_1 \leq C_{16} r$, with C_{16} depending on C_{15} and M, such that

$$P(x, r_1) \leq 2^{n+4}\delta(x, r_1) \qquad \text{and} \qquad \delta(x, r_1) \geq \delta(x, r).$$

Proof. To simplify notation we set $a = 2^{n+4}$. The lemma follows from the following:

Claim. *Under the assumptions of the lemma, either $P(x,r) \leq a\delta(x,r)$ or there exists some t with $r \leq t \leq C_{18} r$ (with C_{18} depending on C_{15} and M) such that $\delta(x,t) \geq 8\,\delta(x,r)$.*

Suppose that the above statement holds. If $P(x,r) > a\delta(x,r)$, then there exists s_1 with $r < s_1 \leq C_{18} r$ such that $\delta(x, s_1) \geq 8\,\delta(x,r)$.

By repeated application of the claim, we deduce that either there exists a sequence $s_1, s_2, s_3, \ldots, s_m$ such that

$$\delta(x, s_m) \geq 8\,\delta(x, s_{m-1}) \geq \ldots \geq 8^{m-1}\,\delta(x, s_1) \geq 8^m\, C_{15}^{-1}, \tag{8.6}$$

or

there exists some s_j, $1 \leq j \leq m-1$, such that $P(x, s_j) \leq a\delta(x, s_j)$. (8.7)

The statement (8.6) fails for m big enough since $\delta(x, s_j) \leq M$ for all j. Thus (8) holds for some j big enough, and so the lemma follows by choosing the minimal such j.

To prove the claim we set

$$\begin{aligned} P(x,r) &= \left(\int_{|x-y|\leq r} + \sum_{k\geq 1}\int_{2^{k-1}r<|x-y|\leq 2^k r}\right) \frac{r}{\left(|x-y|^2+r^2\right)^{(n+1)/2}}\,d\mu(y) \\ &\leq \frac{\mu(B(x,r))}{r^n} + \sum_{k\geq 1}\frac{r}{(2^{k-1}r)^{n+1}}\,\mu(B(x,2^k r)) \\ &\leq \delta(x,r) + \sum_{k=1}^{N} 2^{n+1-k}\delta(x,2^k r) + M\sum_{k\geq N+1} 2^{n+1-k} \\ &= \delta(x,r) + \sum_{k=1}^{N} 2^{n+1-k}\delta(x,2^k r) + M2^{n+1-N}. \end{aligned}$$

Since $P(x,r) \geq a\delta(x,r)$ we infer that

$$(a-1)\delta(x,r) \leq \sum_{k=1}^{N} 2^{n-k}\delta(x,2^k r) + M2^{n+1-N}.$$

For N big enough we have $M2^{n+1-N} \leq C_{15}^{-1} \leq \delta(x,r)$, and so

$$(a-2)\delta(x,r) \leq 2^{n+1}\sum_{k=1}^{N} 2^{-k}\delta(x,2^k r),$$

which implies that there exists some $k \in [1,N]$ such that

$$\delta(x,2^k r) \geq 2^{-n-1}(a-2)\delta(x,r) \geq 8\,\delta(x,r)$$

(recall that $a = 2^{n+4}$). □

Lemma 8.5. *Let μ be a Borel measure on $\mathbb{R}^d$, $F \subset \mathbb{R}^d$ and $B = B(x,r)$ such that*

$$|\widetilde{R}_\varepsilon \mu(y) - \widetilde{R}_\varepsilon \mu(z)| \leq \delta \qquad \text{for all } y,z \in F\cap 3B \text{ and } r \leq \varepsilon \leq \delta^{-1}r, \tag{8.8}$$

and

$$\mu(F\cap B) \geq C_{15}^{-1}\, r^n, \qquad \mu(B(x,t)) \leq Mt^n \text{ for all } t \geq r.$$

Then we have

$$\beta_{\infty,F}(B) \leq \varepsilon_1,$$

with ε_1 depending on C_{15}, δ, M, and $\varepsilon_1 \to 0$ as $\delta \to 0$ for each fixed C_{15}, M.

Proof. Let $\Delta_0, \ldots, \Delta_n$ be balls of radius t like the ones in Lemma 8.2 with $C^{-1}r \leq t \leq r$ and $\mu(F\cap\Delta_i) \gtrsim t$ (we apply Lemma 8.2 to $\mu_{|F\cap B}$). Consider $z_i \in F\cap\Delta_i$ for each $i = 0,\ldots,n$. Given any ℓ with $r \leq \ell \leq \delta^{-1}r$, by (8.8) and Lemma 8.3, for any $y \in F\cap 3B$ we have

$$\operatorname{dist}(y,L) \leq C\frac{\ell}{P_2(x,\ell)}\delta + \frac{C\,P(x,\ell)}{P_2(x,\ell)}\,\frac{r^2}{\ell}, \tag{8.9}$$

where L is the n-plane passing through $z_0, \ldots, z_n$.

Given $\varepsilon_1 > 0$, take $s \geq r$ such that

$$\frac{C\,r^2}{s} \leq \frac{\varepsilon_1 r}{2}.$$

Notice that $\delta(x,s) \geq C(\varepsilon_1)\delta(x,r)$. By Lemma 8.4, we can choose $\ell \geq s$ such that $s \leq \ell \leq C_{16}s$ (with C_{16} depending on ε_1) and

$$P(x,\ell) \leq 2^{n+4}\delta(x,\ell) \qquad \text{and} \qquad P_2(x,\ell) \geq C^{-1}\delta(x,\ell) \geq C(\varepsilon_1)^{-1}\delta(x,r).$$

Moreover, if δ is small enough then we also have $\ell \leq \delta^{-1}r$, so that (8.9) holds, and then we deduce that

$$\operatorname{dist}(y,L) \lesssim C(\varepsilon_1)\delta\ell + \frac{\varepsilon_1 r}{2} \leq C\varepsilon_1 r,$$

if $\delta \ll C(\varepsilon_1)^{-1}$. □

9. Construction of the Lipschitz graph for the proof of Main Lemma 7.1

9.1. **Léger's theorem.** To construct the Lipschitz graph Γ we will follow quite closely the arguments of [Lé2]. Recall that in this paper the author proves that if $E \subset \mathbb{R}^d$ has finite length and finite curvature, then E is rectifiable (i.e. 1-rectifiable). A more precise result is the following (see [Lé2, Proposition 1.1]:

Theorem 9.1. *For any constant $C_{17} \geq 10$, there exists a number $\eta > 0$ such that if σ is a Borel measure on $\mathbb{R}^d$ verifying*

- $\sigma(B(0,2)) \geq 1$, $\operatorname{supp}\sigma \subset B(0,2)$,
- *for any ball B, $\sigma(B) \leq C_{19}\operatorname{diam}(B)$,*
- $c^2(\sigma) \leq \eta$,

then there exists a Lipschitz graph Γ such that $\sigma(\Gamma) \geq \frac{99}{100}\,\sigma(\mathbb{R}^n)$.

Let us remark that, although Léger's theorem is a 1-dimensional result, it easily generalizes to higher dimensions, as the author claims in [Lé2].

Instead of an estimate on the curvature of μ, to prove the Main Lemma 7.1 we will use $L^2(\mu)$ estimates of Riesz transforms (by means of Theorem 1.3).

9.2. **The stopping regions for the construction of the Lipschitz graph.** In the rest of the paper we assume that μ, B_0 and F satisfy the assumptions of Main Lemma 7.1.

Notice that by Lemma 8.5 we know that there exists some n-plane D_0 such that

$$\operatorname{dist}(x, D_0) \leq C\delta r_0 \quad \text{for all } x \in F.$$

Without loss of generality we will assume that $D_0 = \mathbb{R}^n \times \{(0,\ldots,0)\} \equiv \mathbb{R}^n$.

As stated above, to construct the Lipschitz graph, we follow very closely the arguments from [Lé2]. First we need to define a family of stopping time

regions, which are the same as the ones defined in [Lé2, Subsection 3.1]. Given positive constants $\delta_0, \varepsilon, \alpha$ to be fixed below, we set

$$S_{total} = \left\{ (x,t) \in (F \cap B_0) \times (0, 8r_0), \begin{array}{ll} \text{(i)} & \delta_F(x,t) \geq \frac{1}{2}\delta_0 \\ \text{(ii)} & \beta_{1,F}(x,t) < 2\varepsilon \\ \text{(iii)} & \exists D_{x,t} \text{ s.t. } \left\{ \begin{array}{l} \beta_{1,F}^{D_{x,t}}(x,t) \leq 2\varepsilon, \text{ and} \\ \measuredangle(D_{x,t}, D_0) \leq \alpha \end{array} \right. \end{array} \right\}.$$

In the definition above to simplify notation we have denoted $\delta_F(x,r) \equiv \delta_{\mu_{|F}}(x,r)$ and $\beta_{1,F}(x,r) \equiv \beta_{1,\mu_{|F}}(B(x,r))$. Also. $D_{x,t}$ are n-planes depending on x and t and

$$\beta_{1,F}^{D_{x,t}}(x,t) = \frac{1}{t^n} \int_{y \in F : |x-y| \leq 3t} \frac{\operatorname{dist}(y, D_{x,t})}{t}\, d\mu(y).$$

Let us remark that $\delta_0, \varepsilon, \alpha$ will be chosen so that $0 < \varepsilon \ll \alpha \ll \delta_0 \ll 1$.

For $x \in F \cap B_0$ we set

(9.1)

$$h(x) = \sup \left\{ t > 0 : \exists y \in F, \exists \tau, \frac{t}{3} \geq \tau \geq \frac{t}{4}, x \in B\big(y, \frac{\tau}{3}\big) \text{ and } (y,\tau) \notin S_{total} \right\},$$

and

$$S = \{(x,t) \in S_{total} : t \geq h(x)\}.$$

Notice that if $(x,t) \in S$, then $(x,t') \in S$ for $t' > t$.

Now we consider the following partition of $F \cap B_0$:

$$\mathcal{Z} = \{x \in F \cap B_0 : h(x) = 0\},$$

$$F_1 = \left\{ x \in F \cap B_0 \setminus \mathcal{Z} : \exists y \in F, \exists \tau \in \big[\tfrac{h(x)}{5}, \tfrac{h(x)}{2}\big], x \in B(y, \tfrac{\tau}{2}),\, \delta(y,\tau) \leq \delta_0 \right\},$$

$$F_2 = \Big\{ x \in F \cap B_0 \setminus (\mathcal{Z} \cup F_1) : \exists y \in F, \exists \tau \in \big[\tfrac{h(x)}{5}, \tfrac{h(x)}{2}\big], x \in B(y, \tfrac{\tau}{2}),\, \beta_{1,F}(y,\tau) \geq \varepsilon \Big\},$$

$$F_3 = \Big\{ x \in F \cap B_0 \setminus (\mathcal{Z} \cup F_1 \cup F_2) : \exists y \in F, \exists \tau \in \big[\tfrac{h(x)}{5}, \tfrac{h(x)}{2}\big], x \in B(y, \tfrac{\tau}{2}),\, \measuredangle(D_{y,t}, D_0) \geq \tfrac{3}{4}\alpha \Big\}.$$

Remark 9.2. It is easy to check that if $x \in F_3$, then for $h(x) \leq t \leq 100h(x)$ we have $\measuredangle(D_{x,h(x)}, D_0) \geq \alpha/2$, due to the fact that $\varepsilon \ll \alpha$. See [Lé2, Remark 3.3].

The only difference between the definitions above and the ones in [Lé2, Subsection 3.1] is that we work with n-dimensional densities, β's, and planes, while in [Lé2] the dimension is $n = 1$.

9.3. F_2 is void.

Lemma 9.3. *If δ_2 is small enough in Main Lemma 7.1, then F_2 is void. Moreover, $\beta_{\infty,F}(x,r) \leq \varepsilon^2$ for all $x \in F$ and $r > 3h(x)$.*

Proof. By definition, since $r > 3h(x)$, then $(x, r) \in S_{total}$, and then $\delta_F(x, r) \geq \delta_0$. We set $s := M_1 r_0/\delta_2$. For $y \in F$ with $|x - y| \leq 3r$ and $0 < \tau \leq r_0/\delta_2$ we have

$$|\widetilde{R}_\tau \mu(x) - \widetilde{R}_\tau \mu(y)| \leq |\widetilde{R}_{\tau,s} \mu(x)| + |\widetilde{R}_{\tau,s} \mu(y)| + |\widetilde{R}_s \mu(x) - \widetilde{R}_s \mu(y)|$$

(notice that $\tau < s$). By the smoothness of the kernel of $\widetilde{R}_s$ and the assumption (b) in Main Lemma 7.1, it is easy to check that

$$|\widetilde{R}_s \mu(x) - \widetilde{R}_s \mu(y)| \lesssim \frac{M_1 |x - y|}{s} \lesssim \frac{M_1 r_0}{s} = \delta_2.$$

Also, by (d) in Main Lemma 7.1, since $s \leq r_0/\delta_2^2$ (for δ_2 small enough), we have

$$|\widetilde{R}_{\tau,s} \mu(x)| + |\widetilde{R}_{\tau,s} \mu(y)| \leq 2\delta_2.$$

Therefore,

$$|\widetilde{R}_\tau \mu(x) - \widetilde{R}_\tau \mu(y)| \leq C\delta_2$$

$0 < \tau \leq r_0/\delta_2$, and so from Lemma 8.5, we derive $\beta_{\infty,F}(x, r) \leq \varepsilon^2$ for all $x \in F$ and $r \geq 2h(x)$, assuming δ_2 small enough (notice that δ_2 may depend on δ_0). In particular, this implies that F_2 is void. □

Let us remark that we have preferred to maintain the definition of F_2 in the preceding subsection in order to keep the analogy with the construction in [Lé2], although here F_2 turns out to be void.

9.4. **The Lipschitz graph and the size of F_1.** For $x \in \mathbb{R}^d$ we set

$$d(x) = \inf_{(X,t) \in S} (|X - x| + t),$$

and for $p \in D_0$,

$$D(p) = \inf_{x \in \Pi^{-1}(p)} d(x) = \inf_{(X,t) \in S} (|\Pi(X) - p| + t).$$

Notice that d and D are 1-Lipschitz functions. Moreover, $h(x) \geq d(x)$ for $x \in F \cap B_0$, and

$$\mathcal{Z} = \{x \in F \cap B_0 : d(x) = 0\}.$$

Observe also that $d(\cdot)$ is defined on $\mathbb{R}^d$, and not only on $F \cap B_0$. Moreover, $d(x) \geq r_0$ if $x \notin 2B_0$, since $(X, t) \in S$ implies that $X \in F \cap B_0$.

The construction of the Lipschitz graph Γ is basically the same as the one in [Lé2]. The only difference is that in our case the dimension is $n > 1$. So, we have:

Lemma 9.4. *There exists a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ supported on $\Pi(3B_0)$ with $\|\nabla A\|_\infty \leq C\alpha$ such that if we set $\widetilde{A}(p) = (p, A(p))$ for $p \in \mathbb{R}^n$ and*

$$\widetilde{F} = \{x \in F : \operatorname{dist}(x, \widetilde{A}(\Pi(x))) \leq \varepsilon^{1/2} d(x)\},$$

then we have

$$\mu(F \setminus \widetilde{F}) \leq C\varepsilon^{1/2} \mu(F).$$

Moreover,

$$|\nabla^2 A(p)| \leq \frac{C\varepsilon}{D(p)}, \qquad p \in \mathbb{R}^n. \tag{9.2}$$

See Lemma 3.13 and Proposition 3.8 of [Lé2] for the details.

Notice that if $x \notin 2B_0$, then $d(x) > r_0$, and taking into account that $\beta_{\infty,F}(10B_0) \leq \varepsilon^2$, it turns out that $F \setminus 2B_0 \subset \widetilde{F}$ (recall also that $F \subset 10B_0$).

To tell the truth, the Lipschitz graph that is constructed in [Lé2] needs not to be supported on $\Pi(3B_0)$, however it is not difficult to show that if one has a Lipschitz graph A_0 satisfying the assumptions above except the one on the support, then one can take $A = A_0\eta$ where $\eta : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function such that $\chi_{\Pi(2B_0)}\eta \leq \chi_{\Pi(3B_0)}$.

Remark 9.5. To prove Main Lemma 7.1 we will show that if parameters δ_0, α and ε are chosen small enough, then $\mu(\widetilde{F} \cap B_0) \geq \frac{99}{100}\, c_n r_0^n$ (see Lemma 10.5) and the sets F_1 and F_3 are much smaller that $\mu(\widetilde{F} \cap B_0)$. By the preceding construction and definitions, we have $\widetilde{F} \cap B_0 \setminus (F_1 \cup F_2 \cup F_3) \subset \Gamma$.

Arguing as in [Lé2, Proposition 3.19], if δ_0 and ε are small enough, we get

Lemma 9.6.

$$\mu(F_1) \leq 10^{-6}\mu(F \cap B_0).$$

9.5. **A technical lemma.** The following is a technical result that will be used below.

Lemma 9.7. *If $x \in F$ and $y \in \mathbb{R}^d$ satisfy $\Pi(x) = \Pi(y)$, then*

$$d(x) \lesssim d(y).$$

and so

$$d(x) \approx D(\Pi(x)).$$

Proof. The second assertion is a straightforward consequence of the first one. So we only have to prove that $d(x) \lesssim d(y)$. Set $\ell = |x - y|$. We distinguish several cases:

• If $\ell \leq d(x)/2$, since $d(\cdot)$ is 1-Lipschitz, it follows that $|d(x) - d(y)| \leq d(x)/2$, and so $d(x) \approx d(y)$.

• Suppose that $d(x)/2 < \ell \leq r_0$ and that $d(y) \leq d(x)/8$. By the definition of $d(x)$ it turns out that there exists some $(X, t) \in S$ such that

$$|X - x| + t \leq 2d(x) < 4\ell.$$

Notice that we also have $(X, 6\ell) \in S$ (because $\ell \leq r_0$), and thus

$$\beta_{\infty,F}(X, 6\ell) \leq \varepsilon^2$$

by Lemma 9.3. If $D_{X,6\ell}$ stands for the n-plane that minimizes $\beta_{\infty,F}(X, 6\ell)$, since $\measuredangle(D_0, D_{X,6\ell}) \ll 1$ and $\Pi(x) = \Pi(y)$,

$$|x - y| \leq 2\big[\mathrm{dist}(x, D_{X,6\ell}) + \mathrm{dist}(y, D_{X,6\ell})\big] \leq 12\varepsilon^2\ell + 2\mathrm{dist}(y, D_{X,6\ell}), \tag{9.3}$$

since $x \in F$.

By the definition of $d(y)$, there exists $(Y, u) \in S$ such that

$$|Y - y| + u \leq 2d(y) \leq \frac{d(x)}{4} \leq \frac{\ell}{2}. \tag{9.4}$$

Since

$$|Y - X| \leq |Y - y| + |y - x| + |x - X| \leq \ell + \ell + 4\ell = 6\ell$$

and $Y \in F$, we also have

$$\operatorname{dist}(Y, D_{X,6\ell}) \leq 6\varepsilon^2 \ell,$$

by Lemma 9.7 again. Therefore, by (9.4),

$$\operatorname{dist}(y, D_{X,6\ell}) \leq |y - Y| + \operatorname{dist}(Y, D_{X,6\ell}) \leq \frac{\ell}{2} + 12\varepsilon^2 \ell.$$

Thus by (9.3),

$$|x - y| \leq \frac{\ell}{2} + 24\varepsilon^2 \ell < \ell$$

if ε is small enough, which is a contradiction.

• Suppose now that $\ell > r_0$. Since $F \subset 10B_0$, $\beta_{\infty,F}(10B_0) \ll 1$, $\Pi(x) = \Pi(y)$, and $|x - y| \geq r_0$, by geometric arguments it easily follows that $\operatorname{dist}(y, F) \gtrsim r_0$. This implies that $d(y) \gtrsim r_0$ by the definition of $d(y)$, and so $d(y) \gtrsim d(x)$. □

10. The proof that F_3 is small

10.1. **The strategy.** For $x \in \mathbb{R}^d$, we set

$$\ell(x) := \frac{1}{10}\, D(\Pi(x)).$$

Also, for any measure σ we denote

$$R^\perp_{\ell(\cdot),r_0}\sigma(x) := \widehat{R}^\perp_{\ell(x)}\sigma(x) - \widehat{R}^\perp_{r_0}\sigma(x).$$

For simplicity we have preferred the notation $R^\perp_{\ell(\cdot),r_0}\sigma(x)$ instead of $\widehat{R}^\perp_{\ell(\cdot),r_0}\sigma(x)$, although the latter seems more natural.

Roughly speaking, the arguments to show that F_3 cannot be too big are the following:

$$\begin{aligned} F_3 \text{ big } &\Rightarrow \|\nabla A\|_2 \text{ big } \Rightarrow \|R^\perp \mathcal{H}^n_{|\Gamma}\|_{L^2(\Gamma)} \text{ big} \\ &\Rightarrow \|R^\perp_{\ell(\cdot),r_0} \mathcal{H}^n_{|\Gamma\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0)} \text{ big} \\ &\Rightarrow \|R^\perp_{\ell(\cdot),r_0} \mu_{|\widetilde{F}}\|_{L^2(\Gamma\cap 4B_0)} \text{ big } \Rightarrow \|R^\perp_{\ell(\cdot),r_0}\mu\|_{L^2(\mu_{|F})} \text{ big,} \end{aligned}$$

which contradicts the assumptions of Main Lemma 7.1.

Let us explain some more details. The fact that $\|\nabla A\|_2$ must be big if F_3 is big follows from the definition of F_3. Loosely speaking, if $x \in F_3$, then the approximating Lipschitz graph has slope $\gtrsim \alpha$ near x, by construction. As a consequence, we should expect $\|\nabla A\|_2 \gtrsim \alpha\mu(F_3)^{1/2}$ (or a similar inequality) to hold.

The implication

$$\|\nabla A\|_2 \text{ big } \Rightarrow \|R^\perp \mathcal{H}^n_{|\Gamma}\|_{L^2(\Gamma)} \text{ big}$$

is a direct consequence of Theorem 1.3. Finally, the implications

$$\|R^\perp \mathcal{H}^n_{|\Gamma}\|_{L^2(\Gamma)} \text{ big } \Rightarrow \cdots \Rightarrow \|R^\perp_{\ell(\cdot),r_0}\mu\|_{L^2(\mu_{|F})} \text{ big}$$

follow, basically, by approximation. For these arguments to work one has to control the "errors" in this approximation. In particular, the errors must be smaller than $C\alpha\mu(F_3)^{1/2}$. A key point here is that these errors depend mostly on the parameter ε in the definition of F_2 and we have chosen $\varepsilon \ll \alpha$.

10.2. **The implication F_3 big $\Rightarrow$ $\|\nabla A\|_2$ big.**

Lemma 10.1. *We have*

$$\mu(F_3) \leq C\alpha^{-2}\|\nabla A\|_2^2 + C\varepsilon^{1/2}\mu(F).$$

Proof. For a fixed $x \in F_3$, consider the ball $B = B(x,r)$, with $r = 2h(x)$ (recall that $h(x)$ was defined in (9.1)). Suppose that $\mu(B\cap\widetilde{F}) \geq \mu(B\cap F)/2$. By Lemma 8.2 there are $n+1$ balls $\Delta_0,\ldots,\Delta_n$ with radius t/C_{12} such that $\mu(\widetilde{F}\cap\Delta_i) \geq C(\delta)^{-1}r^n$ and for all $(x_0,\ldots,x_n) \in \Delta_0\times\ldots\times\Delta_n$ we have

$$\mathrm{vol}^n((x_0,\ldots,x_n)) \geq C^{-1}r^n.$$

By Remark 9.2, we have $\measuredangle(D_{x,r},D_0) \geq \alpha/2$. Then, it is easy to check that

$$m_B(|A - m_B(A)|) \geq C^{-1}\alpha\, r,$$

since, for each i, $\Delta_i \cap \widetilde{F}$ is very close to the graph of A and also very close to $D_{x,r}$, and moreover $\varepsilon^{1/2} \ll \alpha$. As a consequence, by Poincaré inequality,

$$m_B(|\nabla A|) \geq C^{-1}\frac{m_B(|A-m_B(A)|)}{r} \geq C^{-1}\alpha.$$

Thus, for this ball we have

$$\|\chi_B\nabla A\|_2^2 \geq C^{-1}\alpha^2 r^n.$$

Take now a Besicovitch covering of F_3 with balls $B_i = B(x_i,r_i)$ as above (i.e. $x_i \in F_3$ and $r_i = 2h(x_i)$). Denote by I_1 the collection of balls B_i such that $\mu(B_i\cap\widetilde{F}) \geq \mu(B_i\cap F)/2$. We have

$$\alpha^2\sum_{i\in I_1}\mu(B_i\cap F) \leq C\sum_{i\in I_1}\|\chi_{B_i}\nabla A\|_2^2 \leq C\|\nabla A\|_2^2. \tag{10.1}$$

For the balls B_i in the other collection, that we denote by I_2, we have $\mu(B_i\cap\widetilde{F}) < \mu(B_i\cap F)/2$. Thus,

$$\mu(B_i\cap F\setminus\widetilde{F}) \geq \frac{1}{2}\,\mu(B_i\cap F), \qquad i\in I_2.$$

So we get

$$\sum_{i\in I_2}\mu(B_i\cap F) \leq 2\sum_i \mu(B_i\cap F\setminus\widetilde{F}) \leq C\mu(F\setminus\widetilde{F}) \leq C\varepsilon^{1/2}\mu(F). \tag{10.2}$$

The lemma follows from (10.1) and (10.2). □

10.3. **The implication $\|\nabla A\|_2$ big $\Rightarrow \|R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)}$ big.** This is a direct consequence of Corollary 1.4. Indeed, recall that we showed that

$$\|R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)} \approx \|\nabla A\|_2, \tag{10.3}$$

assuming that $\|\nabla A\|_\infty$ is small enough, which is true in our construction if $\alpha \ll 1$.

10.4. **The implication $\|R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)}$ big $\Rightarrow \|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0)}$ big.**

Lemma 10.2.

$$\|R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)} \leq \|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0)} + C\alpha^2\, r_0^{n/2}.$$

Proof. Recall that $\operatorname{supp}(A) \subset 3B_0$. We set

$$\begin{aligned}\|R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)} &\leq \|\chi_{4B_0} R^\perp(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma)} \\ &\quad + \|\chi_{4B_0} R^\perp(\mathcal{H}^n_{\Gamma\setminus 5B_0})\|_{L^2(\Gamma)} + \|\chi_{\Gamma\setminus 4B_0} R^\perp(\mathcal{H}^n_\Gamma)\|_{L^2(\Gamma)} \\ &= I + II + III.\end{aligned}$$

Let us see that the terms II and III are small. We consider first II. Given $x \in 4B_0$, we have

$$\begin{aligned}|R^\perp(\mathcal{H}^n_{\Gamma\setminus 5B_0})(x)| &\leq \int_{y\in\Gamma:|y-x|\geq r_0} \frac{|x^\perp - y^\perp|}{|x-y|^{n+1}}\, d\mathcal{H}^n(y) \\ &= \int_{y\in D_0:|y-x|\geq r_0} \frac{\operatorname{dist}(x,D_0)}{|x-y|^{n+1}}\, d\mathcal{H}^n(y) \lesssim \frac{\operatorname{dist}(x,D_0)}{r_0}.\end{aligned}$$

If we square and integrate the last estimate on $4B_0$, we get

$$II^2 \lesssim \beta_{2,\Gamma}(2B_0)^2 r_0^n \lesssim \varepsilon^2 r_0^n.$$

To estimate the term III we take $x \in \Gamma \setminus 4B_0 = D_0 \setminus 4B_0$ (so $x^\perp = 0$), and we set

$$\begin{aligned}|R^\perp(\mathcal{H}^n_\Gamma)(x)| &\leq \int_{y\in\Gamma} \frac{\operatorname{dist}(y,D_0)}{|x-y|^{n+1}}\, d\mathcal{H}^n(y) = \int_{y\in\Gamma\cap 3B_0} \frac{\operatorname{dist}(y,D_0)}{|x-y|^{n+1}}\, d\mathcal{H}^n(y) \\ &\approx \frac{1}{\big(r_0 + |x-x_0|\big)^{n+1}} \int_{y\in\Gamma\cap 3B_0} \operatorname{dist}(y,D_0)\, d\mathcal{H}^n(y) \\ &\lesssim \beta_{1,\Gamma}(2B_0)\, \frac{r_0^{n+1}}{\big(r_0 + |x-x_0|\big)^{n+1}}.\end{aligned}$$

Squaring and integrating on $D_0 \setminus 4B_0$, we obtain

$$III^2 \lesssim \beta_{1,\Gamma}(2B_0)^2 r_0^n \leq \varepsilon^2 r_0^n.$$

To deal with the term I, given $x \in \Gamma \cap 4B_0$, we set

$$\begin{aligned}|R^\perp(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| &\leq |R^\perp_{0,\ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| + |R^\perp_{\ell(x),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| \\ &\quad + |\widehat{R}^\perp_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)|.\end{aligned}$$

We consider first the term $|\widehat{R}^{\perp}_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)|$, for $x\in\Gamma\cap 4B_0$:

$$\begin{aligned}|\widehat{R}^{\perp}_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| &\leq \int_{y\in\Gamma\cap 5B_0:|y-x|>r_0/2} \frac{|x^\perp-y^\perp|}{|x-y|^{n+1}}\,d\mathcal{H}^n_\Gamma(y)\\ &\lesssim \int_{y\in\Gamma\cap 5B_0} \frac{\operatorname{dist}(x,D_0)+\operatorname{dist}(y,D_0)}{r_0^{n+1}}\,d\mathcal{H}^n_\Gamma(y)\lesssim \beta_{\infty,\Gamma}(2B_0).\end{aligned}$$

So we get

$$\|\chi_{4B_0}\widehat{R}^{\perp}_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|^2_{L^2(\Gamma)}\lesssim \beta_{\infty,\Gamma}(2B_0)^2 r_0^n\lesssim \varepsilon^2 r_0^n. \tag{10.4}$$

To estimate $|R^{\perp}_{0,\ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)|$ we will use the smoothness of Γ on the stopping cubes. That is, we will use the estimate (9.2). Notice first that

$$R^{\perp}_{0,\ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)=R^{\perp}_{0,\ell(x)}(\mathcal{H}^n_{\Gamma})(x)$$

for $x\in 4B_0$, since $\ell(x)<r_0$. So if we set $x=\widetilde{A}(p)$, $y=\widetilde{A}(q)$, with $p,q\in\mathbb{R}^n$, we have

$$R^{\perp}_{0,\ell(x)}\mathcal{H}^n_{\Gamma\cap 5B_0}(x)=\int\left(1-\psi\Big(\frac{\widetilde{A}(p)-\widetilde{A}(q)}{D(p)/10}\Big)\right)\frac{A(p)-A(q)}{\big|\widetilde{A}(p)-\widetilde{A}(q)\big|^{n+1}}\,J(\widetilde{A})(q)dq, \tag{10.5}$$

where dq stands for the n-dimensional Lebesgue measure. We denote by $S(x)$ the integral on the right hand side of (10.5), and we set

$$\begin{aligned}S(x)&=\int\left(1-\psi\Big(\frac{p-q}{D(p)/10}\Big)\right)\frac{A(p)-A(q)}{\big|\widetilde{A}(p)-\widetilde{A}(q)\big|^{n+1}}\,dq\\ &\quad+\int\left(\psi\Big(\frac{p-q}{D(p)/10}\Big)-\psi\Big(\frac{\widetilde{A}(p)-\widetilde{A}(q)}{D(p)/10}\Big)\right)\frac{A(p)-A(q)}{\big|\widetilde{A}(p)-\widetilde{A}(q)\big|^{n+1}}\,dq\\ &\quad+\int\left(1-\psi\Big(\frac{\widetilde{A}(p)-\widetilde{A}(q)}{D(p)/10}\Big)\right)\frac{A(p)-A(q)}{\big|\widetilde{A}(p)-\widetilde{A}(q)\big|^{n+1}}\,\big(J(\widetilde{A})(q)-1\big)dq\\ &=S_1(x)+S_2(x)+S_3(x).\end{aligned}$$

Recall that by Remark 6.2 we have

$$\|J(\widetilde{A})-1\|_2\lesssim\|\nabla A\|_\infty\|\nabla A\|_2.$$

So, by the L^2 boundedness of Riesz transforms on Lipschitz graphs we get

$$\|S_3\|_2\lesssim\|\nabla A\|_\infty\|\nabla A\|_2. \tag{10.6}$$

To deal with S_2 notice that

$$\big||\widetilde{A}(p)-\widetilde{A}(q)|-|p-q|\big|\leq|A(p)-A(q)|\leq C\alpha|p-q|\leq\frac{1}{2}|p-q|, \tag{10.7}$$

and so

$$\frac{1}{2}|p-q|\leq|\widetilde{A}(p)-\widetilde{A}(q)|\leq 2|p-q|.$$

Since $\psi(z) = 0$ if $|z| \leq 1/2$ and $\psi(z) = 1$ if $|z| \geq 1$, we deduce that

$$\psi\Big(\frac{p-q}{D(p)/10}\Big) - \psi\Big(\frac{\widetilde{A}(p) - \widetilde{A}(q)}{D(p)/10}\Big) = 0$$

if $|p-q| \leq D(p)/40$ or $|p-q| \geq D(p)/5$. Moreover, from the mean value theorem and (10.7),

$$\left|\psi\Big(\frac{p-q}{D(p)/10}\Big) - \psi\Big(\frac{\widetilde{A}(p) - \widetilde{A}(q)}{D(p)/10}\Big)\right| \leq \frac{C\alpha|p-q|}{D(p)}.$$

Thus,

$$\begin{aligned}|S_2(x)| &\lesssim \int_{D(p)/40\leq|p-q|\leq D(p)/5} \frac{\alpha|p-q|}{D(p)}\,\frac{|A(p)-A(q)|}{|p-q|^{n+1}}\,dq \\ &\lesssim \frac{\alpha^2}{D(p)}\int_{D(p)/40\leq|p-q|\leq D(p)/5} \frac{1}{|p-q|^{n-1}}\,dq \lesssim \alpha^2.\end{aligned}$$

Therefore,

$$\|S_2\|_2 \lesssim \alpha^2\, r_0^{n/2}. \tag{10.8}$$

We are left with the term $S_1(x)$. By Taylor's formula, we have

$$\begin{aligned}&\frac{A(p)-A(q)}{\big(|p-q|^2 + |A(p)-A(q)|^2\big)^{(n+1)/2}} \\ &\qquad = \sum_{k=0}^{\infty}(-1)^k \frac{(n+2k-1)!!}{2^k k!}\cdot\frac{|A(p)-A(q)|^{2k}}{|p-q|^{n+2k+1}}\,(A(p)-A(q)).\end{aligned}$$

The series is uniformly convergent since $|A(p)-A(q)|/|p-q| \leq C\alpha \ll 1$. Notice that the integrand in S_1 vanishes if $|p-q| > D(p)/10$. On the other hand, by Taylor's formula and (9.2) we also have

$$A(p) - A(q) = \nabla A(p)(p-q) + E(p,q),$$

with

$$\begin{aligned}|E(p,q)| &\leq C \sup_{z\in B(p,D(p)/10)} |\nabla^2 A(z)|\,|p-q|^2 \\ &\leq \varepsilon|p-q|^2 \sup_{z\in B(p,D(p)/10)} \frac{1}{D(z)} \lesssim \frac{\varepsilon|p-q|^2}{D(p)},\end{aligned} \tag{10.9}$$

since $D(\cdot)$ is 1-Lipschitz. Then it turns out that

$$(A(p)-A(q))\,|A(p)-A(q)|^{2k} = \nabla A(p)(p-q)\,|\nabla A(p)(p-q)|^{2k} + E_k(p,q),$$

with[2]

$$|E_k(p,q)| \leq C\varepsilon 2^{-k}\,\frac{|p-q|^{2k+2}}{D(p)}.$$

[2]For this estimate we take into account that $\|\nabla A\|_\infty + \varepsilon \leq 1/4$ and we use the fact that $(a+b)^m = a^m + c$, with $|c| \leq 2^k|b|\max(|a|,|b|)^{m-1}$.

We have

$$\begin{aligned} S_1(x) =& \sum_{k=0}^{\infty} (-1)^k \frac{(n+2k-1)!!}{2^k k!} \\ &\times \int \left(1 - \psi\Big(\frac{p-q}{D(p)/10}\Big)\right) \frac{\nabla A(p)(p-q)\,|\nabla A(p)(p-q)|^{2k}}{|p-q|^{n+2k+1}}\,dq \\ &+ \sum_{k=0}^{\infty} (-1)^k \frac{(n+2k-1)!!}{2^k k!} \int_{|p-q|\leq D(p)/10} \frac{E_k(p,q)}{|p-q|^{n+2k+1}}\,dq. \end{aligned}$$

Notice that the first sum on the right side above vanishes because each integral in the sum equals zero by the antisymmetry of the integrand. We obtain

$$\begin{aligned} |S_1(x)| &\leq \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{2^k k!} \int_{|p-q|\leq D(p)/10} \frac{|E_k(p,q)|}{|p-q|^{n+2k+1}}\,dq \\ &\lesssim \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{4^k k!} \int_{|p-q|\leq D(p)/10} \frac{\varepsilon}{D(p)|p-q|^{n-1}}\,dq \\ &\approx \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{4^k k!}\,\varepsilon \approx \varepsilon. \end{aligned} \tag{10.10}$$

From (10.5), (10.6), (10.8), and (10.10) we deduce that

$$\|\chi_{4B_0} R^{\perp}_{0,\ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma)} \lesssim \|\nabla A\|_\infty \|\nabla A\|_2 + \alpha^2 r_0^{n/2} \lesssim \alpha^2 r_0^{n/2}$$

because $\alpha^2 \ll \varepsilon$ and $\|\nabla A\|_2 \lesssim \alpha r_0^{n/2}$, since A is supported on $\Pi(3B_0)$. Therefore, by (10.4),

$$I \leq C(\varepsilon + \alpha^2) r_0^{n/2} \lesssim \alpha^2 r_0^{n/2}.$$

The lemma follows from the preceding estimate and the ones obtained above for the terms II and III. □

10.5. **The implication $\|R^{\perp}_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0)}$ big $\Rightarrow$ $\|R^{\perp}_{\ell(\cdot),r_0}\mu_{\widetilde{F}}\|_{L^2(\Gamma)}$ big.** This implication is one of the most delicate steps of the proof that F_3 is a small set. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth radially non increasing function with $\|\varphi\|_1 = 1$ such that $\operatorname{supp}(\varphi) \subset B_n(0,1)$ and φ equals 1 on $B_n(0,c_0)$ for some $0 < c_0 < 1$ which may depend on n. As usual, for $t > 0$ we denote,

$$\varphi_t(x) = \frac{1}{t^n}\,\varphi\Big(\frac{x}{t}\Big), \qquad x \in \mathbb{R}^n.$$

Then we consider the function $g : \mathbb{R}^n \to \mathbb{R}$ given by

$$g(x) = \varphi_{\varepsilon^{1/4} D(x)} * \Pi_{\#}(\mu_{|\widetilde{F}})(x).$$

We will show below that $g(x)\,dx$ is very close to the measure dx on $B(x_0, 6r_0)$, in a sense.

First we need the following preliminary result:

Lemma 10.3. *For all $x, y \in \mathbb{R}^n$,*

$$|\varphi_{\varepsilon^{1/4}D(x)}(x-y) - \varphi_{\varepsilon^{1/4}D(y)}(x-y)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(y))^n}\,\chi_{B(0,C\varepsilon^{1/4}D(y))}(x-y).$$

Proof. For any $z \in \mathbb{R}^n$ and $s, t > 0$ with $s \approx t$,

$$\begin{aligned}|\varphi_s(z) - \varphi_t(z)| &\leq \Big|\frac{1}{s^n} - \frac{1}{t^n}\Big|\,\varphi\Big(\frac{z}{s}\Big) + \frac{1}{t^n}\,\Big|\varphi\Big(\frac{z}{s}\Big) - \varphi\Big(\frac{z}{t}\Big)\Big| \\ &\leq \frac{C|s-t|}{s^{n+1}}\,\varphi\Big(\frac{z}{s}\Big) + \frac{C}{t^n}\,\Big|\frac{z}{s} - \frac{z}{t}\Big| \leq \frac{C|s-t|}{s^{n+1}},\end{aligned}$$

since we may assume that $|z| \lesssim s$. As a consequence,

$$|\varphi_s(z) - \varphi_t(z)| \leq \frac{C|s-t|}{s^{n+1}}\,\chi_{B(0,Cs)}(z).$$

We set $s = \varepsilon^{1/4}D(y)$ and $t = \varepsilon^{1/4}D(x)$. Notice that $\varphi_{\varepsilon^{1/4}D(x)}(x-y) \neq 0$ implies that $|x-y| \leq \varepsilon^{1/4}D(x)$, and then it turns out that $D(x) \approx D(y)$. Of course, the same happens if $\varphi_{\varepsilon^{1/4}D(y)}(x-y) \neq 0$. In both cases we have

$$\frac{|s-t|}{s} = \frac{|D(x) - D(y)|}{D(y)} \leq \frac{|x-y|}{D(y)} \lesssim \varepsilon^{1/4}.$$

Therefore,

$$|\varphi_{\varepsilon^{1/4}D(x)}(x-y) - \varphi_{\varepsilon^{1/4}D(y)}(x-y)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(y))^n}\,\chi_{B(0,C\varepsilon^{1/4}D(y))}(x-y).$$

□

Lemma 10.4. *Let ν be a Borel measure on $\mathbb{R}^n$ such that*

$$\nu(B(x,r)) \leq r^n \qquad \text{for all } x \in \operatorname{supp}(\nu) \text{ and } r \geq \eta r_0.$$

For any $\delta > 0$, if $\eta > 0$ is small enough, we have

$$\nu(B(x,r)) \leq (1+\delta)r^n \qquad \text{for all } x \in \mathbb{R}^n \text{ and } r \geq r_0.$$

Moreover, η only depends on δ.

Proof. Notice first that any ball $B(x,r)$ with radius $r \geq \eta r_0$ satisfies

$$\nu(B(x,r)) \leq Cr, \tag{10.11}$$

where C depends only on n.

Let t be a small constant such that $0 < t \leq 1/4$ to be fixed below. Consider a ball $B := B(x,r)$ with $r \geq r_0$. Set $V_1 = B \setminus B(x,(1-t)r)$. We can cover $\operatorname{supp}(\nu) \cap V_1$ by balls contained in B centered at points in $\operatorname{supp}(\nu) \cap V_1$, with radii $r_1 = tr/2$. By Besicovitch covering theorem, we can extract a subfamily of disjoint balls $\{B_i\}_{i \in I_1}$ such that

$$\nu(V_1) \leq N\,\nu\Big(\bigcup_{i \in I_1} B_i \cap V_1\Big),$$

where N is some fixed integer depending only on n.

Define now $V_2 = V_1 \setminus \bigcup_{i\in I_1}(1+t)B_i$, so that $\nu(V_2) \leq (1-1/N)\nu(V_1)$. Again by Besicovitch covering theorem, there exists a family of disjoint balls $\{B_i\}_{i\in I_2}$ centered at points in $\operatorname{supp}(\nu)\cap V_2$ and radii $r_2 = t^2 r/2$ (and so they are contained in $B \setminus \bigcup_{i\in I_1} B_i$) such that

$$\nu(V_2) \leq N\,\nu\Big(\bigcup_{i\in I_2} B_i \cap V_2\Big).$$

Consider now $V_3 = V_2 \setminus \bigcup_{i\in I_2}(1+t)B_i$, and take again a family of disjoint balls $\{B_i\}_{i\in I_3}$ centered at points in $\operatorname{supp}(\nu)\cap V_3$ and radii $r_3 = t^3 r/2$ such that

$$\nu(V_3) \leq N\,\nu\Big(\bigcup_{i\in I_3} B_i \cap V_3\Big).$$

If we perform this procedure several times, we obtain N_0 sets V_k and families of balls $\{B_i\}_{i\in I_k}$, $k = 1,\ldots,N_0$, so that the union

$$\bigcup_{1\leq k\leq N_0}\bigcup_{i\in I_k} B_i$$

is pairwise disjoint, with $\nu(V_k) \leq (1-1/N)\nu(V_{k-1})$. Moreover, all the balls B_i are contained in B, centered at $\operatorname{supp}(\nu)$, and B_i has radius $r_k = t^k r/2$ if $i \in I_k$. By construction, we have

$$B = \big(B \setminus (1-t)B\big) \cup \bigcup_{1\leq k\leq N_0}\bigcup_{i\in I_k}(1+t)B_i \cup V_{N_0+1}.$$

Thus,

$$\nu(B) \leq \nu\big(B \setminus (1-t)B\big) + \sum_{1\leq k\leq N_0}\sum_{i\in I_k}\nu((1+t)B_i) + \nu(V_{N_0+1}).$$

Since $B\setminus(1-t)B$ can be covered by a fixed number m_0 (depending only on n) of balls with radii $2tr$, by (10.11) we get

$$\nu\big(B \setminus (1-t)B\big) \leq C m_0 t^n r^n,$$

assuming $\eta \ll t$. If η is again small enough, for $i \in I_k$ we have $\nu((1+t)B_i) \leq (1+t)^n r_k^n$. Since the balls B_i, $i\in I_k$, $1\leq k\leq N_0$, are pairwise disjoint and contained in B,

$$\sum_{1\leq k\leq N_0}\sum_{i\in I_k}\nu((1+t)B_i) \leq (1+t)^n \sum_{1\leq k\leq N_0}\sum_{i\in I_k} r_k^n \leq (1+t)^n r^n.$$

Also, by construction $\nu(V_{N_0+1}) \leq (1-1/N)^{N_0}\nu(B) \leq (1-1/N)^{N_0} r^n$. Therefore,

$$\nu(B) \leq \big((1-1/N)^{N_0} + C m_0 t^n + (1+t)^n\big)\, r^n.$$

If N_0 is chosen big enough and t small enough, the lemma follows.

□

In next lemma we show that g is very close to the function identically 1 on $8B_0$. We also prove that $\mu(\widetilde{F}\cap B_0)$ is big, which was already mentioned in Remark 9.5.

Lemma 10.5. *If ε has been chosen small enough and $\delta_1 \leq \alpha^2$ (where δ_1 is the constant from (a) and (b) in Main Lemma 7.1), then we have*

(10.12)
$$\Pi_\#(\mu_{|\widetilde{F}})(B(p,r)) \leq c_n(1+c\alpha^2)r \quad \text{for all } p \in \mathbb{R}^n \text{ and } r \geq \varepsilon^{1/2}D(p),$$

$$0 \leq g(p) \leq 1 + C_{20}\alpha^2 \quad \text{for all } p \in \mathbb{R}^n, \tag{10.13}$$

$$\|\chi_{8B_0}(g-1)\|_1 \leq C\alpha^2\, r_0^n, \tag{10.14}$$

and

$$\|\chi_{8B_0}(g-1)\|_2 \leq C\alpha\, r_0^{n/2}. \tag{10.15}$$

Also,

$$\mu(\widetilde{F} \cap B_0) \geq \frac{99}{100}\, c_n r_0^n. \tag{10.16}$$

Proof. First we will show (10.12). Since for all $x \in \Pi^{-1}(B_n(p,t)) \cap \widetilde{F}$ (recall that $B_n(p,r)$ is an n-dimensional ball in $\mathbb{R}^n$), we have $\beta_{\infty,F}(x,t) \leq C\varepsilon$ for $t \geq D(p)$, we infer that there exists some n-plane L such that

$$\Pi^{-1}(B_n(p,r)) \cap \widetilde{F} \subset U_{C\varepsilon^{1/2}r}(L) \quad \text{if } r \geq \varepsilon^{1/2}D(p).$$

Further, by construction, the n-plane L satisfies $\measuredangle(L,\mathbb{R}^n) \leq C\alpha$. All together, this implies that there exists some ball $B(z,R) \subset \mathbb{R}^d$, with

$$R \leq (1 + C\sin(\alpha)^2)^{1/2}r + C\varepsilon^{1/2}r \leq (1 + C\alpha^2 + C\varepsilon^{1/2})r,$$

such that

$$\Pi^{-1}(B_n(p,r)) \cap \widetilde{F} \subset B(z,R).$$

If $p \in \Pi(\widetilde{F})$, then we may take $z \in \widetilde{F}$, and so by the assumption (b) in the Main Lemma 7.1,

$$\begin{aligned}\Pi_\#\mu_{|\widetilde{F}}(B_n(p,r)) &\leq \mu(B(z,R)) \leq c_n(1+\delta_1)(1 + C\alpha^2 + C\varepsilon^{1/2})^n r^n \\ &\leq c_n(1 + \delta_1 + C_{21}\alpha^2)r^n \end{aligned} \tag{10.17}$$

for $r \geq \varepsilon^{1/2}D(p)$ (recall that $\varepsilon^{1/2} \ll \alpha^2$).

Consider now the case $p \notin \Pi(\widetilde{F})$. Suppose that $\varepsilon^{1/4}D(p) \leq r \leq D(p)$ and let $\nu = \Pi_\#\mu_{|\widetilde{F}\cap B(p,D(p)/10)}$. By (10.17),

$$\nu(B(z,r)) \leq c_n(1 + \delta_1 + C_{21}\alpha^2)r^n$$

for all $z \in \operatorname{supp}(\nu)$ and $r \geq \varepsilon^{1/2}D(z) \approx \varepsilon^{1/2}D(p)$. From Lemma 10.4 we deduce that

$$\nu(B(p,r)) \leq c_n(1 + \delta_1 + 2C_{21}\alpha^2)r^n$$

if $r \geq c_0\varepsilon^{1/4}D(p)$ and ε is small enough (recall that c_0 was defined at the beginning of the current subsection).

To prove (10.13) for a given $p \in \mathbb{R}^n$, let $\psi : \mathbb{R} \to \mathbb{R}$ be such that $\psi(|q|) = \varphi_{\varepsilon^{1/4}D(p)}(q)$ and denote $\sigma = \Pi_\#(\mu_{|\widetilde{F}})$. We have

$$\begin{aligned} g(p) &= \int \psi(|p-q|)\,d\sigma(p) = -\int_0^\infty \int_{|p-q|}^\infty \psi'(r)\,dr\,d\sigma(p) \\ &= -\int_0^\infty \Pi_\#\mu_{|\widetilde{F}}(B_n(p,r))\,\psi'(r)\,dr. \end{aligned} \tag{10.18}$$

Notice that

$$\sigma(B_n(p,r)) = \Pi_\#\mu_{|\widetilde{F}}(B_n(p,r)) = \mu(\Pi^{-1}(B_n(p,r)) \cap \widetilde{F}).$$

Moreover, $\operatorname{supp}(\psi') \subset [c_0\varepsilon^{1/4}D(p), \varepsilon^{1/4}D(p)]$, and so

$$g(p) = -\int_{c_0\varepsilon^{1/4}D(p)}^{\varepsilon^{1/4}D(p)} \mu(\Pi^{-1}(B_n(p,r)) \cap \widetilde{F})\,\psi'(r)\,dr.$$

Thus,

$$|g(p)| \leq c_n(1+\delta_1+2C_{21}\alpha^2)\int_{c_0\varepsilon^{1/4}D(p)}^{\varepsilon^{1/4}D(p)} r^n\,|\psi'(r)|\,dr = 1+\delta_1+2C_{21}\alpha^2,$$

and (10.13) follows.

Now we turn our attention to (10.14). First we will show that

$$\int_{B_n(x_0,8r_0)} g(p)dp \geq (1-C\varepsilon^{1/4})\mathcal{L}^n(8B_0 \cap \mathbb{R}^n). \tag{10.19}$$

Since $D(p) \leq 9r_0$ for all $p \in \Pi(8B_0)$, we have

(10.20)

$$\begin{aligned} \int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} g(p)\,dp &= \int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} \varphi_{\varepsilon^{1/4}D(p)} * \sigma(p)\,dp \\ &= \int_{p\in B_n(x_0,(8+9\varepsilon^{1/4})r_0)} \int \varphi_{\varepsilon^{1/4}D(p)}(p-q)\,d\sigma(q)\,dp \\ &\geq \int_{q\in B_n(x_0,8r_0)} \int \varphi_{\varepsilon^{1/4}D(p)}(p-q)\,dp\,d\sigma(q). \end{aligned}$$

Recall now that by Lemma 10.3,

$$|\varphi_{\varepsilon^{1/4}D(p)}(p-q) - \varphi_{\varepsilon^{1/4}D(q)}(p-q)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(q))^n}\,\chi_{B(q,C\varepsilon^{1/4}D(q))}(p).$$

From this inequality and (10.20) we get

(10.21)
$$\begin{aligned}\int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} g(p)\,dp &\geq \int_{q\in B_n(x_0,8r_0)}\int \varphi_{\varepsilon^{1/4}D(q)}(p-q)\,dp\,d\sigma(q)\\ &\quad - \int_{q\in B_n(x_0,8r_0)} \frac{\varepsilon^{1/4}\mathcal{L}^n(B(q,C\varepsilon^{1/4}D(q)))}{(\varepsilon^{1/4}D(q))^n}\,d\sigma(q)\\ &= (1-C\varepsilon^{1/4})\,\sigma(B_n(x_0,8r_0))\\ &\geq (1-C\varepsilon^{1/4})\mathcal{L}^n(B_n(x_0,8r_0)).\end{aligned}$$

The last inequality follows from the assumption (a) of Main Lemma 7.1 and the fact that $\mu(F\setminus\widetilde{F})\lesssim \varepsilon^{1/2}\mu(F)$. Inequality (10.19) is a consequence of (10.21) and the estimate $\|g\|_\infty\leq 2$ (by(10.13)).

The estimate (10.14) is a direct consequence of (10.13) and (10.19):

$$\begin{aligned}\int_{\Pi(8B_0)} |(1+C_{20}\alpha^2)-g(p)|\,dp &= \int_{\Pi(8B_0)} \big((1+C_{20}\alpha^2)-g(p)\big)\,dp\\ &= (1+C_{20}\alpha^2)\mathcal{L}^n(\Pi(8B_0)) - \int_{\Pi(8B_0)} g(p)\,dp\\ &\leq (C_{20}\alpha^2+C\varepsilon^{1/4})\mathcal{L}^n(\Pi(8B_0)).\end{aligned}$$

Thus,

$$\int_{\Pi(8B_0)} |1-g(p)|\,dp \leq (2C_{20}\alpha^2+C\varepsilon^{1/4})\mathcal{L}^n(\Pi(8B_0)),$$

and so we get (10.14) if ε is small enough.

On the other hand, (10.15) is a direct consequence of (10.14):

$$\int_{\Pi(8B_0)} |1-g(p)|^2\,dp \leq (1+\|g\|_\infty)\int_{\Pi(8B_0)} |1-g(p)|\,dp \leq C\alpha^2\,r_0^n.$$

Finally we deal with (10.16): if we argue as in (10.20) and (10.21), with $B_n(x_0,8r_0)\setminus B_n(x_0,\frac{999}{1000}r_0)$ instead of $B_n(x_0,8r_0)$, we get

$$\begin{aligned}\sigma(\Pi(8B_0)\setminus\Pi(\tfrac{999}{1000}B_0)) &\leq \int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)\setminus B_n(x_0,(1-9\varepsilon^{1/4})\frac{999}{1000}r_0)} g(p)\,dp + C\varepsilon^{1/4}r_0^n\\ &\leq c_n(1+C\alpha^2)(8^n-\tfrac{999}{1000})r_0^n + C\varepsilon^{1/4}r_0^n.\end{aligned}$$

Since $\mu(\widetilde{F}\cap B_0)\geq\sigma(\Pi(\frac{999}{1000}B_0))$ if $\beta_{\infty,F}(B_0)$ is small enough, we have

$$\begin{aligned}\mu(\widetilde{F}\cap B_0) &\geq \sigma(\Pi(8B_0)) - \sigma(\Pi(8B_0)\setminus\Pi(\tfrac{999}{1000}B_0))\\ &\geq c_n 8^n r_0^n - C\varepsilon^{1/2}r_0^n - c_n(1+C\alpha^2)(8^n-\tfrac{999}{1000})r_0^n - C\varepsilon^{1/4}r_0^n\\ &\geq \frac{99}{100}\,c_n r_0^n,\end{aligned}$$

if α and ε are small enough. □

Recall that Π stands for the orthogonal projection of $\mathbb{R}^d$ onto $D_0 \equiv \mathbb{R}^n$, and $\sigma = \Pi_\# \mu_{|\widetilde{F}}$. We also denote by P the projection from $\mathbb{R}^d$ onto Γ which is orthogonal to $D_0 \equiv \mathbb{R}^n$. Moreover, for $x \in \Gamma$ we set

$$h(x) = \frac{g(\Pi(x))}{J\widetilde{A}(\Pi(x))},$$

so that $h(x)\, d\mathcal{H}^n_{|\Gamma}(x)$ is the image measure of $g(x)\, dx$ by P.

Lemma 10.6. *If $f : \mathbb{R}^d \to \mathbb{R}$ is a function with $\operatorname{supp}(f) \subset 5B_0$, then we have*

(10.22)

$$\begin{aligned}\left|\int_{P(5B_0)} f(x)\, h(x)\, d\mathcal{H}^n(x) - \int_{5B_0\cap\widetilde{F}} f(x)\, d\mu(x)\right| &\\ \leq \iint_{\substack{p\in\Pi(6B_0)\\ |p-q|\leq \varepsilon^{1/4} D(q)}} \frac{C}{\left(\varepsilon^{1/4} D(q)\right)^n} \left|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right| d\sigma(q) dp &\\ + \left|\int_{p\in\Pi(6B_0)} f(\widetilde{A}(p))\, b(p)\, dp\right| + \int_{5B_0\cap\widetilde{F}} \left|f(P(x)) - f(x)\right| d\mu(x),&\end{aligned}$$

where $b(p)$ is some function satisfying $\|b\|_\infty \lesssim \varepsilon^{1/4}$.

Proof. We have

$$\begin{aligned}\int_{P(5B_0)} f\, h\, d\mathcal{H}^n - \int_{5B_0\cap\widetilde{F}} f\, d\mu &\\ = \left(\int_{\Pi(5B_0)} f(\widetilde{A}(p))\, g(p)\, dp - \int_{\Pi(5B_0)} f(\widetilde{A}(p))\, d\Pi_\#\mu_{|\widetilde{F}}(p)\right) &\\ + \left(\int_{5B_0\cap\widetilde{F}} f(P(x))\, d\mu(x) - \int_{5B_0\cap\widetilde{F}} f(x)\, d\mu(x)\right) =: S + T.&\end{aligned}$$

For this identity we took into account that

$$\int_{\Pi(5B_0)} f(\widetilde{A}(p))\, d\Pi_\#\mu_{|\widetilde{F}}(p) = \int_{P(5B_0)} f(x)\, dP_\#\mu_{|\widetilde{F}}(x) = \int_{5B_0\cap\widetilde{F}} f(P(x))\, d\mu(x).$$

To estimate the term S we recall that

$$g(p) = \varphi_{\varepsilon^{1/4}D(p)} * \Pi_\#(\mu_{|\widetilde{F}})(p) = \varphi_{\varepsilon^{1/4}D(p)} * \sigma(p),$$

and so

$$\begin{aligned}S &= \iint_{p\in\Pi(5B_0)} f(\widetilde{A}(p))\, \varphi_{\varepsilon^{1/4}D(p)}(p-q)\, d\sigma(q) dp - \int_{q\in\Pi(6B_0)} f(\widetilde{A}(q))\, d\sigma(q)\\ &= \iint_{p\in\Pi(6B_0)} \left[f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right] \varphi_{\varepsilon^{1/4}D(q)}(p-q)\, d\sigma(q) dp\\ &\quad + \iint_{p\in\Pi(6B_0)} f(\widetilde{A}(p)) \left[\varphi_{\varepsilon^{1/4}D(p)}(p-q) - \varphi_{\varepsilon^{1/4}D(q)}(p-q)\right] d\sigma(q) dp\\ &=: S_1 + S_2,\end{aligned}$$

since

$$\int_{p\in\Pi(6B_0)} \varphi_{\varepsilon^{1/4}D(q)}(p-q)\,dp = 1 \quad \text{for } q\in \operatorname{supp}(f\circ\widetilde{A}).$$

Clearly, we have

$$|S_1| \lesssim \iint_{\substack{p\in\Pi(6B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{1}{\big(\varepsilon^{1/4}D(q)\big)^n}\big|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\big|\,d\sigma(q)dp.$$

To deal with S_2 we denote

$$b(p) = \int \big[\varphi_{\varepsilon^{1/4}D(p)}(p-q) - \varphi_{\varepsilon^{1/4}D(q)}(p-q)\big]\,d\sigma(q).$$

By Lemma 10.3

$$|b(p)| \lesssim \frac{\varepsilon^{1/4}\sigma(B(p, C\varepsilon^{1/4}D(q)))}{(\varepsilon^{1/4}D(q))^n} \lesssim \varepsilon^{1/4}.$$

Concerning the term T, we have

$$|T| \leq \int_{5B_0\cap\widetilde{F}} \big|f(P(x)) - f(x)\big|\,d\mu(x).$$

□

Lemma 10.7.

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0}) - R^\perp_{\ell(\cdot),r_0}(h\,\mathcal{H}^n_{|\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0)} \lesssim \varepsilon^{1/4} r_0^{n/2}.$$

Proof. For any $x\in\Gamma\cap 4B_0$ we have

$$\begin{aligned} B(x) &:= R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})(x) - R^\perp_{\ell(\cdot),r_0}(h\,\mathcal{H}^n_{|\Gamma\cap 5B_0})(x)\\ &= \int_{y\in\widetilde{F}\cap 5B_0} K^\perp_{\ell(x),r_0}(x-y)d\mu(y) - \int_{y\in P(5B_0)} K^\perp_{\ell(x),r_0}(x-y)\,h(y)\,d\mathcal{H}^n(y).\end{aligned}$$

To estimate $B(x)$ we apply Lemma 10.6 with $f(y) = K^\perp_{\ell(x),r_0}(x-y)$, for each fixed $x\in\Gamma\cap 4B_0$. To simplify notation, we set $F(x,p) := K^\perp_{\ell(x),r_0}(x-\widetilde{A}(p))$. The first term on the right side of (10.22) for this choice of f is

$$B_1(x) := \iint_{\substack{p\in\Pi(6B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{C}{\big(\varepsilon^{1/4}D(q)\big)^n}\big|F(x,p) - F(x,q)\big|\,d\sigma(q)dp.$$

For p,q satisfying $|p-q|\leq \varepsilon^{1/4}D(p)\lesssim \varepsilon^{1/4}D(q)$ we have

$$\big|F(x,p) - F(x,q)\big| \lesssim \frac{\varepsilon^{1/4}D(q)}{\big(|x-\widetilde{A}(q))| + \ell(x)\big)^{n+1}}.$$

Moreover,

$$D(q)\leq 10\ell(x) + |D(q) - 10\ell(x)| \leq 10\ell(x) + 10|\Pi(x) - q| \lesssim \ell(x) + |x-\widetilde{A}(q)|,$$

since $D(\Pi(x)) = 10\ell(x)$ and D is 1-Lipschitz. So we get

$$\big|F(x,p) - F(x,q)\big| \lesssim \frac{\varepsilon^{1/4}D(q)}{\big(D(q) + |x-\widetilde{A}(q)|\big)^{n+1}}.$$

Thus,

$$|B_1(x)| \lesssim \int_{q\in\Pi(6B_0)} \frac{\varepsilon^{1/4} D(q)}{\big(D(q)+|x-\widetilde{A}(q)|\big)^{n+1}}\,d\sigma(q)$$
$$\lesssim \int_{z\in\widetilde{F}\cap 7B_0} \frac{\varepsilon^{1/4}\ell(z)}{\big(\ell(z)+|x-z|\big)^{n+1}}\,d\mu(z).$$

Consider now the operator

$$Sf(x) = \int_{z\in\widetilde{F}\cap 7B_0} \frac{\varepsilon^{1/4}\ell(z)}{\big(\ell(z)+|x-z|\big)^{n+1}}\,f(z)\,d\mu(z), \qquad x\in\Gamma\cap 4B_0. \tag{10.23}$$

It is easy to check that its adjoint satisfies

$$|S^*f(z)| \lesssim \varepsilon^{1/4} Mf(z),$$

where M stands for the Hardy-Littlewood maximal operator

$$Mf(z) = \sup_{r>0} \frac{1}{r^n} \int_{B(z,r)\cap\Gamma\cap 4B_0} |f|\,d\mathcal{H}^n,$$

which is bounded from $L^2(\Gamma\cap 4B_0)$ into $L^2(\mu_{|\widetilde{F}\cap 7B_0})$, and so

$$\|S^*\|_{L^2(\Gamma\cap 4B_0),L^2(\mu_{|\widetilde{F}\cap 7B_0})} \lesssim \varepsilon^{1/4}.$$

Therefore, $S: L^2(\mu_{|\widetilde{F}\cap 7B_0}) \to L^2(\Gamma\cap 4B_0)$ is bounded with norm $\lesssim \varepsilon^{1/4}$ and then

$$\int_{x\in\Gamma\cap 4B_0} \left(\int_{y\in\widetilde{F}\cap 7B_0} \frac{\varepsilon^{1/4} d(y)}{d(y)+|y-x|^{n+1}}\,d\mu(y)\right)^2 d\mathcal{H}^n(x) \lesssim \varepsilon^{1/2} r_0^n.$$

Thus,

$$\|B_1\|_{L^2(\Gamma\cap 4B_0)} \lesssim \varepsilon^{1/4} r_0^{n/2}.$$

We deal now with the second term on the right hand side of (10.22), with $f(y) = K^\perp_{\ell(x),r_0}(x-y)$:

$$B_2(x) := \left|\int_{x\in\Pi(6B_0)} K^\perp_{\ell(x),r_0}(x-\widetilde{A}(p))\,b(p)\,dp\right|.$$

By the L^2-boundedness of Riesz transforms on $L^2(\Gamma)$ and the fact that $\|b\|_\infty \lesssim \varepsilon^{1/4}$, we get

$$\|B_2\|_{L^2(\Gamma\cap 4B_0)} \lesssim \|\chi_{\Pi(6B_0)} b\|_2 \le \varepsilon^{1/4} r_0^{n/2}.$$

Finally we deal with the third term on the right side of (10.22):

$$B_3(x) = \int_{5B_0\cap\widetilde{F}} \big|K^\perp_{\ell(x),r_0}(x-P(y)) - K^\perp_{\ell(x),r_0}(x-y)\big|\,d\mu(x).$$

Since

$$|\nabla K^\perp_{\ell(x),r_0}(z)| \lesssim \frac{1}{\big(\ell(x)+|z|\big)^{n+1}}$$

for all $z \in \mathbb{R}^d$, and $|y - P(y)| \leq C\mathrm{dist}(y,\Gamma) \leq C\varepsilon^{1/2} d(y)$ for $y \in \widetilde{F}$, we deduce that

$$\begin{aligned}\left|K^\perp_{\ell(x),r_0}(x-y) - K^\perp_{\ell(x),r_0}(x-P(y))\right| &\lesssim \frac{|y-P(y)|}{\big(\ell(x)+|y-x|\big)^{n+1}}\\ &\lesssim \frac{\varepsilon^{1/2} d(y)}{\big(\ell(x)+|y-x|\big)^{n+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}|R^\perp_{\ell(x),r_0}(\mu_{|\widetilde{F}\cap 5B_0})(x) - R^\perp_{\ell(x),r_0}(P_\#\mu_{|\widetilde{F}\cap 5B_0})(x)|&\\ \lesssim \int_{y\in\widetilde{F}\cap 5B_0} \frac{\varepsilon^{1/2} d(y)}{\big(\ell(x)+|y-x|\big)^{n+1}}\,d\mu(y).&\end{aligned}$$

Recall that $\ell(x) = 10D(\Pi(x))$, and since $D(\cdot)$ is 1-Lipschitz,

$$D(\Pi(y)) \lesssim D(\Pi(x)) + |x-y| = 10\ell(x) + |x-y|.$$

For $y \in F$, by Lemma 9.7 we infer that $d(y) \approx D(\Pi(y))$, and so

$$d(y) \lesssim \ell(x) + |x-y|.$$

Thus,

$$\|B_3\|^2_{L^2(\Gamma\cap 4B_0)} \lesssim \int_{x\in\Gamma\cap 4B_0} \left(\int_{y\in\widetilde{F}\cap 5B_0} \frac{\varepsilon^{1/2} d(y)}{\big(d(y)+|y-x|\big)^{n+1}}\,d\mu(y)\right)^2 d\mathcal{H}^n(x).$$

Let $T : L^2(\mu_{|\widetilde{F}\cap 5B_0}) \to L^2(\Gamma\cap 4B_0)$ be the following operator

$$Tf(x) = \int_{y\in\widetilde{F}\cap 5B_0} \frac{\varepsilon^{1/2} d(y)}{\big(d(y)+|y-x|\big)^{n+1}}\,d\mu(y). \tag{10.24}$$

Arguing as in the case of the operator S from (10.23), it is easy to check that $T : L^2(\mu_{|\widetilde{F}\cap 5B_0}) \to L^2(\Gamma\cap 4B_0)$ is bounded with norm $\lesssim \varepsilon^{1/2}$ and then

$$\begin{aligned}\int_{x\in\Gamma\cap 4B_0} \left(\int_{y\in\widetilde{F}\cap 5B_0} \frac{\varepsilon^{1/2} d(y)}{\big(d(y)+|y-x|\big)^{n+1}}\,d\mu(y)\right)^2 d\mathcal{H}^n(x) &\lesssim \varepsilon\mathcal{H}^n(\Gamma\cap 4B_0)\\ &\lesssim \varepsilon r_0^n.\end{aligned}$$

Thus we obtain

$$\|B_3\|_{L^2(\Gamma\cap 4B_0)} \lesssim \varepsilon^{1/2} r_0^{n/2}.$$

If we add the estimates obtained for B_1, B_2 and B_3, the lemma follows. □

Lemma 10.8. *We have*

$$\|R^\perp_{\ell(\cdot),r_0}(h\,d\mathcal{H}^n_{|\Gamma\cap 5B_0}) - R^\perp_{\ell(\cdot),r_0}\mathcal{H}^n_{|\Gamma\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0)} \lesssim \alpha^2\|\nabla A\|_2 + \alpha^2 r_0^{n/2}. \tag{10.25}$$

Let us remark that, for the arguments in Lemma 10.11 below, it is important that the last term on the right side of (10.25) is $\alpha^2 r_0^{n/2}$ instead of $\alpha r_0^{n/2}$, say.

Proof. By Lemma 6.1 we have

$$\|R^\perp_{\ell(\cdot),r_0}\|_{L^2(\Gamma\cap 4B_0),L^2(\Gamma\cap 4B_0)} \lesssim \|\nabla A\|_\infty.$$

Thus

$$\|R^\perp_{\ell(\cdot),r_0}(h\,d\mathcal{H}^n_{|\Gamma\cap 5B_0}) - R^\perp_{\ell(\cdot),r_0}\mathcal{H}^n_{|\Gamma\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0)} \lesssim \|\nabla A\|_\infty \|h-1\|_{L^2(\Gamma\cap 6B_0)}.$$

On the other hand, writing $p=\Pi(x)$ we have

$$|h(x)-1| = \left|\frac{g(p)}{J(\widetilde{A})(p)} - 1\right| \leq \left|\frac{g(p)}{J(\widetilde{A})(p)} - g(p)\right| + |g(p)-1|.$$

Recalling that $\|J(\widetilde{A})-1\|_2 \leq \|\nabla A\|_\infty \|\nabla A\|_2$ and $\|\chi_{\Pi(8B_0)}(g-1)\|_2 \lesssim \alpha r_0^{n/2}$, we get

$$\|h-1\|_{L^2(\Gamma\cap 6B_0)} \lesssim \alpha\|\nabla A\|_2 + \alpha r_0^{n/2},$$

and thus

$$\|R^\perp_{\ell(\cdot),r_0}(h\,d\mathcal{H}^n_{|\Gamma\cap 5B_0}) - R^\perp_{\ell(\cdot),r_0}\mathcal{H}^n_{|\Gamma\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0)} \lesssim \alpha^2\|\nabla A\|_2 + \alpha^2 r_0^{n/2}.$$

□

10.6. **The implication $\|R^\perp_{\ell(\cdot),r_0}\mu_{\widetilde{F}}\|_{L^2(\Gamma)}$ big $\Rightarrow$ $\|R^\perp_{\ell(\cdot),r_0}\mu\|_{L^2(\mu_{|F})}$ big.** Recall that on $4B_0$, the image measure of $P_\#\mu_{|\widetilde{F}}$ by Π coincides with σ and that $h\,d\mathcal{H}^n_{|\Gamma\cap 5B_0} = P_\#\big(g(x)\,dx\big)$, with $g(x) = \big(\varphi_{\varepsilon^{1/4}D(x)} * \sigma\big)(x)$. We denote

$$G_1 = \{p\in\Pi(8B_0) : g(p) > 1/2\},$$

and

$$G_0 = P(G_1).$$

Lemma 10.9. *We have*

$$\mathcal{H}^n(\Gamma\cap 6B_0\setminus G_0) \lesssim \alpha^2\, r_0^n.$$

Proof. By (10.14) we have

$$\int_{\Pi(8B_0)} |g-1|\,dx \leq C\alpha^2\, r_0^n.$$

Thus,

$$\begin{aligned}\mathcal{L}^n(\Pi(8B_0)\setminus G_1) &\leq \mathcal{L}^n\{p\in\Pi(8B_0) : |g(p)-1|>1/2\}\\ &\leq 2\int_{\Pi(8B_0)} |g-1|\,dp \leq C\alpha^2\, r_0^n.\end{aligned}$$

It is clear that then we also have

$$\mathcal{H}^n(\Gamma\cap 6B_0\setminus P(G_1)) \lesssim \alpha^2\, r_0^n.$$

□

Lemma 10.10.

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0\cap G_0)} \lesssim \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|_{L^2(\mu_{|\widetilde{F}\cap 4B_0})} + \varepsilon^{1/8} r_0^{n/2}.$$

Proof. We denote $f(x) = R^\perp_{\ell(\cdot),r_0}(\mu_{|5B_0\cap\widetilde{F}})(x)$. Since $h(x) > 1/3$ on G_0, we have

$$\begin{aligned}\|f\|^2_{L^2(\Gamma\cap 4B_0\cap G_0)} &\leq 3\int_{\Gamma\cap 4B_0} |f|^2\,h\,d\mathcal{H}^n \\ &\leq 3\left|\int_{\Gamma\cap 4B_0} |f|^2\,h\,d\mathcal{H}^n - \int_{\widetilde{F}\cap 4B_0} |f|^2\,d\mu\right| + 3\int_{\widetilde{F}\cap 4B_0} |f|^2\,d\mu.\end{aligned}$$

To prove the lemma it is enough to show that

$$I := \left|\int_{\Gamma\cap 6B_0} |f|^2\,h\,d\mathcal{H}^n - \int_{\widetilde{F}\cap 6B_0} |f|^2\,d\mu\right| \lesssim \varepsilon^{1/8} r_0^{n/2}. \tag{10.26}$$

To this end we will use Lemma 10.6, with $|f|^2$ instead of f, and with $6B_0$ replacing $5B_0$, and $7B_0$ replacing $6B_0$. Notice that $\operatorname{supp}(f) \subset 6B_0$. It is clear that Lemma 10.6 also holds in this situation. So we have

$$\begin{aligned}I &\leq C\iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq\varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n}\left||f(\widetilde{A}(p))|^2 - |f(\widetilde{A}(q))|^2\right| d\sigma(q)dp \\ &\quad + \left|\int_{p\in\Pi(7B_0)} |f(\widetilde{A}(p))|^2\,b(p)\,dp\right| + \int_{7B_0\cap\widetilde{F}}\left||f(P(x))|^2 - |f(x)|^2\right| d\mu(x) \\ &=: C\,I_1 + I_2 + I_3,\end{aligned}$$

where $b(p)$ is some function satisfying $\|b\|_\infty \lesssim \varepsilon^{1/4}$.

First we estimate I_1. Setting

$$\left||f(\widetilde{A}(p))|^2 - |f(\widetilde{A}(q))|^2\right| \leq \left|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right| \times \left(|f(\widetilde{A}(p))| + |f(\widetilde{A}(q))|\right) \tag{10.27}$$

and applying Cauchy-Schwartz, we get

$$\begin{aligned}I_1 &\leq \left(\iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq\varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n}\left|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right|^2 d\sigma(q)dp\right)^{1/2} \\ &\quad\times \left(\iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq\varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n}\left(|f(\widetilde{A}(p))| + |f(\widetilde{A}(q))|\right)^2 d\sigma(q)dp\right)^{1/2} \\ &=: I_{1,1}^{1/2}\times I_{1,2}^{1/2}.\end{aligned} \tag{10.28}$$

To estimate $I_{1,1}$ notice that if $|p-q| \leq \varepsilon^{1/4}D(q)$, then

$$\left|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right| = \left|R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(p)) - R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(q))\right| \lesssim \varepsilon^{1/4}. \tag{10.29}$$

For this inequality notice $D(p) \approx D(q)$ because $|p-q| \leq \varepsilon^{1/4} D(q)$, and recall also that $\ell(x) = 10D(\Pi(x))$. We leave the details for the reader. Therefore,

$$I_{1,1} = \iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n} \left|f(\widetilde{A}(p)) - f(\widetilde{A}(q))\right|^2 d\sigma(q)dp$$
$$\lesssim \varepsilon^{1/2} \iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n} d\sigma(q)dp \lesssim \varepsilon^{1/2}\, r_0^n.$$

To deal with $I_{1,2}$ we set

(10.30)

$$I_{1,2}^{1/2} \leq \left(\iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n} |R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(p))|^2\, d\sigma(q)dp\right)^{1/2}$$
$$+ \left(\iint_{\substack{p\in\Pi(7B_0)\\ |p-q|\leq \varepsilon^{1/4}D(q)}} \frac{1}{\left(\varepsilon^{1/4}D(q)\right)^n} |R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(q))|^2\, d\sigma(q)dp\right)^{1/2}$$
$$=: I_{1,2,a}^{1/2} + I_{1,2,b}^{1/2}.$$

Concerning $I_{1,2,a}$, we have

$$I_{1,2,a}^{1/2} \lesssim \left(\int_{p\in\Pi(7B_0)} |R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(p))|^2\, dp\right)^{1/2} \lesssim r_0^{n/2},$$

by the L^2 boundedness of Riesz transforms from $L^2(\mu_{|\widetilde{F}})$ into $L^2(\Gamma)$. For the last integral in the right side of (10.30) we take into account that $D(p) \approx D(q)$, and then we get

(10.31)

$$I_{1,2,b}^{1/2} \lesssim \left(\int_{q\in\Pi(7.5B_0)} |R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(\widetilde{A}(q))|^2\, d\sigma(q)\right)^{1/2}$$
$$= C\left(\int_{q\in\Pi(7.5B_0)} |f(\widetilde{A}(q))|^2\, d\sigma(q)\right)^{1/2} \leq C\left(\int_{y\in 8B_0} |f(P(y))|^2\, d\mu_{\widetilde{F}}(y)\right)^{1/2}$$
$$\leq C\left(\int_{y\in 8B_0} |f(P(y)) - f(y)|^2\, d\mu_{\widetilde{F}}(y)\right)^{1/2} + C\left(\int_{y\in 8B_0} |f(y)|^2\, d\mu_{\widetilde{F}}(y)\right)^{1/2}.$$

Using the $L^2(\mu_{|\widetilde{F}})$ boundedness of Riesz transforms, the last integral is $\lesssim r_0^n$. For the first one we argue as in (10.29): given $y \in \widetilde{F}$, we have $|y - P(y)| \lesssim \varepsilon^{1/2}\, d(y) \approx \varepsilon^{1/2}\ell(y)$, and then it easily follows that

$$\left|f(P(y)) - f(y)\right| = \left|R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(P(y)) - R^\perp_{\ell(\cdot),r_0}\mu_{|5B_0\cap\widetilde{F}}(y)\right| \lesssim \varepsilon^{1/2}.$$

Therefore, the first term on the right side of (10.31) is bounded above by $\varepsilon^{1/2} r_0^{n/2}$, and so $I_{1,2,b}^{1/2} \lesssim r_0^{n/2}$, and thus $I_{1,2}^{1/2} \lesssim r_0^{n/2}$. Recalling that $I_{1,1} \leq \varepsilon^{1/2} r_0^n$, we deduce that

$$I_1 \lesssim \varepsilon^{1/4} r_0^n.$$

To estimate the integral I_2 we use the fact that $\|b\|_\infty \lesssim \varepsilon^{1/4}$, and so

$$I_2 \lesssim \varepsilon^{1/4} \int_{x\in\Pi(7B_0)} |f(\widetilde{A}(p))|^2\,dp.$$

The last integral is similar to $I_{1,2,a}$, and thus we have

$$I_2 \lesssim \varepsilon^{1/4}\, r_0^n.$$

To deal with I_3 we argue as in (10.27) and, similarly to (10.28), we infer that

$$\begin{aligned} I_3 &\leq \left(\int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(P(x)) - f(x)\bigr|^2\,d\mu(x)\right)^{1/2} \\ &\quad\times \left(\int_{x\in 7B_0\cap\widetilde{F}} \bigl||f(P(x))| + |f(x)|\bigr|^2\,d\mu(x)\right)^{1/2} =: I_{3,1}^{1/2}\times I_{3,2}^{1/2}. \end{aligned} \tag{10.32}$$

The integral $I_{3,1}$ is similar to the first one on right side of (10.31), and so we have $I_{3,1}\lesssim \varepsilon^{1/4} r_0^n$. For $I_{3,2}$ we set

$$I_{3,2} \leq \left(\int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(x)\bigr|^2\,d\mu(x)\right)^{1/2} + \left(\int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(P(x))\bigr|^2\,d\mu(x)\right)^{1/2}.$$

The first term on the right side is bounded above by $C\,r_0^{n/2}$, by the $L^2(\mu_{|\widetilde{F}})$ boundedness of Riesz transforms, and for the second one we write

$$\begin{aligned} S &:= \left(\int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(P(x))\bigr|^2\,d\mu(x)\right)^{1/2} \\ &\leq \int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(x)\bigr|^2\,d\mu(x)\right)^{1/2} + \int_{x\in 7B_0\cap\widetilde{F}} \bigl|f(P(x)) - f(x)\bigr|^2\,d\mu(x)\right)^{1/2}. \end{aligned}$$

As above, the first term satisfies $\lesssim r_0^{n/2}$, and the second one coincides with $I_{3,1}^{1/2}$, and so we have $S\lesssim r_0^{n/2}$. Thus, $I_{3,2}\lesssim r_0^{n/2}$, and then $I_3\lesssim \varepsilon^{1/8} r_0^n$.

If we gather the estimates obtained for I_1, I_2 and I_3, (10.26) follows and we are done. □

10.7. **The proof that F_3 is small.**

Lemma 10.11. *We have*

$$\mu(F_3)\leq \alpha^{1/2}\mu(F).$$

Proof. We will use all the results obtained Subsections 10.2-10.6. From (10.3) and Lemma 10.2, we deduce

$$\|\nabla A\|_2 \lesssim \|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0)} + C\alpha^2\, r_0^{n/2}. \tag{10.33}$$

By Lemma 10.9 and since $R^\perp_{\ell(\cdot),r_0}$ is bounded in $L^4(\Gamma)$ with norm $\lesssim \|\nabla A\|_\infty \lesssim \alpha$, we deduce

$$\begin{aligned}\|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|^2_{L^2(\Gamma\cap 4B_0\setminus G_0)}&\\ &\leq \|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|^2_{L^4(\Gamma\cap 4B_0)}\,\mathcal{H}^n(\Gamma\cap 4B_0\setminus G_0)^{1/2}\\ &\lesssim \alpha^2\mathcal{H}^n(\Gamma\cap 5B_0)^{1/2}\,\mathcal{H}^n(\Gamma\cap 4B_0\setminus G_0)^{1/2}\lesssim \alpha^3\, r_0^n.\end{aligned}$$

From this inequality and (10.33) we derive

$$\|\nabla A\|_2 \lesssim \|R^\perp_{\ell(\cdot),r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma\cap 4B_0\cap G_0)} + \alpha^{3/2}\,\mu(F)^{1/2},$$

since $\alpha \ll 1$. This estimate and Lemmas 10.7 and 10.8 imply that

$$\|\nabla A\|_2 \lesssim \|R^\perp_{\ell(\cdot),r_0}\mu_{|\widetilde{F}\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0\cap G_0)}+\alpha^2\|\nabla A\|_2+(\alpha^{3/2}+\alpha^2+\varepsilon^{1/4})\mu(F)^{1/2}.$$

Thus, if α is small enough and $\varepsilon^{1/4}\leq \alpha^{3/2}$, we get

$$\|\nabla A\|_2 \lesssim \|R^\perp_{\ell(\cdot),r_0}\mu_{|\widetilde{F}\cap 5B_0}\|_{L^2(\Gamma\cap 4B_0\cap G_0)} + \alpha^{3/2}\,\mu(F)^{1/2}.$$

Together with Lemma 10.1 this implies that

(10.34)
$$\mu(F_3)\lesssim \alpha^{-2}\|\nabla A\|_2^2+\varepsilon^{1/2}\mu(F)\lesssim \alpha^{-2}\|R^\perp_{\ell(\cdot),r_0}\mu_{|\widetilde{F}\cap 5B_0}\|^2_{L^2(\Gamma\cap 4B_0\cap G_0)}+\alpha\,\mu(F).$$

Recall that by Lemma 10.10, we have

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|5B_0\cap\widetilde{F}})\|_{L^2(\Gamma\cap 4B_0\cap G_0)}\lesssim \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|_{L^2(\mu_{|\widetilde{F}\cap 4B_0})}+\varepsilon^{1/8}r_0^{n/2}.$$

From this estimate and (10.34) we deduce that

$$\mu(F_3)\lesssim \alpha^{-2}\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0})}+\alpha\,\mu(F) \tag{10.35}$$

(assuming always $\varepsilon\ll\alpha\ll 1$).

Now we denote

$$B_1=\{x\in\widetilde{F}: R_*\mu_{|5B_0\setminus\widetilde{F}}(x)>\varepsilon^{1/4}\}.$$

By the boundedness of Riesz transforms from $M(\mathbb{R}^d)$ (the space of finite Borel measures on $\mathbb{R}^d$) into $L^{1,\infty}(\mu_{|\widetilde{F}})$, we get

$$\mu(B_1)\lesssim \frac{\mu(5B_0\setminus\widetilde{F})}{\varepsilon^{1/4}}\lesssim \frac{\varepsilon^{1/2}\mu(\widetilde{F})}{\varepsilon^{1/4}}=\varepsilon^{1/4}\mu(\widetilde{F}).$$

By Cauchy-Schwartz and the $L^4(\mu_{|\widetilde{F}})$ boundedness of Riesz transforms we get

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|B_1})}\leq \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^4(\mu_{|B_1})}\mu(B_1)^{1/2}\lesssim \varepsilon^{1/8}\,\mu(F).$$

On the other hand, from (10.35) we infer that

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0})}\geq C^{-1}\alpha^2\big[\mu(F_3)-C\alpha\,\mu(F)\big].$$

Suppose that $\mu(F_3) > \alpha^{1/2}\mu(F)$. Then $\mu(F_3) - C\alpha\,\mu(F) \gtrsim \alpha^{1/2}\mu(F)$, and by the preceding estimates we get

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0})} \geq C^{-1}\alpha^{5/2}\mu(F) \geq 2\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|B_1})},$$

because $\varepsilon^{1/8} \ll \alpha^{5/2}$. Therefore,

$$\begin{aligned}\|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0\setminus B_1})}&\\ &= \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0})} - \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|B_1})}\\ &\geq \|R^\perp_{\ell(\cdot),r_0}(\mu_{|\widetilde{F}\cap 5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0})} \gtrsim \alpha^{5/2}\mu(F).\end{aligned}$$

Since $R_*\mu_{|5B_0\setminus\widetilde{F}}(x) \leq \varepsilon^{1/4}$ on $\widetilde{F}\setminus B_1$, we have

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|5B_0\setminus\widetilde{F}})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0\setminus B_1})} \leq \varepsilon^{1/2}\mu(4B_0).$$

Thus we deduce that

$$\|R^\perp_{\ell(\cdot),r_0}(\mu_{|5B_0})\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0\setminus B_1})} \geq (C^{-1}\alpha^{5/2} - C\varepsilon^{1/2})\mu(F) \gtrsim \alpha^{5/2}\mu(F).$$

Since $R^\perp_{\ell(\cdot),r_0}(\mu_{|5B_0})(x) = R^\perp_{\ell(\cdot),r_0}\mu(x)$ for any $x \in 4B_0$, we have

$$\|R^\perp_{\ell(\cdot),r_0}\mu\|^2_{L^2(\mu_{|\widetilde{F}\cap 4B_0\setminus B_1})} \gtrsim \alpha^{5/2}\mu(F),$$

which contradicts the assumption (d) in Main Lemma 7.1. □

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