# Analytic capacity and quasiconformal mappings with $W^{1,2}$ Beltrami coefficient

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**Abstract**

We show that if $\phi$ is a quasiconformal mapping with compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}$, then $\phi$ preserves sets with vanishing analytic capacity. It then follows that a compact set $E$ is removable for bounded analytic functions if and only if it is removable for bounded quasiregular mappings with compactly supported Beltrami coefficient in $W^{1,2}$.

## 1 Introduction

A Beltrami coefficient is a measurable function $\mu$ such that $\|\mu\|_\infty < 1$. Given an open set $\Omega \subset \mathbb{C}$, we say that $f : \Omega \to \mathbb{C}$ is $\mu$-quasiregular if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the Beltrami equation

$$\overline{\partial} f(z) = \mu(z)\, \partial f(z), \qquad a.e. z \in \Omega.$$

If moreover $f$ is a homeomorphism, then we call it *$\mu$-quasiconformal.* For any $K \geq 1$, we say that $f$ is *$K$-quasiregular* (or *$K$-quasiconformal* if $f$ is homeomorphism) for some Beltrami coefficient $\mu$ satisfying $\|\mu\|_\infty \leq \frac{K-1}{K+1}$.

Several works have focussed in the question of how these mappings distort measures and capacities. For instance, Ahlfors (see [Ah1]) proved that they always preserve sets of zero area. In a remarkable paper, Astala [As] obtained deep estimates for the area distortion under $K$-quasiconformal mappings. More precisely, if $\phi$ is any (conveniently normalized) $K$-quasiconformal mapping, then

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one has the estimate

$$|\phi(E)| \leq C\,|E|^{\frac{1}{K}}$$

where the constant $C$ depends only on $K$. As a consequence, the author obtained also sharp results on integrability of $K$-quasiconformal mappings, which in turn led to the bounds on $K$-quasiconformal distortion of Hausdorff dimension. Namely, for any $K$-quasiconformal mapping $\phi$ and any compact set $E$,

$$\frac{1}{K}\left(\frac{1}{\dim(E)}-\frac{1}{2}\right) \leq \frac{1}{\dim(\phi(E))}-\frac{1}{2} \leq K\left(\frac{1}{\dim(E)}-\frac{1}{2}\right). \tag{1}$$

Moreover, in [As] the author shows the sharpness of both inequalities.

It is well known that sometimes the regularity of the Beltrami coefficient $\mu$ is inherited by the mapping itself. For instance, when $\mu$ is a compactly supported $\mathcal{C}^\infty$ function, then every $\mu$-quasiconformal mapping $\phi$ is also $\mathcal{C}^\infty$. As a consequence, $\phi$ is locally bilipschitz, and then some set functions like Hausdorff measures, Riesz and Bessel capacities, are preserved.

Nevertheless, there are other situations which, even far from $\mu \in \mathcal{C}^\infty$, give interesting improvements of equation (1). For instance, when $\mu$ belongs to the class $VMO$ of functions of vanishing mean oscillation, then

$$\dim(\phi(E)) = \dim(E).$$

That is, the corresponding $\mu$-quasiconformal mappings $\phi$ do not distort Hausdorff dimension (see for instance [Iw]). However, nothing is known on the ratio between the Hausdorff measures $\mathcal{H}^t(E)$ and $\mathcal{H}^t(\phi(E))$, for any $t \in [0,2]$.

In this context, of special interest is the assumption that $\mu$ is a compactly supported function in the Sobolev class $W^{1,2}$ (notice that this implies $\mu \in VMO$). For such Beltrami coefficients, it is shown in [CFMOZ] that the corresponding $\mu$-quasiconformal mappings $\phi$ preserve sets with zero length, that is

$$\mathcal{H}^1(E) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0, \tag{2}$$

The proof of this fact uses some $BMO$ removability techniques, related to both the Cauchy-Riemann (i.e. $\overline{\partial}$) and the Beltrami ($\overline{\partial} - \mu\,\partial$) differential operators. The main tool is an extended version of Weyl's lemma. Recall that classical Weyl's Lemma asserts that distributional solutions to the Cauchy-Riemann equation are actually analytic functions. In the more general case of the Beltrami equation [CFMOZ, Theorem 1], an analogous result can be given provided that the Beltrami coefficient belongs to $W^{1,2}$.

**Theorem.** *Let $\mu$ be a compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}(\mathbb{C})$. Let $f \in L^p_{loc}(\mathbb{C})$ for some $p > 2$, and suppose that*

$$\langle \overline{\partial} f - \mu\, \partial f, \varphi \rangle = 0$$

*whenever $\varphi \in \mathcal{C}^\infty$ is compactly supported. Then, $f$ is $\mu$-quasiregular.*

In [CFMOZ], similar arguments to those in (2), replacing $BMO$ by $VMO$, allowed the authors to prove that if $\mu \in W^{1,2}$ is any compactly supported Beltrami coefficient, and $\phi$ is $\mu$-quasiconformal, then

$$\mathcal{H}^1(E) \text{ is } \sigma\text{-finite} \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) \text{ is } \sigma\text{-finite.} \tag{3}$$

Furthermore, these mappings $\phi$ are shown to map 1-rectifiable sets to 1-rectifiable sets (and purely 1-unrectifiable sets to purely 1-unrectifiable sets).

As we shall see in this paper, all these facts have interesting consequences when studying removability problems for bounded $\mu$-quasiregular mappings, that is, the $\mu$-quasiregular counterpart for the problem of Painlevé. Recall that a compact set $E$ is said to be *removable* (*for bounded analytic functions*) if for any open set $\Omega \supset E$, every bounded function $f : \Omega \to \mathbb{C}$, analytic on $\Omega \setminus E$, admits an analytic extension to the whole of $\Omega$. The *problem of Painlevé* consists of giving metric and geometric characterizations of these sets.

When studying removable sets, it is natural to talk about analytic capacity. Recall that given a compact set $E$, the *analytic capacity* of $E$ is defined as

$$\gamma(E) = \sup \{ |f'(\infty)|; f \in H^\infty(\mathbb{C} \setminus E), \|f\|_\infty \leq 1 \}.$$

Here, by $H^\infty(\Omega)$ we mean the space of bounded analytic functions on the open set $\Omega$, and $f'(\infty) = \lim_{z\to\infty} z\, (f(\infty) - f(z))$. For a set $A \subset \mathbb{C}$ which may be non compact, one defines

$$\gamma(A) = \sup_{E \subset A \text{ compact}} \gamma(E).$$

Ahlfors [Ah2] proved that $E$ is removable for bounded analytic functions if and only if $\gamma(E) = 0$. Furthermore, it is not difficult to show that $\gamma(E) \leq C\, \mathcal{H}^1(E)$, while $\dim(E) > 1$ implies $\gamma(E) > 0$. It took long time to have a precise geometric characterization of the zero sets for $\gamma$. In [Da1], G. David proved that if $E$ has finite length then

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad E \text{ is purely 1-unrectifiable.}$$

Later, in [To2], X. Tolsa characterized sets with vanishing analytic capacity in terms of Menger curvature (see Theorem 5 below for more details).

In this paper, as well as in [CFMOZ], our main objects of study are the removable singularities for bounded solutions to a fixed Beltrami equation. Namely, we say that a compact set $E$ is *removable for bounded $\mu$-quasiregular mappings*, or simply *$\mu$-removable*, if for any open set $\Omega$, any bounded function $f : \Omega \to \mathbb{C}$, $\mu$-quasiregular on $\Omega \setminus E$, admits a $\mu$-quasiregular extension to the whole of $\Omega$. By means of Stoilow's factorization Theorem, one easily shows that $E$ is $\mu$-removable if and only if $\gamma(\phi(E)) = 0$ for any $\mu$-quasiconformal mapping $\phi$. In connection with this question, the following result is proved in [CFMOZ].

**Theorem.** *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a $\mu$-quasiconformal mapping. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0 \tag{4}$$

*for any compact set $E$ with $\sigma$-finite $\mathcal{H}^1(E)$.*

Pekka Koskela suggested us that the $\sigma$-finiteness assumption might be removed in the preceding result. In this paper we do the job.

**Theorem 1.** *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a $\mu$-quasiconformal mapping. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0$$

*for any compact set $E$.*

It follows from Theorem 1 that if $\mu \in W^{1,2}$ is compactly supported, then being removable and being $\mu$-removable are equivalent notions.

**Corollary 2.** *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Then, a compact set $E$ is removable for bounded $\mu$-quasiregular mappings if and only if $\gamma(E) = 0$.*

Theorem 1 implies that, given a compactly supported Beltrami coefficient $\mu \in W^{1,2}(\mathbb{C})$, the corresponding $\mu$-quasiconformal mappings preserve the removable sets for bounded analytic functions. This fact is closely related to a question of J. Verdera [Ve1] on the preservation of removable sets under some planar homeomorphisms. More precisely, the author wondered how analytic capacity is distorted under bilipschitz mappings. Recall that a mapping $\phi : \mathbb{C} \to \mathbb{C}$ is said to be $L$-bilipschitz if

$$\frac{1}{L}|z - w| \le |\phi(z) - \phi(w)| \le L|z - w|$$

for any pair of points $z, w \in \mathbb{C}$. This question was solved in [To2]:

**Theorem.** *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an $L$-bilipschitz mapping. Then,*

$$\gamma(\phi(E)) \simeq \gamma(E) \tag{5}$$

*with constants that depend only on $L$.*

Furthermore, it is shown in [To2] that any planar homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ satisfying (5) must be a bilipschitz mapping. It is well known that $L$-bilipschitz mappings are $\mu$-quasiconformal for some Beltrami coefficient $\mu$ with $\|\mu\|_\infty$ depending only on $L$, but in general this does not imply any $W^{1,2}$ regularity for $\mu$. In fact, there is not a precise description of the class of compactly supported Beltrami coefficients $\mu$ that produce bilipschitz $\mu$-quasiconformal mappings. It was remarked in [CFMOZ, Example 4] that there are non bilipschitz $\mu$-quasiconformal mappings with compactly supported $\mu \in W^{1,2}$. At the same time, the example $\mu(z) = \frac{1}{2}\,\chi_{\mathbb{D}}(z)$ gives a bilipschitz $\mu$-quasiconformal mapping with $\mu \notin W^{1,2}$. Thus, there is no relation between bilipschitz $\mu$-quasiconformal mappings and Beltrami coefficients $\mu \in W^{1,2}$.

For the proof of the Theorem 1, our main tool is the following improved version of the preceding theorem on the bilipschitz invariance of analytic capacity.

**Theorem 3.** *Given $E, F \subset \mathbb{C}$, let $\phi : E \to F$ be a bilipschitz homeomorphism. That is, there exists $L > 0$ such that*

$$\frac{1}{L}\,|z - w| \leq |\phi(z) - \phi(w)| \leq L\,|z - w| \tag{6}$$

*whenever $z, w \in E$. Then there exists some constant $C$ depending only on L such that*

$$\frac{1}{C}\,\gamma(F) \leq \gamma(E) \leq C\gamma(F).$$

Notice that in this result we assume the mapping $\phi$ to be bilipschitz only on $E$, not in the whole complex plane. From this theorem we deduce

**Corollary 4.** *Assume that $\phi : \mathbb{C} \to \mathbb{C}$ is a planar homeomorphism, locally bilipschitz in a measurable set $\Omega \subset \mathbb{C}$. That is, there are constants $C > 0$ and $\delta > 0$ such that*

$$\frac{1}{C}\,|z - w| \leq |\phi(z) - \phi(w)| \leq C\,|z - w|$$

*whenever $z, w \in \Omega$ and $|z - w| < \delta$. Then*

$$\gamma(E \cap \Omega) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E \cap \Omega)) = 0$$

*for any compact set $E \subset \mathbb{C}$.*

As explained above, the $\mu$-quasiconformal mappings we deal with are not bilipschitz in the whole plane, in general. However, they are locally bilipschitz on the level sets of its Jacobian determinant, modulo some *small* set of bad points. This follows from the quasisymmetry and from the fact that these $\mu$-quasiconformal mappings belong to some second order Sobolev spaces. Moreover, it turns out that they are strongly differentiable everywhere except on a set of Hausdorff dimension 0.

The paper is structured as follows. In Section 2 we prove Theorem 3, and in Section 3 its Corollary 4. In Section 4, we use this result to prove Theorem 1.

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# 2 Proof of Theorem 3

## 2.1 Analytic capacity and curvature

We need to recall the notion of curvature of a measure. Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the circumference passing through $x, y, z$ (with $R(x, y, z) = \infty$, $c(x, y, z) = 0$ if $x, y, z$ lie on the same line). If two among these points coincide, we set $c(x, y, z) = 0$. For a positive finite Borel measure $\sigma$ on $\mathbb{C}$, we define the *curvature of* $\sigma$ as

$$c^2(\sigma) = \iiint c(x, y, z)^2 \, d\sigma(x) d\sigma(y) d\sigma(z). \tag{7}$$

We recall the characterization of $\gamma$ in terms of curvature from [To2]:

**Theorem 5.** *For any compact $E \subset \mathbb{C}$ we have*

$$\gamma(E) \simeq \sup \sigma(E),$$

*where the supremum is taken over all Borel measures $\sigma$ supported on $E$ such that $\sigma(B(x, r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\sigma) \leq \sigma(E)$.*

We will prove the following result.

**Theorem 6.** *Let $\sigma$ be a Borel measure supported on a compact set $E \subset \mathbb{C}$, such that $\sigma(B(x, r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\sigma) < \infty$. Let $\phi : E \to \phi(E)$ be* *a bilipschitz mapping. There exists a positive constant $C$ depending only on the bilipschitz constant of $\phi$ such that*

$$c^2(\phi_{\#}\sigma) \leq C\big(\sigma(E) + c^2(\sigma)\big),$$

*where $\phi_{\#}\sigma$ stands for the image measure of $\sigma$ by $\phi$.*

It is easy straightforward check that Theorem 3 is a direct consequence of Theorems 5 and 6. We remark that Theorem 6 was proved in [To2, Theorem 1.3] under the stronger assumption that $\phi$ is bilipschitz on the whole complex plane. The next Subsections 2.2, 2.3 and 2.4 deal with the proof of this result.

## 2.2 Additional notation and terminology

By a square we mean a square with sides parallel to the axes. Moreover, we assume the squares to be half closed - half open. The side length of a square $Q$ is denoted by $\ell(Q)$. Given $a > 0$, $aQ$ denotes the square concentric with $Q$ with side length $a\ell(Q)$. The average (linear) density of a Borel measure $\sigma$ on $Q$ is

$$\theta_\sigma(Q) := \frac{\sigma(Q)}{\ell(Q)}. \tag{8}$$

We say that $\sigma$ has linear growth if there exists some constant $C$ such that

$$\sigma(B(x,r)) \leq Cr \qquad \text{for all } x \in \mathbb{C},\ r > 0.$$

A square $Q \subset \mathbb{C}$ is called 4-dyadic if it is of the form $[j2^{-n},\, (j+4)2^{-n}) \times [k2^{-n},\, (k+4)2^{-n})$, with $j, k, n \in \mathbb{Z}$. So a 4-dyadic square with side length $4 \cdot 2^{-n}$ is made up of 16 dyadic squares with side length $2^{-n}$. Given $a, b > 1$, we say that $Q$ is $(a,b)$-doubling if $\sigma(aQ) \leq b\sigma(Q)$. If we don't want to specify the constant $b$, we say that $Q$ is $a$-doubling. Given two squares $Q \subset R$, we set

$$\delta_\sigma(Q,R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|}\, d\sigma(y),$$

where $x_Q$ stands for the center of $Q$, and $R_Q$ is the smallest square concentric with $Q$ that contains $R$. Given a bilipschitz mapping $\phi : \mathbb{C} \to \mathbb{C}$ and a square $Q$, in [To2] one says that that $\phi(Q)$ is a $\phi$-square, and then one defines its side length as $\ell(\phi(Q)) := \ell(Q)$. Now we only know that $\phi$ bilipschitz from $E$ onto $\phi(E)$, and thus $\phi(Q)$ is not defined in general. So we have to change the notion of $\phi$-square. A first attempt would consist in saying that $P$ is a $\phi$-square if $P = \phi(Q \cap E)$ for some square $Q$. This definition has a serious drawback: we cannot set $\ell(P) := \ell(Q)$ because it may happen that $P = \phi(Q \cap E) = \phi(R \cap E)$ for two different squares $Q, R$. However there is an easy solution: a $\phi$-square is not a subset of $\mathbb{C}$, but a pair of the form $P = (Q, \phi(Q \cap E))$, for some square $Q \subset \mathbb{C}$. We denote $\ell(P) := \ell(Q)$. On the other hand, abusing language sometimes we will identify the $\phi$-square $P = (Q, \phi(Q \cap E))$ with the set $\phi(Q \cap E)$, and so we will use notations such as $\operatorname{diam}(P)$ (notice that $\operatorname{diam}(P) \lesssim \ell(P)$), or $\phi_{\#}\sigma(P)$. If $Q_0$ is a dyadic (or 4-dyadic) square, we say that $(Q_0, \phi(Q_0 \cap E))$ is a dyadic (or 4-dyadic) $\phi$-square. If $Q = (Q_0, \phi(Q_0 \cap E))$ is a $\phi$-square, we denote $\lambda Q = (\lambda Q_0, \phi(\lambda Q_0 \cap E))$, for $\lambda > 0$. To simplify notation, we set $\tau := \phi_{\#}\sigma$ and $F := \phi(E)$. A $\phi$-square $Q$ is said to be $\lambda$-doubling if $\tau(\lambda Q) \leq C\tau(Q)$ for some fixed $C \geq 1$. We also set

$$\theta_\tau(Q) := \frac{\tau(Q)}{\ell(Q)}$$

and if $R$ is another $\phi$-square which contains $Q$, we put

$$\delta_\tau(Q, R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|} \, d\tau(y),$$

where $x_Q$ stands for some fixed (arbitrary) point of $Q$ and $R_Q$ is the smallest $\phi$-square concentric with $Q$ that contains $R$. That is to say, if $Q = (Q_0, \phi(Q_0 \cap E))$, $R = (R_0, \phi(R_0 \cap E))$, and $S$ is the smallest square concentric with $Q_0$ that contains $R_0$, we set $R_Q = (S, \phi(S \cap E))$. An Ahlfors regular curve is a curve $\Gamma$ such that $\mathcal{H}^1(\Gamma \cap B(x,r)) \leq Cr$ for all $x \in \Gamma$, $r > 0$, and some fixed $C > 0$. We say that $\Gamma$ is a chord arc curve if it is a bilipschitz image of an interval in $\mathbb{R}$. If the bilipschitz constant of the map is $L$, we say that $\Gamma$ is an $L$-chord arc curve.

## 2.3 The corona decomposition

Theorem 6 will be proved by means of a corona type decomposition for $\sigma$ similar to the one in [To2]. In Lemma 7 below, where we prove the existence of this decomposition, we will introduce a family $\operatorname{Top}(E)$ of 4-dyadic squares (the top squares) satisfying some precise properties. Given any square $Q \in \operatorname{Top}(E)$, we denote by $\operatorname{Stop}(Q)$ the subfamily of the squares $P \in \operatorname{Top}(E)$ satisfying

(a) $P \cap 3Q \neq \varnothing$,

(b) $\ell(P) \leq \frac{1}{8}\ell(Q)$,

(c) $P$ is maximal, in the sense that there doesn't exist another square $P' \in \operatorname{Top}(E)$ satisfying (a) and (b) which contains $P$.

We also denote by $Z(\sigma)$ the set of points $x \in \mathbb{C}$ such that there does not exist a sequence of $(70, 5000)$-doubling squares $\{Q_n\}_n$ centered at $x$ with $\ell(Q_n) \to 0$ as $n \to \infty$, so that moreover $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$. We have $\sigma(Z(\sigma)) = 0$

(see [To2, Remark 2.1]). The set of good points for $Q$ is defined as

$$G(Q) := 3Q \cap \operatorname{supp}(\sigma) \setminus \Big[Z(\sigma) \cup \bigcup_{P\in \operatorname{Stop}(Q)} P\Big].$$

**Lemma 7 (The corona decomposition).** *Let $\sigma$ be a Borel measure supported on $E \subset \mathbb{C}$ such that $\sigma(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$ and $c^2(\sigma) < \infty$. There exists a family* $\operatorname{Top}(E)$ *of* 4*-dyadic* $(16, 5000)$*-doubling squares (called* top squares*) which satisfy the packing condition*

$$\sum_{Q\in \operatorname{Top}(E)} \theta_\sigma(Q)^2 \sigma(Q) \leq C\big(\sigma(E) + c^2(\sigma)\big), \tag{9}$$

*and such that for each square $Q \in \operatorname{Top}(E)$ there is a family of $C_1$-chord arc curves $\Gamma_Q^i$, $i = 1, \dots, N_0$, such that if we set $\Gamma_Q = \bigcup_{i=1}^{N_0} \Gamma_Q^i$, we have*

*(a)* $G(Q) \subset \Gamma_Q \cap E$.

*(b) For each $P \in \operatorname{Stop}(Q)$ there exists some square $\widetilde{P}$ containing $P$ such that $\delta_\sigma(P, \widetilde{P}) \leq C_2\theta_\sigma(Q)$ and $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$.*

*(c) If $P$ is a square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another square $P' \in \operatorname{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\sigma(P) \leq C_3\, \theta_\sigma(Q)\, \ell(P)$.*

*Moreover,* $\operatorname{Top}(E)$ *contains some* 4*-dyadic square $R_0$ such that $E \subset R_0$. The constants $C_1, C_2, C_3, N_0$ are absolute.*

Notice that the chord arc constant of the curves $\Gamma_Q^i$ in the lemma is uniformly bounded above by $C_1$.

*Proof.* The proof of this lemma is very similar to the one of Main Lemma 3.1 in [To2], and so we will only describe in detail the required modifications for the proof. Notice that, in Lemma 7, $\Gamma_Q$ is made up of a finite union of chord arc curves, while in [To2, Main Lemma 3.1] $\Gamma_Q$ is an Ahlfors regular curve. On the other hand, in the statement (b) above we ask $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$, while in (b) of Main Lemma 3.1 in [To2] one asks only $\widetilde{P} \cap \Gamma_Q \neq \varnothing$. These are the only differences between both lemmas.

First we will explain the arguments to show that $\Gamma_Q$ is made up of a finite number of chord arc curves. First we need some notation. Given a set $K \subset \mathbb{C}$ and a square $Q$, let $V_Q$ be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $w(V_Q)$ denote the width of $V_Q$. Denote

$$\beta_K(Q) = \frac{w(V_Q)}{\ell(Q)}.$$

Recall the following version of Jones' traveling salesman theorem [Jo]:

**Theorem 8 (P. Jones).** *A set $K \subset \mathbb{C}$ is contained in an Ahlfors regular curve if and only if there exists some constant $C_4$ such that for every dyadic square $Q$*

$$\sum_{P\in\mathcal{D},P\subset Q} \beta_K(P)^2\ell(P) \leq C_4\ell(Q).$$

*The regularity constant of the curve depends on $C_4$.*

Moreover, in [Jo] the author claims that there exists some constant $\eta > 0$ small enough such that if $\beta_K(Q) \leq \eta$ for every square $Q \in \mathcal{D}$, then $K$ is contained in quasicircle. So we have

**Theorem 9.** *There exists some absolute constant $\eta > 0$ such that if, for every dyadic square $Q$, $K \subset \mathbb{C}$ satisfies*

$$\sum_{P\in\mathcal{D},P\subset Q} \beta_K(P)^2\ell(P) \leq \eta\ell(Q),$$

*then $K$ is contained in a chord arc curve.*

Although the preceding result is not proved in [Jo], it follows from easy modifications of the author's arguments (we suggest the reader to look also at [GM, Chapter 10, Theorem 2.3]).
Given $R \in \mathrm{Top}(E)$, in [To2] the existence of the curve $\Gamma_R$ follows from an application of Theorem 8. To this end, in [To2, Lemma 4.5] one constructs a set $K$ which contains $G(R)$ and which, in a sense, approximates $\mathrm{supp}(\sigma)$ on the squares $P \in \mathrm{Stop}(R)$. Then one proves in [To2, Subsection 4.4] that for any square $Q$ with $\ell(Q) \leq C_5\ell(R)$ (where $C_5 < 1$ is some small positive constant),

$$\sum_{P\in\mathcal{D},P\subset Q} \beta_K(P)^2\ell(P) \leq C\theta_\sigma(R)^{-3} \iiint_{(x,y,z)\in(3Q)^3\cap R^\ell} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z). \tag{10}$$

See p. 1266 of [To2] for the precise definition of $R^\ell$. For the reader's convenience, let us say that the triple integral on the right hand side is some truncated version of the curvature $c^2(\sigma_{|3Q})$. Moreover, from the construction of the stopping squares, it turns out that

$$C\theta_\sigma(R)^{-3} \iiint_{(x,y,z)\in(3Q)^3\cap R^\ell} c(x,y,z)^2 d\sigma(x)d\sigma(y)d\sigma(z) \leq C_6\ell(Q). \tag{11}$$

¿From (10), (11), and Theorem 8 one infers the existence of the regular curve $\Gamma_Q$.
However, a careful examination of the proof of Main Lemma 3.1 in [To2] shows

that the constant $C_6$ in (11) can be taken so that $C_6 \leq \eta$ (in fact, as small as we want). To this end, one has to take the parameter $\varepsilon_0$ small enough in the construction of the high curvature squares (see p. 1252 of [To2]). We leave the details for the reader. Then Theorem 9 can be applied for the squares $Q$ with $Q \cap 3R \neq \varnothing$ such that with $\ell(Q) \leq C_5 \ell(R)$. As a consequence, one deduces that $K$ is contained in a finite number $N_0$ of chord arc curves (we only have to cover $3R$ by a finite number of squares with side length $C_5 \ell(R)$ and then we apply Theorem 9).

On the other hand, it is easy to check that the curve $\Gamma_Q$ constructed in the proof of Main Lemma 3.1 satisfies $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$, as required above in (b). Of course, the same happens with the "new" curves $\Gamma_Q$ described above since the method of construction has not changed. This is due to the fact that the set $K$ obtained in [To2, Lemma 4.5] is contained in $\operatorname{supp}(\mu)$. This is not stated in [To2, Lemma 4.5], but it is easily seen. □

**Remark.** *It is easy to check that one can always assume* $\widetilde{P} \subset 16Q$ *in the statement (b) of Lemma 7.*

## 2.4 The curvature of $\phi_\# \sigma$

In Lemma 7 we have shown how to construct a corona type decomposition for a measure $\sigma$ with linear growth and finite curvature. We will see below that $\phi$ sends this corona type decomposition into another corona decomposition in terms of $\phi$-squares, and we will prove that that the existence of such a decomposition implies that the curvature is finite. These will be the basic ingredients for the proof of Theorem 6.

First we introduce some notation. Given a family $\operatorname{Top}(F)$ of 4-dyadic $\phi$-squares and a fixed $Q \in \operatorname{Top}(F)$, we denote by $\operatorname{Stop}(Q)$ the subfamily of $\phi$-squares which satisfy the properties (a), (b), (c) stated at the beginning of Subsection 2.3 (with squares replaced by $\phi$-squares). The set $G(Q)$ is also defined as in Subsection 2.3, with $\phi$-squares instead of squares.

**Lemma 10.** *Let $\tau$ be a Borel measure supported on a compact set $F \subset \mathbb{C}$. Suppose that $\tau(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. Let $\operatorname{Top}(F)$ be a family of 4-dyadic 16-doubling $\phi$-squares which contains some 4-dyadic $\phi$-square $R_0$ such that $F = R_0$, and such that for each $Q \in \operatorname{Top}(F)$ there exists a $C_7$-AD regular curve $\Gamma_Q$ satisfying:*

*(a) $\tau$-almost every point in $G(Q)$ belongs to $\Gamma_Q$.*

*(b) For each $P \in \operatorname{Stop}(Q)$ there exists some $\phi$-square $\widetilde{P}$ containing $P$ such that $\delta_\tau(P, \widetilde{P}) \leq C\theta_\tau(Q)$ and $\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*

*(c) If $P$ is a $\phi$-square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another $\phi$-square $P' \in \operatorname{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\tau(P) \leq C\, \theta_\tau(Q)\, \ell(P)$.*

*Then,*

$$c^2(\tau) \leq C \sum_{Q \in \operatorname{Top}(F)} \theta_\tau(Q)^2 \tau(Q).$$

The proof of this lemma is almost the same as the one of [To2, Main Lemma 8.1], and so we omit the details.

In order to show that $\phi$ transforms a corona type decomposition like the one in Lemma 7 into another like the one of the preceding lemma we need the following result of MacManus [MM]:

**Theorem 11.** *Any $M$-bilipschitz map of a subset of a line or a circle into the plane has an extension to a $C(M)$-bilipschitz map from the plane onto itself.*

We are ready to prove Theorem 6 now:

**Proof of Theorem 6.** We consider the measure $\sigma$ and its corona type decomposition given by Lemma 7. It is straightforward to check that $\tau := \phi_\# \sigma$ has linear growth. We take the family $\operatorname{Top}(F) = \phi(\operatorname{Top}(E))$, and for $Q \in \operatorname{Top}(F)$ with $Q = (Q_0, \phi(Q_0 \cap E))$, we define $\operatorname{Stop}(Q) = \phi(\operatorname{Stop}(Q_0))$.
To construct a regular curve $\Gamma_Q$ such as the one required in Lemma 10, we consider the union of chord arc curves $\Gamma_{Q_0} = \bigcup_{i=1}^{N_0} \Gamma^i_{Q_0}$ of Lemma 7. Notice that we cannot set $\Gamma_Q = \phi(\Gamma_{Q_0})$ because $\phi$ is not defined on the whole set $\Gamma_{Q_0}$. By MacManus' theorem we can solve this problem easily. For each $i = 1, \ldots, N_0$, let $\rho_i : \mathbb{R} \supset I \to \Gamma^i_{Q_0}$ be a bilipschitz parametrization of the chord arc curve $\Gamma^i_{Q_0}$. Consider the subset $\Gamma^i_{Q_0} \cap E$ and the bilipschitz map $\phi \circ \rho_i$, defined on $\rho_i^{-1}(\Gamma^i_{Q_0} \cap E) \subset \mathbb{R}$. By Theorem 11, $\phi \circ \rho_i$ has a bilipschitz extension $f_i$ onto the whole complex plane. We consider the chord arc curve $\Gamma_i := f_i(\mathbb{R})$, and we set $\Gamma_Q := \bigcup_{i=1}^{N_0} \Gamma_i$ (and we add a finite number of segments if necessary to ensure that that $\Gamma_Q$ is a regular curve).

We have to show that the assumptions of Lemma 10 hold for the family $\operatorname{Top}(F)$, their corresponding stopping squares, and the curves $\Gamma_Q$, $Q \in \operatorname{Top}(F)$. Indeed, (a) and (c) are the translation of the corresponding statements (a) and (c) of Lemma 7. On the other hand, (b) is a consequence of the fact that if

$P_0 \in \mathrm{Stop}(Q_0)$ for some $Q_0 \in \mathrm{Top}(E)$, and $\widetilde{P}_0 \subset 16Q_0$ (recall Remark 2.3), and moreover we have

$$\delta_\sigma(P_0, \widetilde{P}_0) \le C_2 \theta_\sigma(Q_0),$$

then $P = (P_0, \phi(P_0 \cap E))$ and $\widetilde{P} = (\widetilde{P}_0, \phi(\widetilde{P}_0 \cap E))$ satisfy

$$\delta_\tau(P, \widetilde{P}) \lesssim \theta_\tau(Q). \tag{12}$$

To prove this estimate, recall that

$$\delta_\tau(P, \widetilde{P}) := \int_{\widetilde{P}_P \setminus 2P} \frac{1}{|y - z_P|}\, d\tau(y),$$

where $z_P$ is some fixed point of $P$ and $\widetilde{P}_P$ is the smallest $\phi$-square concentric with $P$ that contains $\widetilde{P}$. Then, if we set $z_{P_0} = \phi^{-1}(z_P)$, we have

$$\delta_\tau(P, \widetilde{P}) = \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|\phi(y) - \phi(z_{P_0})|}\, d\sigma(y) \simeq \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - z_{P_0}|}\, d\sigma(y).$$

Since $z_{P_0} \in P_0$, from the property (c) of Lemma 7, it follows easily that

$$\begin{aligned} \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - z_{P_0}|}\, d\sigma(y) &\le \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - x_{P_0}|}\, d\sigma(y) + C\theta_\sigma(Q_0) \\ &= \delta(P_0, \widetilde{P}_0) + C\theta_\sigma(Q_0) \le C\theta_\sigma(Q_0) \simeq \theta_\tau(Q). \end{aligned}$$

(recall that $x_{P_0}$ is the center of $P_0$, which may not coincide with $z_{P_0}$), and so (12) holds.

Thus from Lemmas 10 and 7 we infer that

$$c^2(\tau) \lesssim C \sum_{Q \in \mathrm{Top}(F)} \theta_\tau(Q)^2 \tau(Q) \simeq \sum_{Q_0 \in \mathrm{Top}(E)} \theta_\sigma(Q_0)^2 \sigma(Q_0) \lesssim \sigma(E) + c^2(\sigma).$$

□

# 3 Proof of Corollary 4

This is an immediate consequence of Theorem 3. Let $F \subset E \cap \Omega$ be compact. Obviously, $\gamma(F) = 0$. Cover $F$ by a finite number of closed balls $B_i$, $1 \le i \le N$, of diameter $\delta/2$. Since $\phi : F \cap B_i \to \phi(F \cap B_i)$ is bilipschitz, we have

$$\gamma(\phi(F \cap B_i)) \simeq \gamma(F \cap B_i) = 0$$

for all $i$. This implies that $\gamma(\phi(F)) = 0$ (this is a consequence of the semiadditivity of $\gamma$, but it can be proven by much simpler arguments). Since this holds for any compact set $\phi(F) \subset \phi(E \cap \Omega)$, we have $\gamma(\phi(E \cap \Omega)) = 0$. □

# 4 Proof of Theorem 1

In all this section, $\mu$ is a compactly supported Beltrami coefficient belonging to the Sobolev space $W^{1,2}(\mathbb{C})$, and $\phi:\mathbb{C}\to\mathbb{C}$ is a $\mu$-quasiconformal mapping. There is no restriction if we normalize $\phi$ in such a way that $\phi(z)-z=O(1/|z|)$ as $|z|\to\infty$. We can assume also that $\text{supp}(\mu)\subset\mathbb{D}$.

We start by giving some auxiliary results. The first one gives some information on the distortion of Riesz capacities under $\mu$-quasiconformal mappings. Recall that if $E$ is any compact set on the plane, the $(1,p)$-*Riesz capacity* of $E$ is defined as

$$C_{1,p}(E)=\inf\left\{\|D\psi\|_p\right\}$$

where the infimum is taken over all compactly supported $\psi\in\mathcal{C}^\infty(\mathbb{C})$ with $\psi\geq\chi_E$. One obtains the same quantity if we simply assume $\psi\in W^{1,p}(\mathbb{C})$. For more details about Riesz capacities, see [AH], for example.

**Lemma 12.** *Let $\mu\in W^{1,2}$ be a compactly supported Beltrami coefficient, and let $\phi:\mathbb{C}\to\mathbb{C}$ be a $\mu$-quasiconformal mapping. Then,*

$$C_{1,q}(\phi(E))^{\frac{1}{q}}\lesssim C_{1,p}(E)^{\frac{1}{p}}$$

*for any compact set $E\subset\mathbb{D}$, and any $1<q<p<2$.*

*Proof.* It is not hard to see that if $\psi$ is a $W^{1,p}$ function, continuous and compactly supported inside of $\mathbb{D}$, and $\psi\geq 1$ on $E$, then $\psi\circ\phi^{-1}$ is a continuous, compactly supported function of class $W^{1,q}$, whenever $q<p$. Indeed,

$$\left(\int|D(\psi\circ\phi^{-1})|^q\right)^{\frac{1}{q}}\leq\left(\int|D\psi|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{D}}J(\cdot,\phi)^{\frac{1-\frac{q}{2}}{1-\frac{q}{p}}}\right)^{\frac{1}{q}-\frac{1}{p}}$$

and here we must remark that the last integral converges since $q<p<2$ and $\mu\in W^{1,2}$. Therefore,

$$C_{1,q}(\phi(E))^{\frac{1}{q}}\leq\left(\int|D\psi|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{D}}J(\cdot,\phi)^{\frac{1-\frac{q}{2}}{1-\frac{q}{p}}}\right)^{\frac{1}{q}-\frac{1}{p}}$$

Thus,

$$C_{1,q}(\phi(E))^{\frac{1}{q}}\leq C_{1,p}(E)^{\frac{1}{p}}\left(\int_{\mathbb{D}}J(\cdot,\phi)^{\frac{1-\frac{q}{2}}{1-\frac{q}{p}}}\right)^{\frac{1}{q}-\frac{1}{p}}$$

and the result follows. □

The following result establishes that the means of the Jacobian determinant of a quasiconformal mapping behave precisely as an incremental quotient.

**Lemma 13.** *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a $K$-quasiconformal mapping. Let $z, w \in \mathbb{C}$ be such that $z \neq w$, and $D = D(z, |z-w|)$. Then,*

$$\frac{|\phi(z)-\phi(w)|}{|z-w|} \simeq \left(\frac{1}{|D|}\int_D J(z,\phi)\,dA(z)\right)^{\frac{1}{2}}$$

*with constants that depend only on $K$.*

*Proof.* Let $r = |w-z|$. Then,

$$\begin{aligned}|\phi(w)-\phi(z)| &\leq \max_{|\zeta-z|=r}|\phi(\zeta)-\phi(z)| \leq C_K \min_{|\zeta-z|=r}|\phi(\zeta)-\phi(z)| \\ &\leq C_K\left(\frac{|\phi(D(z,r))|}{\pi}\right)^{\frac{1}{2}} \\ &= C_K\,|z-w|\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} J(\zeta,\phi)\,dA(\zeta)\right)^{\frac{1}{2}}\end{aligned}$$

The converse inequality can be obtained analogously. □

**Proof of Theorem 1.** Since $\mu \in W^{1,2}(\mathbb{C})$ is compactly supported, we know (see for instance [CFMOZ, Proposition 3]) that $\phi \in W^{2,p}_{loc}$ for every $p < 2$. Thus, writing the Jacobian determinant as $J(\cdot,\phi) = |\partial\phi|^2 - |\overline{\partial}\phi|^2$, we get $J(\cdot,\phi) \in W^{1,p}_{loc}$ for every $p < 2$, and in particular $J(\cdot,\phi)$ admits a $C_{1,p}$-quasicontinuous representative, for each $p < 2$. Moreover, modulo a set of small $C_{1,p}$-capacity, every point is a Lebesgue point for $J(\cdot,\phi)$ (for more details see [AH, Chapter 6]). Then, given $\varepsilon > 0$ we can find an open set $U \subset \mathbb{D}$ such that:

- $C_{1,p}(U) < \varepsilon$.
- $J(\cdot,\phi)$ is continuous on $\overline{\mathbb{D}} \setminus U$.
- $\displaystyle\lim_{r\to 0}\frac{1}{|D(z,r)|}\int_{D(z,r)} J(w,\phi)\,dA(w) = J(z,\phi)$ uniformly on $z \in \overline{\mathbb{D}} \setminus U$.

As a consequence of the third point, together with Lemma 13, we can find $\delta = \delta(\varepsilon) > 0$ such that whenever $z, w \in \overline{\mathbb{D}} \setminus U$ satisfy $|z-w| < \delta$, we have

$$J(z,\phi)^{\frac{1}{2}} \simeq \frac{|\phi(w)-\phi(z)|}{|w-z|}$$

with constants that depend on $K$ and $\varepsilon$. At this point, we could feel tempted to say that $\phi$ is locally bilipschitz on $\overline{\mathbb{D}} \setminus U$, but this may fail because $J(\cdot,\phi)$ may vanish at some points on the boundary of $\overline{\mathbb{D}} \setminus U$. To solve this, for each integer $n \in \mathbb{N}$ we set

$$F_n = \left\{ z \in \overline{\mathbb{D}} \setminus U : \frac{1}{n} < J(z,\phi) < n \right\}$$

Denote by $Z$ the set of points $z \in \overline{\mathbb{D}} \setminus U$ such that either $\phi$ is not differentiable at $z$, or $\phi^{-1}$ is not differentiable at $\phi(z)$. Then, it is straightforward to show that

$$\overline{\mathbb{D}} \setminus U = Z \cup \bigcup_{n=1}^{\infty} F_n$$

Since $J(\cdot, \phi)$ is continuous on $\overline{\mathbb{D}} \setminus U$, $F_n$ is open in the topology of $\overline{\mathbb{D}} \setminus U$, so that $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is closed in the same topology. Therefore, since $\overline{\mathbb{D}} \setminus U$ is closed as a subset of $\mathbb{C}$, also $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is closed. Thus, $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is a decreasing sequence of compact sets and then we have

$$\begin{aligned}
\lim_{n\to\infty} C_{1,p}\left((\overline{\mathbb{D}} \setminus U) \setminus F_n\right) &= C_{1,p}\left(\bigcap_{n=1}^{\infty} (\overline{\mathbb{D}} \setminus U) \setminus F_n\right) \\
&= C_{1,p}\left((\overline{\mathbb{D}} \setminus U) \setminus \bigcup_{n=1}^{\infty} F_n\right) \\
&\leq C_{1,p}(Z)
\end{aligned}$$

But both $\phi$ and $\phi^{-1}$ are differentiable $C_{1,p}$-almost everywhere, for any $p < 2$, so that $\dim(Z) = 0$ and therefore the above limit also vanishes, that is,

$$\lim_{n\to\infty} C_{1,p}\left((\overline{\mathbb{D}} \setminus U) \setminus F_n\right) = 0$$

Now assume that $E \subset \mathbb{D}$ is a compact set with $\gamma(E) = 0$. We can split $E$ into several small pieces. First of all, we consider $E \cap U$. By Lemma 12, and since $1 < q < 2$, one easily shows that

$$\gamma(\phi(E\cap U)) \lesssim C_{1,q}(\phi(E\cap U))^{\frac{1}{q}} \lesssim C_{1,p}(E\cap U)^{\frac{1}{p}} \leq C_{1,p}(U)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$$

with constants that may depend on $\phi$, but not on $\epsilon$. Secondly, since $\phi$ does not distort Hausdorff dimension,

$$\dim(\phi(E\cap Z)) = \dim(E\cap Z) = 0$$

For the remaining part, it follows from Lemma 13 that $\phi$ is locally bilipschitz on every set $F_n$. In fact, we can split $F_n$ into a finite union of subsets $F_{n,k}$ with diameter small enough so that $\phi$ is bilipschitz on $F_{n,k}$ for each $k$. Thus, applying Corollary 4 we get

$$\gamma(\phi(E\cap F_n)) = 0$$

because $\gamma(E\cap F_n) \leq \gamma(E) = 0$. Summarizing, we get for any $n \in \mathbb{N}$

$$\begin{aligned}
\gamma(\phi(E\cap((\overline{\mathbb{D}} \setminus U) \setminus F_n))) &\leq C_{1,q}(\phi(E\cap((\overline{\mathbb{D}} \setminus U) \setminus F_n)))^{\frac{1}{q}} \\
&\lesssim C_{1,p}(E\cap((\overline{\mathbb{D}} \setminus U) \setminus F_n))^{\frac{1}{p}} \\
&\leq C_{1,p}((\overline{\mathbb{D}} \setminus U) \setminus F_n)^{\frac{1}{p}}
\end{aligned}$$

Therefore, by the semiadditivity of analytic capacity [To2],

$$\begin{aligned}\gamma(\phi(E)) &\lesssim \gamma(\phi(E\cap U)) + \gamma(\phi(E\cap(\overline{\mathbb{D}}\setminus U)))\\ &\lesssim \gamma(\phi(E\cap U)) + \gamma(\phi(E\cap Z)) + \gamma(\phi(E\cap F_n)) + \gamma(\phi(E\cap((\overline{\mathbb{D}}\setminus U)\setminus F_n)))\\ &\lesssim \varepsilon^{\frac{1}{p}} + 0 + 0 + C_{1,p}(E\cap((\overline{\mathbb{D}}\setminus U)\setminus F_n))^{\frac{1}{p}}\end{aligned}$$

for each $n\in\mathbb{N}$. In the right hand side, the last term converges to 0 as $n\to\infty$. Thus, $\gamma(\phi(E))$ can be made as small as we wish. □

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