ANALYTIC CAPACITY AND CALDERÓN-ZYGMUND THEORY WITH NON DOUBLING MEASURES

XAVIER TOLSA

ABSTRACT. These notes are the lecture notes of a series of talks given at the Universidad de Sevilla in December 2003. We survey some results of Calderón-Zygmund theory with non doubling measures, and we apply them to prove the semiaddivity of the analytic capacity γ_+. We provide a quite elementary proof which does not use the $T(1)$ theorem. We also review other recent results in connection with the comparability between analytic capacity and the capacity γ_+.

1. INTRODUCTION

The main purpose of this expository paper is to discuss and review several results on Calderón-Zygmund theory with non doubling measures (also known as *non homogeneous* Calderón-Zygmund theory) and to show how these results can be applied to problems related to analytic capacity.

In recent years it was shown that many results on Calderón-Zygmund theory remain valid if one does not assume that the underlying measure of the space is doubling. Recall that a Borel measure μ on $\mathbb{R}^d$ is said to be doubling if there exists some constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \text{for all } x \in \operatorname{supp}(\mu),\ r > 0.$$

One of the main motivations for extending the classical theory to the non doubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture or Painlevé's problem. In this type of problems, one considers an arbitrary compact set E in the complex plane and one is interested in finding a Radon measure μ supported on it such that the Cauchy transform $\mathcal{C}_\mu$ (see Section 2 for the precise definition) is bounded on $L^2(\mu)$. It may happen that the only non zero measures with these properties are non doubling.

In order to study n-dimensional Calderón-Zygmund operators (CZO's) in $\mathbb{R}^d$, with $0 < n \leq d$, we will consider measures μ satisfying the growth condition

$$\mu(B(x, r)) \leq C_0\, r^n \quad \text{for all } x \in \mathbb{R}^d,\ r > 0. \tag{1}$$

Let us remark that this is a quite natural condition, because it is necessary for the $L^2(\mu)$ boundedness of any CZO whose kernel $k(x, y)$ satisfies $|k(x, y)| \geq C|x - y|^{-n}$ (see [Dd2, Theorem III.1.4]).

One of the main difficulties that arises when one deals with a non doubling measure μ is due to the fact that the non centered maximal Hardy-Littlewood

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Partially supported by grants BFM2000-0361 and MTM2004-00519 (Spain), 2001-SGR-00431 (Generalitat de Catalunya), and HPRN-2000-0116 (European Union).

operator

$$M_\mu^{nc} f(x) := \sup\left\{ \frac{1}{\mu(\bar{B})} \int_{\bar{B}} |f|\, d\mu : \; \bar{B} \text{ closed ball, } x \in \bar{B} \right\}$$

may fail to be of weak type $(1,1)$ (the superindex "nc" stands for non centered). Sometimes the centered version of the operator, that is

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu,$$

is a good substitute of $M_\mu^{nc} f$, because using Besicovitch's covering theorem one can show that M_μ is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$, and in $L^p(\mu)$, for $1 < p \leq \infty$. However, one cannot always use the centered maximal Hardy-Littlewood operator instead of the non centered one. In these cases, other arguments (usually more involved) are required.

This paper is not intended to be a complete survey neither on Calderón-Zygmund theory with non doubling measures nor on analytic capacity. We recommend the interested reader to have a look at the surveys [Dd3], [Ma2], [Ve3], [MTV2], for example.

Regarding non homogeneous Calderón-Zygmund theory, we will focus our attention on some of the results more directly connected to analytic capacity. In Section 3, for example, we will review the proof of the weak $(1,1)$ boundedness of CZO's which are bounded in $L^2(\mu)$, using a Calderón-Zygmund type decomposition adapted to the non doubling context. We will also give the detailed proof of Cotlar's inequality, which we think that is particularly simple and illuminating. We will state and discuss (but not prove) the $T(1)$ theorem. On the other hand, for reasons of brevity and simplicity we will not pay much attention to $T(b)$ type theorems, although they are important results which play a very important role in connection with analytic capacity. We ask the reader the to forgive us about this question. Similarly, we will only make some brief comments about other results dealing with the space RBMO, Hardy spaces, commutators, weights, etc.

The second part of the paper is dedicated to analytic capacity. In Section 4, we review some properties of analytic capacity and its connection with the Cauchy transform, Menger curvature, and rectifiability. In Section 5 we obtain several characterizations of the analytic capacity γ_+ using some of the results proved or described previously about non homogeneous Calderón-Zygmund theory. In particular, from one of these characterizations the semiadditivity of γ_+ follows in a straightforward way. Moreover, we provide a quite elementary proof of the semiadditivity of γ_+ which does not use the $T(1)$ theorem (although we also explain the argument which uses the $T(1)$ theorem).

The proof of the semiadditivity of γ and its comparability with γ_+ requires much more work and it is out of the scope of this paper. Nevertheless, the last section contains some comments about this topic and related results.

Acknowledgements. I would like to thank Genaro López for his kind invitation to give the lectures from which these notes arose, and the entire Department of Mathematical Analysis of the Universidad de Sevilla for his hospitality during my visit in December 2003.

2. Preliminaries

An open ball centered at x with radius r is denoted by $B(x,r)$, and a closed ball by $\bar{B}(x,r)$. By a cube Q we mean a closed cube with sides parallel to the axes. We denote its side length by $\ell(Q)$ and its center by x_Q.

A Radon measure on $\mathbb{R}^d$ has growth of degree n (or is of degree n) if there exists some constant C_0 such that $\mu(B(x,r)) \leq C_0 r^n$ for all $x \in \mathbb{R}^d$, $r > 0$. When $n = 1$, we say that μ has linear growth. If there exists some constant C such that

$$C^{-1} r \leq \mu(B(x,r)) \leq Cr \quad \text{for all } x \in \operatorname{supp}(\mu),\ 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)),$$

then we say that μ is n-dimensional AD-regular

The space of finite complex Radon measures on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. This is a Banach space with the norm of the total variation: $\|\mu\| = |\mu|(\mathbb{R}^d)$.

We say that $k(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\} \to \mathbb{C}$ is an n-dimensional Calderón-Zygmund kernel if there exist constants $C > 0$ and η, with $0 < \eta \leq 1$, such that the following inequalities hold for all $x, y \in \mathbb{R}^d$, $x \neq y$:

(2)

$$\begin{aligned} &|k(x,y)| \leq \frac{C}{|x-y|^n}, \qquad \text{and} \\ &|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq \frac{C|x-x'|^\eta}{|x-y|^{n+\eta}} \qquad \text{if } |x-x'| \leq |x-y|/2. \end{aligned}$$

Given a positive or complex Radon measure μ on $\mathbb{R}^d$, we define

$$T\mu(x) := \int k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu). \tag{3}$$

We say that T is an n-dimensional Calderón-Zygmund operator (CZO) with kernel $k(\cdot,\cdot)$. The integral in the definition may not be absolutely convergent if $x \in \operatorname{supp}(\mu)$. For this reason, we consider the following ε-truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon \mu(x) := \int_{|x-y|>\varepsilon} k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

Observe that now the integral on the right hand side converges absolutely if, for instance, $|\mu|(\mathbb{R}^d) < \infty$.

Given a fixed positive Radon measure μ on $\mathbb{R}^d$ and $f \in L^1_{loc}(\mu)$, we denote

$$T_\mu f(x) := T(f\, d\mu)(x) \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(f\, d\mu),$$

and

$$T_{\mu,\varepsilon} f(x) := T_\varepsilon(f\, d\mu)(x).$$

The last definition makes sense for all $x \in \mathbb{R}^d$ if, for example, $f \in L^1(\mu)$. We say that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$. Analogously, with respect to the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. We also say that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ if there exists some constant C such that for all $\nu \in M(\mathbb{R}^d)$ and all $\lambda > 0$,

$$\mu\{x \in \mathbb{R}^d : |T_\varepsilon \nu| > \lambda\} \leq \frac{C\|\nu\|}{\lambda}$$

uniformly on $\varepsilon > 0$.

The Cauchy transform is the CZO on $\mathbb{C}$ originated by the kernel

$$k(x,y) := \frac{1}{y-x}, \quad x,y \in \mathbb{C}.$$

It is denoted by $\mathcal{C}$. That is to say,

$$\mathcal{C}\mu(x) := \int \frac{1}{y-x}\, d\mu(y), \qquad x \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

As usual, in the paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

3. Calderón-Zygmund theory with non doubling measures

In this section we will review some results about Calderón-Zygmund theory for non doubling measures μ in $\mathbb{R}^d$ satisfying the growth condition (1) that will be useful in connection with analytic capacity. First, we will describe a Calderón-Zygmund decomposition suitable for this type of measures, and then we will show how one can use it to prove that a CZO which is bounded on $L^2(\mu)$ is also of weak type $(1,1)$. Further, we will prove Cotlar's inequality, and we will talk about the $T(1)$ theorem, and other results.

Preliminarily, in next subsection, we deal with the existence and properties of the so called *doubling cubes*, which play a very important role in this theory.

3.1. **Doubling cubes.** Given $\alpha > 1$ and $\beta > \alpha^n$, we say that Q is (α,β)-doubling if $\mu(\alpha Q) \leq \beta\, \mu(Q)$, where αQ is the cube concentric with Q with side length $\alpha\, \ell(Q)$. For definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube.

Before proving Theorem 3, we state some remarks about the existence of doubling cubes.

Because μ satisfies the growth condition (1), there are a lot of "big" doubling cubes. To be precise, given any point $x \in \operatorname{supp}(\mu)$ and $c > 0$, there exists some (α,β)-doubling cube Q centered at x with $l(Q) \geq c$. This follows easily from (1) and the fact that $\beta > \alpha^n$. Indeed, if there are no doubling cubes centered at x with $l(Q) \geq c$, then $\mu(\alpha^m Q) > \beta^m \mu(Q)$ for each m, and letting $m \to \infty$ one sees that (1) cannot hold.

There are a lot of "small" doubling cubes too: if $\beta > \alpha^d$, then for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of (α,β)-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. This is a property that any Radon measure on $\mathbb{R}^d$ satisfies (the growth condition (1) is not necessary in this argument). The proof is an easy exercise on geometric measure theory that is left for the reader.

Observe that, by the Lebesgue differentiation theorem, for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of $(2, 2^{d+1})$-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f\, d\mu = f(x).$$

As a consequence, for any fixed $\lambda > 0$, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists a sequence of cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ such that

$$\limsup_{k\to\infty} \frac{1}{\mu(2Q_k)} \int_{Q_k} |f|\, d\mu > \frac{\lambda}{2^{d+1}}.$$

In next lemma we prove a very useful estimate from [DM] involving non doubling squares which relies on the idea that the mass μ which lives on non doubling squares must be small.

Lemma 1. *If $Q \subset R$ are concentric cubes such that there are no (α, β)-doubling cubes (with $\beta > \alpha^n$) of the form $\alpha^k Q$, $k \geq 0$, with $Q \subset \alpha^k Q \subset R$, then,*

$$\int_{R\setminus Q} \frac{1}{|x - x_Q|^n}\, d\mu(x) \leq C_1,$$

where C_1 depends only on α, β, n, d and C_0.

Proof. Let N be the least integer such that $R \subset \alpha^N Q$. For $0 \leq k \leq N$ we have $\mu(\alpha^k Q) \leq \mu(\alpha^N Q)/\beta^{N-k}$. Then,

$$\begin{aligned}
\int_{R\setminus Q} \frac{1}{|x - x_Q|^n}\, d\mu(x) &\leq \sum_{k=1}^{N} \int_{\alpha^k Q \setminus \alpha^{k-1} Q} \frac{1}{|x - x_Q|^n}\, d\mu(x) \\
&\leq C \sum_{k=1}^{N} \frac{\mu(\alpha^k Q)}{\ell(\alpha^k Q)^n} \\
&\leq C \sum_{k=1}^{N} \frac{\beta^{k-N}\, \mu(\alpha^N Q)}{\alpha^{(k-N)n}\, \ell(\alpha^N Q)^n} \\
&\leq C\, \frac{\mu(\alpha^N Q)}{\ell(\alpha^N Q)^n} \sum_{j=0}^{\infty} \left(\frac{\alpha^n}{\beta}\right)^j \leq C.
\end{aligned}$$

□

3.2. **Calderón-Zygmund decomposition.**

Lemma 2 (Calderón-Zygmund decomposition)**.** *Assume that μ satisfies* (1)*. For any $f \in L^1(\mu)$ and any $\lambda > 0$ (with $\lambda > 2^{d+1}\, \|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$) we have:*

(a) *There exists a family of almost disjoint cubes $\{Q_i\}_i$ (that is, $\sum_i \chi_{Q_i} \leq C$) such that*

$$\frac{1}{\mu(2Q_i)} \int_{Q_i} |f|\, d\mu > \frac{\lambda}{2^{d+1}}, \tag{4}$$

$$\frac{1}{\mu(2\eta Q_i)} \int_{\eta Q_i} |f|\, d\mu \leq \frac{\lambda}{2^{d+1}} \quad \text{for } \eta > 2, \tag{5}$$

$$|f| \leq \lambda \quad \text{a.e. } (\mu) \text{ on } \mathbb{R}^d \setminus \textstyle\bigcup_i Q_i. \tag{6}$$

(b) *For each i, let R_i be a $(6, 6^{n+1})$-doubling cube concentric with Q_i, with $l(R_i) > 4l(Q_i)$ and denote $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$. Then, there exists a family of functions φ_i with $\operatorname{supp}(\varphi_i) \subset R_i$ and with constant sign satisfying*

$$\int \varphi_i\, d\mu = \int_{Q_i} f\, w_i\, d\mu, \tag{7}$$

$$\sum_i |\varphi_i| \leq B\,\lambda \tag{8}$$

(where B is some constant), and

$$\|\varphi_i\|_{L^\infty(\mu)}\,\mu(R_i) \leq C \int_{Q_i} |f|\,d\mu. \tag{9}$$

The lemma above was obtained in [To3], where it was used to prove that if a linear operator is bounded from a suitable space of type H^1 into $L^1(\mu)$ and from $L^\infty(\mu)$ into a space of type BMO, then it is bounded in $L^p(\mu)$, for $1 < p < \infty$. This Calderón-Zygmund decomposition has also shown to be useful in a variety of other situations (see, for example, [To4], [HY], [HMY1]).

3.3. **Weak (1,1) boundedness of Calderón-Zygmund operators.** The result below was first obtained in [NTV2], although a previous proof valid only for the Cauchy transform appeared in [To1]. Below we reproduce the proof of [To4], which is different from the one of [NTV2] and it is based on the Calderón-Zygmund decomposition of Lemma 2.

Theorem 3. *Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the growth condition* (1). *If T is an n-dimensional Calderón-Zygmund operator which is bounded in $L^2(\mu)$, then it is also bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In particular, it is of weak type* $(1,1)$. *(as far as we know)*

Proof. We will show that T_μ is of weak type $(1,1)$. By similar arguments, one gets that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In this case, one has to use a version of the Calderón-Zygmund decomposition in the lemma above suitable for complex measures (see the end of the proof for more details).

Let $f \in L^1(\mu)$ and $\lambda > 0$. It is straightforward to check that we may assume $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$. Let $\{Q_i\}_i$ be the almost disjoint family of cubes of Lemma 2. Let R_i be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_i$, $k \geq 1$. Then we can write $f = g + b$, with

$$g = f\,\chi_{\mathbb{R}^d \setminus \bigcup_i Q_i} + \sum_i \varphi_i$$

and

$$b = \sum_i b_i := \sum_i (w_i\, f - \varphi_i)\,,$$

where the functions φ_i satisfy (7), (8) (9) and $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$.

By (4) we have

$$\mu\Big(\bigcup_i 2Q_i\Big) \leq \frac{C}{\lambda} \sum_i \int_{Q_i} |f|\,d\mu \leq \frac{C}{\lambda} \int |f|\,d\mu.$$

So we have to show that

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_{\mu,\varepsilon} f(x)| > \lambda\Big\} \leq \frac{C}{\lambda} \int |f|\,d\mu. \tag{10}$$

Since $\int b_i\,d\mu = 0$, $\operatorname{supp}(b_i) \subset R_i$ and $\|b_i\|_{L^1(\mu)} \leq C \int_{Q_i} |f|\,d\mu$, using condition 2 in the definition of a Calderón-Zygmund kernel (which implies Hörmander's condition), we get

$$\int_{\mathbb{R}^d \setminus 2R_i} |T_{\mu,\varepsilon} b_i|\,d\mu \leq C \int |b_i|\,d\mu \leq C \int_{Q_i} |f|\,d\mu.$$

Let us see that

$$\int_{2R_i\setminus 2Q_i} |T_{\mu,\varepsilon} b_i|\,d\mu \leq C\int_{Q_i} |f|\,d\mu \tag{11}$$

too. On the one hand, by (9) and using the $L^2(\mu)$ boundedness of T and that R_i is $(6,6^{n+1})$-doubling we get

$$\begin{aligned}\int_{2R_i} |T_{\mu,\varepsilon}\varphi_i|\,d\mu &\leq \left(\int_{2R_i} |T_{\mu,\varepsilon}\varphi_i|^2\,d\mu\right)^{1/2} \mu(2R_i)^{1/2}\\ &\leq C\left(\int |\varphi_i|^2\,d\mu\right)^{1/2} \mu(R_i)^{1/2}\\ &\leq C\int_{Q_i} |f|\,d\mu.\end{aligned}$$

On the other hand, since $\operatorname{supp}(w_i f) \subset Q_i$, if $x \in 2R_i \setminus 2Q_i$, then $|T_{\mu,\varepsilon} f(x)| \leq C\int_{Q_i} |f|\,d\mu/|x-x_{Q_i}|^n$, and so

$$\int_{2R_i\setminus 2Q_i} |T_{\mu,\varepsilon}(w_i\,f)|\,d\mu \leq C\int_{2R_i\setminus 2Q_i} \frac{1}{|x-x_{Q_i}|^n}\,d\mu(x) \times \int_{Q_i} |f|\,d\mu,$$

By Lemma 1, the first integral on the right hand side is bounded by some constant independent of Q_i and R_i, since there are no $(6,6^{n+1})$-doubling cubes of the form $6^k Q_i$ between $6Q_i$ and R_i. Therefore, (11) holds.

Then we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\bigcup_k 2Q_k} |T_{\mu,\varepsilon} b|\,d\mu &\leq \sum_i \int_{\mathbb{R}^d\setminus\bigcup_k 2Q_k} |T_{\mu,\varepsilon} b_i|\,d\mu\\ &\leq C\sum_i \int_{Q_i} |f|\,d\mu \leq C\int |f|\,d\mu.\end{aligned}$$

Therefore,

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i :\, |T_{\mu,\varepsilon} b(x)| > \lambda\Big\} \leq \frac{C}{\lambda}\int |f|\,d\mu. \tag{12}$$

The corresponding integral for the function g is easier to estimate. Taking into account that $|g| \leq C\,\lambda$, we get

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i :\, |T_{\mu,\varepsilon} g(x)| > \lambda\Big\} \leq \frac{C}{\lambda^2}\int |g|^2\,d\mu \leq \frac{C}{\lambda}\int |g|\,d\mu. \tag{13}$$

Also, we have

$$\begin{aligned}\int |g|\,d\mu &\leq \int_{\mathbb{R}^d\setminus\bigcup_i Q_i} |f|\,d\mu + \sum_i \int |\varphi_i|\,d\mu\\ &\leq \int |f|\,d\mu + \sum_i \int_{Q_i} |f|\,d\mu \leq C\int |f|\,d\mu.\end{aligned}$$

Now, by (12) and (13) we get (10).

If we want to show that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, then in Lemma 2 and in the arguments above $f\,d\mu$ must be substituted by $d\nu$, with $\nu \in M(\mathbb{R}^d)$, and $|f|\,d\mu$ by $d|\nu|$. Also, condition (6) of Lemma 2 should be stated as "On $\mathbb{R}^d\setminus\bigcup_i Q_i$, ν is absolutely continuous with respect to μ, that is $\nu = f\,d\nu$, and moreover $|f(x)| \leq \lambda$

a.e. (μ) $x \in \mathbb{R}^d \setminus \bigcup_i Q_i$". With other minor changes, the arguments and estimates above work in this situation too. $\square$

3.4. **Cotlar's inequality.** This inequality involves some maximal operators which we proceed to define. The *centered maximal Hardy-Littlewood operator* applied to $\nu \in M(\mathbb{R}^d)$ is, as usual,

$$M_\mu \nu(x) = \sup_{r>0} \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} d|\nu|.$$

A useful variant of this operator is the following:

$$\widetilde{M}_\mu \nu(x) = \sup \left\{ \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} d|\nu| : \, r > 0, \, \mu(\bar{B}(x,5r)) \le 5^{d+1} \mu(\bar{B}(x,r)) \right\}.$$

The non centered version of $\widetilde{M}_\mu$ is

$$N_\mu \nu(x) = \sup \left\{ \frac{1}{\mu(\bar{B})} \int_{\bar{B}} d|\nu| : \, \bar{B} \text{ closed ball, } x \in \bar{B}, \, \mu(5\bar{B}) \le 5^{d+1} \mu(\bar{B}) \right\}.$$

For $f \in L^1_{loc}(\mu)$ we set $M_\mu f \equiv M_\mu(f d\mu)$, $\widetilde{M}_\mu f \equiv \widetilde{M}_\mu(f d\mu)$, and $N_\mu f \equiv N_\mu(f d\mu)$, The operators M_μ and $\widetilde{M}_\mu$ are bounded in $L^p(\mu)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. This fact can be proved using Besicovitch's covering theorem for M_μ and $\widetilde{M}_\mu$, and Vitali's covering theorem with balls $B(x, 5r)$ in the case of N_μ.

If T is a CZO, the *maximal operator* T_* is

$$T_* \nu(x) = \sup_{\varepsilon>0} |T_\varepsilon \nu(x)| \qquad \text{for } \nu \in M(\mathbb{R}^d), \, x \in \mathbb{R}^d,$$

and the δ*-truncated maximal operator* $T_{*,\delta}$ is

$$T_{*,\delta} \nu(x) = \sup_{\varepsilon>\delta} |T_\varepsilon \nu(x)| \qquad \text{for } \nu \in M(\mathbb{R}^d), \, x \in \mathbb{R}^d.$$

We also set $T_{\mu,*} f \equiv T_* (f \, d\mu)$ and $T_{\mu,*,\delta} f \equiv T_{*,\delta} (f \, d\mu)$ for $f \in L^1_{loc}(\mu)$.

Theorem 4 (Cotlar's inequality)**.** *Let μ be a positive Radon measure on $\mathbb{R}^d$ with growth of degree n. If the T is an n-dimensional CZO bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, then for $0 < s \le 1$ we have*

$$T_{*,\delta} \nu(x) \le C_s \left(\widetilde{M}_\mu(|T_\delta \nu|^s)(x)^{1/s} + M_\mu \nu(x) \right), \qquad \text{for } \nu \in M(\mathbb{R}^d), \, x \in \mathbb{R}^d, \tag{14}$$

where C_s depends only on the constant C_0 in (1), s, n, d, and the norm of the T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$.

Cotlar's inequality with non doubling measures is due to Nazarov, Treil and Volberg [NTV2], although not in the form stated above, which is from [To2]

To prove Theorem 4 we will need some lemmas. The first one is Kolmogorov's inequality whose proof can be found in [Ma1, p. 299].

Lemma 5. *Let μ be a positive Radon measure on $\mathbb{R}^d$ and $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ a Borel function in $L^{1,\infty}(\mu)$. Then for $0 < s < 1$ and for any μ-measurable set $A \subset \mathbb{R}^d$ with $\mu(A) < \infty$,*

$$\left(\frac{1}{\mu(A)} \int_A |f|^s d\mu \right)^{1/s} \le (1-s)^{-1/s} \frac{\|f\|_{L^{1,\infty}(\mu)}}{\mu(A)}.$$

Also, we have the following result.

Lemma 6. *Let $0 < r < R$, with $R = 5^N r$. If $5^{d+1}\mu(\bar{B}(x, 5^{k-1}r)) \le \mu(\bar{B}(x, 5^k r))$ for $k = 2, \dots, N$, then we have*

$$|T_R\nu(x) - T_r\nu(x)| \lesssim \frac{\mu(\bar{B}(x, R))}{R}\, M_\mu\nu(x),$$

for each $\nu \in M(\mathbb{R}^d)$.

Compare this result with Lemma 1. In both cases one assumes that there exists a sequence of concentric non doubling balls or squares. Moreover, the proofs are similar.

Proof. We set $\bar{B}_k = \bar{B}(x, 5^k r)$ and $K_0 = \mu(\bar{B}(x, R))/R$. Then we have

$$\begin{aligned} |T_R\nu(x) - T_r\nu(x)| &= \left| \int_{r<|y-x|\le 5^N r} k(x,y)\, d\nu(y) \right| \\ &\lesssim \sum_{k=1}^{N} \int_{5^{k-1}r<|y-x|\le 5^k r} \frac{1}{|y-x|^n}\, d|\nu|(y) \\ &\lesssim \sum_{k=1}^{N} \frac{|\nu|(\bar{B}_k)}{(5^k r)^n} = \sum_{k=1}^{N} \frac{|\nu|(\bar{B}_k)}{(5^{k-N}R)^n}. \end{aligned} \tag{15}$$

Also, notice that

$$\mu(\bar{B}_k) \le 5^{(k-N)(d+1)}\mu(\bar{B}_N) = 5^{(k-N)(d+1)} R K_0.$$

Therefore,

$$\frac{1}{5^{(k-N)n}R^n} \le K_0 \frac{5^{(k-N)(d+1-n)}}{\mu(\bar{B}_k)},$$

and by (15),

$$\begin{aligned} |T_R\nu(x) - T_r\nu(x)| &\lesssim K_0 \sum_{k=1}^{N} \frac{5^{(k-N)(d+1-n)}|\nu|(\bar{B}_k)}{\mu(\bar{B}_k)} \\ &\lesssim K_0 \sum_{k=1}^{N} 5^{(k-N)(d+1-n)} M_\mu\nu(x) \lesssim K_0 M_\mu\nu(x) \end{aligned}$$

□

Combining Lemma 6 with the usual arguments we are going to prove Cotlar's inequality (14).

Proof of Theorem 4 Let $\varepsilon > \delta$ and $x \in \mathbb{R}^d$. Since μ has growth of degree n, there exists some $n \ge 1$ such that

$$\mu(\bar{B}(x, 5^n\varepsilon)) \le 5^{d+1}\mu(\bar{B}(x, 5^{n-1}\varepsilon)) \tag{16}$$

(see Subsection 3.1). We assume that n is the least integer ≥ 1 such that (16) holds. Set $\varepsilon' = 5^n\varepsilon$. By Lemma 6,

$$|T_\varepsilon\nu(x) - T_{\varepsilon'/5}\nu(x)| \le C M_\mu\nu(x).$$

Also, it is straightforward to check that $|T_{\varepsilon'/5}\nu(x) - T_{\varepsilon'}\nu(x)| \le C M_\mu\nu(x)$. Therefore,

$$|T_\varepsilon\nu(x) - T_{\varepsilon'}\nu(x)| \le C M_\mu\nu(x).$$

Thus it only remains to show that

$$|T_{\varepsilon'}\nu(x)| \leq C_s \left(\widetilde{M}_\mu(|T_\delta\nu|^s)(x)^{1/s} + M_\mu\nu(x)\right). \tag{17}$$

Since

$$\mu(\bar{B}(x,\varepsilon')) \leq 5^{d+1}\mu(\bar{B}(x,\varepsilon'/5)), \tag{18}$$

we can apply the usual argument, as in [Ma1], pp. 299-300, to prove (17). We set

$$d\nu_1 = \chi_{\bar{B}(x,\varepsilon')}d\nu, \qquad d\nu_2 = d\nu - d\nu_1.$$

For $y \in \bar{B}(x,\varepsilon'/5)$, since $\varepsilon' > 5\delta$ we have $T_{\varepsilon'}\nu_2(x) = T_\delta\nu_2(x) = T\nu_2(x)$ and $T_\delta\nu_2(y) = T\nu_2(y)$. Using (1) it is easy to check that $|T_\delta\nu_2(y) - T_\delta\nu_2(x)| \leq CM_\mu\nu(x)$. Therefore,

$$|T_{\varepsilon'}\nu(x)| = |T_\delta\nu_2(x)| \leq |T_\delta\nu_2(y)| + C_2M_\mu\nu(x) \leq |T_\delta\nu_1(y)| + |T_\delta\nu(y)| + C_2M_\mu\nu(x). \tag{19}$$

Assume first $s = 1$. If $T_{\varepsilon'}\nu(x) \neq 0$, let $0 < \lambda < |T_{\varepsilon'}\nu(x)|$. For $y \in \bar{B}(x,\varepsilon'/5)$, by (19) either $C_2M_\mu\nu(x) > \lambda/3$ or $|T_\delta\nu(y)| > \lambda/3$ or $|T_\delta\nu_1(y)| > \lambda/3$. Therefore, either

$$\lambda < 3C_2M_\mu\nu(x),$$

or

$$\bar{B}(x,\varepsilon'/5) = \{y \in \bar{B}(x,\varepsilon'/5) : |T_\delta\nu(y)| > \lambda/3\} \cup \{y \in \bar{B}(x,\varepsilon'/5) : |T_\delta\nu_1(y)| > \lambda/3\}.$$

But we have

$$\begin{aligned}\mu\{y \in \bar{B}(x,\varepsilon'/5) : |T_\delta\nu(y)| > \lambda/3\} &\leq \frac{3}{\lambda}\int_{\bar{B}(x,\varepsilon'/5)} |T_\delta\nu|\,d\mu \\ &\leq \frac{3}{\lambda}\,\mu(\bar{B}(x,\varepsilon'/5))\,\widetilde{M}_\mu(T_\delta\nu)(x),\end{aligned}$$

and by the boundedness of T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ and (18),

$$\begin{aligned}\mu\{y \in \bar{B}(x,\varepsilon'/5) : |T_\delta\nu_1(y)| > \lambda/3\} &\lesssim \frac{\|\nu_1\|}{\lambda} = \frac{|\nu|(\bar{B}(x,\varepsilon'))}{\lambda} \\ &\lesssim \frac{\mu(\bar{B}(x,\varepsilon'/5))}{\lambda}\,M_\mu\nu(x).\end{aligned}$$

In any case we obtain $\lambda < 3\widetilde{M}_\mu(T_\delta\nu)(x) + CM_\mu\nu(x)$. Since this holds for $0 < \lambda < |T_{\varepsilon'}\nu(x)|$, (17) follows when $s = 1$.

Assume now $0 < s < 1$. From (19) we get

$$|T_{\varepsilon'}\nu(x)|^s \leq |T_\delta\nu_1(y)|^s + |T_\delta\nu(y)|^s + CM_\mu\nu(x)^s.$$

Integrating with respect to μ and $y \in \bar{B}(x,\varepsilon'/5)$, dividing by $\mu(\bar{B}(x,\varepsilon'/5))$ and raising to the power $1/s$ we obtain

$$\begin{aligned}|T_{\varepsilon'}\nu(x)| \leq C_s &\left[\left(\frac{1}{\mu(\bar{B}(x,\varepsilon'/5))}\int_{\bar{B}(x,\varepsilon'/5)} |T_\delta\nu_1|^s d\mu\right)^{1/s}\right. \\ &\left. + \left(\frac{1}{\mu(\bar{B}(x,\varepsilon'/5))}\int_{\bar{B}(x,\varepsilon'/5)} |T_\delta\nu|^s d\mu\right)^{1/s} + M_\mu\nu(x)\right].\end{aligned} \tag{20}$$

By (18), the second term on the right hand side of (20) can be estimated by $\widetilde{M}_\mu(|T_\delta \nu|^s)(x)^{1/s}$. On the other hand, the first term is estimated using Kolmogorov's inequality, the boundedness of T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, and (18):

$$\left(\frac{1}{\mu(\bar{B}(x,\varepsilon'/5))}\int_{\bar{B}(x,\varepsilon'/5)}|T_\delta\nu_1|^s d\mu\right)^{1/s} \lesssim \frac{\|T_\delta\nu_1\|_{L^{1,\infty}(\mu)}}{\mu(\bar{B}(x,\varepsilon'/5))} \lesssim \frac{\|\nu_1\|}{\mu(\bar{B}(x,\varepsilon'/5))} \lesssim M_\mu\nu(x).$$

Now (17) follows. □

A direct consequence of Cotlar's inequality and Theorem 3 is the following result.

Theorem 7. *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n. If T is an n-dimensional CZO bounded in $L^2(\mu)$, then $T_{\mu,*}$ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$.*

Proof. By Theorem 3, interpolation, and duality, T_μ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. Then, by Cotlar's inequality it is clear that $T_{*,\delta}$ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, uniformly on $\delta > 0$. Hence, by monotone convergence, T_* is also bounded in $L^p(\mu)$, $p \in (1,\infty)$. The boundedness of T_* from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ follows as in the classical doubling case, using Kolmogorov's inequality and taking into account that the non centered version of the maximal operator $\widetilde{M}_\mu$ (which is N_μ) is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. See [To2] for the details. □

3.5. **The $T(1)$ theorem and other results.** Let us introduce some notation and definitions. Given $\rho > 1$, we say that $f \in L^1_{loc}(\mu)$ belongs to the space $BMO_\rho(\mu)$ if

$$\sup_Q \frac{1}{\mu(\rho Q)}\int_Q |f - m_Q(f)|\,d\mu < \infty,$$

where the supremum is taken over all the squares in $\mathbb{R}^d$ and $m_Q(f)$ is the μ-mean of f over Q.

Following [NTV1], a Calderón-Zygmund operator T_μ is said to be weakly bounded if

$$\big|\langle T_{\mu,\varepsilon}\chi_Q, \chi_Q\rangle\big| \le C\mu(Q) \quad \text{for all the cubes } Q \subset \mathbb{R}^d\text{, uniformly on } \varepsilon > 0.$$

Notice that if T_μ is antisymmetric, then the left hand side above equals zero and so T_μ is weakly bounded.

Now we are ready to state the $T(1)$ theorem:

Theorem 8. *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. The following conditions are equivalent:*

(a) *T_μ is bounded on $L^2(\mu)$.*
(b) *T_μ is weakly bounded and, for some $\rho > 1$, we have that $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \in BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$.*
(c) *There exists some constant C_3 such that for all $\varepsilon > 0$ and all the cubes $Q \subset \mathbb{R}^d$,*

$$\|T_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \le C_3\mu(Q)^{1/2} \quad \text{and} \quad \|T^*_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \le C_3\mu(Q)^{1/2}.$$

The classical way of stating the $T(1)$ theorem is the equivalence (a) $\Leftrightarrow$ (b). However, for some applications it is sometimes more practical to state the result in terms of the L^2 boundedness of T_μ and T^*_μ over characteristic functions of cubes, i.e (a) $\Leftrightarrow$ (c).

Theorem 8 is the extension of the classical $T(1)$ theorem of David and Journé to measures of degree n which may be non doubling. The result was proved by Nazarov, Treil and Volberg in [NTV1], although not exactly in the form stated above. An independent proof for the particular case of the Cauchy transform was obtained almost simultaneously in [To1]. For the equivalence of conditions (b) and (c) above, the reader should see [To6, Remark 7.1 and Lemma 7.3]. Other (more recent) proofs of the $T(1)$ theorem for non doubling measures are in [Ve2] (for the particular case of the Cauchy transform) and in [To6].

Let us remark that the boundedness of T_μ on $L^2(\mu)$ does not imply the boundedness of T_μ from $L^\infty(\mu)$ into $BMO(\mu)$ (this is the space $BMO_\rho(\mu)$ with parameter $\rho = 1$), and in general $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \notin BMO(\mu)$ uniformly on $\varepsilon > 0$. See [Ve2] and [MMNO]. On the contrary, one can show that if T_μ is bounded on $L^2(\mu)$, then it is also bounded from $L^\infty(\mu)$ into $BMO_\rho(\mu)$, for $\rho > 1$, by arguments similar to the classical ones for homogeneous spaces. However, the space $BMO_\rho(\mu)$ has some drawbacks. For example, it depends on the parameter ρ and it does not satisfy the John-Nirenberg inequality. To solve these problems, in [To3] a new space called $RBMO(\mu)$ has been introduced. $RBMO(\mu)$ is a subspace of $BMO_\rho(\mu)$ for all $\rho > 1$, and it coincides with $BMO(\mu)$ when μ is an AD-regular measure. Moreover, $RBMO(\mu)$ satisfies a John-Nirenberg type inequality, and all CZO's which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. For these reasons $RBMO(\mu)$ seems to be a good substitute of the classical space BMO for non doubling measures of degree n. For the precise definition of $RBMO(\mu)$ and its properties, see [To3].

Much more results on Calderón-Zygmund theory with non doubling measures have been proved recently. For example, several $T(b)$ type theorems have been obtained in [DM], [Dd4], [NTV3], [NTV4], [NTV5]. There are also results concerning Hardy spaces [To5]; weights [GCM1], [MM], [OP]; commutators [CS], [HMY2], [To3]; multilinear commutators [HMY1]; fractional integrals [GCM2], [GCG1]; Lipschitz spaces [GCG2]; Triebel-Lizorkin spaces [HY]; etc.

4. Analytic capacity

4.1. **Definition.** The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) := \sup |f'(\infty)|, \tag{21}$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

The notion of analytic capacity was introduced by Ahlfors [Ah] in the 1940's in order to study the removability of singularities of bounded analytic functions. A compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. Ahlfors showed that E is removable if and only if $\gamma(E) = 0$.

Painlevé's problem consists of characterizing removable singularities for bounded analytic functions in a metric/geometric way. By Ahlfors' result this is equivalent to describe compact sets with positive analytic capacity in metric/geometric terms.

Vitushkin in the 1950's and 1960's showed that analytic capacity plays a central role in problems of uniform rational approximation on compact sets of the complex plane. Many results obtained by Vitushkin in connection with uniform rational approximation are stated in terms of γ. See [Vi2], or [Ve1] for a more modern approach, for example. Further, because its applications to this type of problems he raised the question of the semiadditivity of γ. Namely, does there exist an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)) \ ?$$

4.2. **Basic properties of analytic capacity.** One should keep in mind that, in a sense, analytic capacity measures the size of a set as a non removable singularity for bounded analytic functions. A direct consequence of the definition is that

$$E \subset F \ \Rightarrow \ \gamma(E) \leq \gamma(F).$$

Moreover, it is also easy to check that analytic capacity is translation invariant:

$$\gamma(z + E) = \gamma(E) \qquad \text{for all } z \in \mathbb{C}.$$

Concerning dilations, we have

$$\gamma(\lambda E) = |\lambda|\gamma(E) \qquad \text{for all } \lambda \in \mathbb{C}.$$

Further, if E is connected, then

$$\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E).$$

The second inequality follows from the fact that the analytic capacity of a closed disk coincides with its radius, and the first one is a consequence of Koebe's 1/4 theorem (see [Ga, Chapter VIII] for the details, for example).

4.3. **Relationship with Hausdorff measure.** The relationship between Hausdorff measure and analytic capacity is the following:

- If $\dim_H(E) > 1$ (here $\dim_H$ stands for the Hausdorff dimension), then $\gamma(E) > 0$. This result follows easily from Frostman's Lemma.
- $\gamma(E) \leq \mathcal{H}^1(E)$, where $\mathcal{H}^1$ is the one dimensional Hausdorff measure, or length. This follows from Cauchy's integral formula, and it was proved by Painlevé about one hundred years ago. Observe that, in particular we deduce that if $\dim_H(E) < 1$, then $\gamma(E) = 0$.

By the statements above, it turns out that dimension 1 is the critical dimension in connection with analytic capacity. Moreover, a natural question arises: is it true that $\gamma(E) > 0$ if and only if $\mathcal{H}^1(E) > 0$?

Vitushkin showed that the answer is *no*. Indeed, he constructed a compact set in $\mathbb{C}$ with positive length and vanishing analytic capacity. This set was purely unrectifiable. That is, it intersects any rectifiable curve at most in a set of zero length. Motivated by this example (and others, I guess) Vitushkin conjectured that pure unrectifiability is a necessary and sufficient condition for vanishing analytic capacity, for sets with finite length.

Guy David [Dd4] showed in 1998 that Vitushkin's conjecture is true:

Theorem 9. *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then, $\gamma(E) = 0$ if and only if E is purely unrectifiable.*

Let us remark that the "if" part of the theorem is not due to David (it follows from Calderón's theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs). The "only if" part of the theorem, which is more difficult, is the one proved by David. See also [MMV], [DM] and [Lé] for some preliminary contributions to the proof.

Theorem 9 is the solution of Painlevé's problem for sets with finite length. The analogous result is false for sets with infinite length. For this type of sets there is no such a nice geometric solution of Painlevé's problem, and we have to content ourselves with a characterization such as the one in Corollary 18 below (at least, for the moment).

4.4. **The capacity γ_+ and the Cauchy transform.** The capacity γ_+ of a compact set $E \subset \mathbb{C}$ is

$$\gamma_+(E) := \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E, \, \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\}. \tag{22}$$

That is, γ_+ is defined as γ in (21) with the additional constraint that f should coincide with $\mathcal{C}\mu$, where μ is some positive Radon measure supported on E (observe that $(\mathcal{C}\mu)'(\infty) = -\mu(\mathbb{C})$ for any Radon measure μ). To be precise, there is another little difference: in (21) we asked $\|f\|_{L^\infty(\mathbb{C}\setminus E)} \leq 1$, while in (22) $\|f\|_{L^\infty(\mathbb{C})} \leq 1$ (for $f = \mathcal{C}\mu$). Trivially, we have $\gamma_+(E) \leq \gamma(E)$.

The following lemma relates weak $(1,1)$ estimates for the Cauchy integral operator with L^∞ estimates (which in its turn are connected with γ_+ and γ).

Lemma 10. *let μ be a Radon measure with linear growth on $\mathbb{C}$. The following statements are equivalent:*

(a) *The Cauchy transform is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$.*
(b) *For any set $A \subset \mathbb{C}$ there exists some function h supported on A, with $0 \leq h \leq 1$, such that $\int h \, d\mu \geq C^{-1}\mu(A)$ and $\|\mathcal{C}_\varepsilon(h \, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ for all $\varepsilon > 0$.*

The constant C in (b) *depends only on the norm of the Cauchy transform is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, and conversely.*

This lemma is a particular case of a result which applies to more general linear operators. The statement (b) should be understood as a weak substitute of the $L^\infty(\mu)$ boundedness of the Cauchy integral operator, which does not hold in general.

We will prove the easy implication of the lemma, that is, (b) $\Rightarrow$ (a). For the other implication, which is due to Davie and Øksendal [DØ] the reader is referred to [Ch, Chapter VII].

Proof of (b) $\Rightarrow$ (a). It is enough to show that for any complex measure $\nu \in M(\mathbb{C})$ and any $\lambda > 0$,

$$\mu\{x \in \mathbb{C} : \operatorname{Re}(\mathcal{C}_\varepsilon\nu(x)) > \lambda\} \leq \frac{C\|\nu\|}{\lambda}.$$

To this end, let us denote by A the set on the left side above, and let h be a function supported on A fulfilling the properties in the statement (b) of the lemma. Then we have

$$\mu(A) \leq C \int h \, d\mu \leq \frac{C}{\lambda}\operatorname{Re}\left(\int (\mathcal{C}_\varepsilon\nu)\, h \, d\mu\right) = \frac{-C}{\lambda}\operatorname{Re}\left(\int \mathcal{C}_\varepsilon(h \, d\mu)\, d\nu\right) \leq \frac{C\|\nu\|}{\lambda}.$$

□

Remark 11. Notice that if E supports a non zero Radon measure μ with linear growth such that the Cauchy integral operator $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$, then there exists some nonzero function h with $0 \leq h \leq \chi_E$ such that $\|\mathcal{C}_\varepsilon(h \, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ uniformly on ε, by Theorem 3 and the preceding lemma. Letting $\varepsilon \to 0$, we infer that $|\mathcal{C}(h \, d\mu)(z)| \leq C$ for all $z \notin E$, and so $\gamma(E) > 0$.

A more precise result will be proved in Theorem 14 below.

4.5. **The curvature of a measure.** Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x,y,z (with $R(x,y,z)=\infty$, $c(x,y,z)=0$ if x,y,z lie on a same line). If two among these points coincide, we let $c(x,y,z)=0$. For a positive Radon measure μ, we set

$$c_\mu^2(x) = \iint c(x,y,z)^2\, d\mu(y)d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c_\mu^2(x)\, d\mu(x) = \iiint c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z). \tag{23}$$

The notion of curvature of measures was introduced by Melnikov [Me] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the big recent advances in connection with analytic capacity. The notion of curvature is connected to the Cauchy transform by the following result, proved by Melnikov and Verdera.

Proposition 12. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. We have*

$$\|\mathcal{C}_\varepsilon\mu\|_{L^2(\mu)}^2 = \frac{1}{6}c_\varepsilon^2(\mu) + O(\mu(\mathbb{C})), \tag{24}$$

where $c_\varepsilon^2(\mu)$ is the ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (23)*, but with the triple integral over $\{x,y,z\in\mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$), and $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$.*

The identity (24) is remarkable because it relates an analytic notion (the Cauchy transform of a measure) with a metric-geometric one (curvature). We give a sketch of the proof.

Sketch of the proof of Proposition 12. If we don't worry about truncations and the absolute convergence of the integrals, we can write

$$\|\mathcal{C}\mu\|_{L^2(\mu)}^2 = \int \left| \int \frac{1}{y-x}\, d\mu(y) \right|^2 d\mu(x) = \iiint \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x).$$

By Fubini (assuming that it can be applied correctly), permuting x, y, z, we get,

$$\|\mathcal{C}\mu\|_{L^2(\mu)}^2 = \frac{1}{6} \iiint \sum_{s\in S_3} \frac{1}{(z_{s_2}-z_{s_1})\overline{(z_{s_3}-z_{s_1})}}\, d\mu(z_1)d\mu(z_2)d\mu(z_3),$$

where S_3 is the group of permutations of three elements. An elementary calculation shows that

$$\sum_{s\in S_3} \frac{1}{(z_{s_2}-z_{s_1})\overline{(z_{s_3}-z_{s_1})}} = c(z_1,z_2,z_3)^2.$$

So we get

$$\|\mathcal{C}\mu\|_{L^2(\mu)}^2 = \frac{1}{6}\, c^2(\mu).$$

To argue rigorously, above one should use the truncated Cauchy transform $\mathcal{C}_\varepsilon\mu$ instead of $\mathcal{C}\mu$. Then we obtain

$$\begin{aligned}\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} &= \iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\,d\mu(y)d\mu(z)d\mu(x)\\ &= \iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon\\|y-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\,d\mu(y)d\mu(z)d\mu(x) + O(\mu(\mathbb{C})). \qquad (25)\end{aligned}$$

By the linear growth of μ, it is easy to check that $|O(\mu(\mathbb{C}))| \leq \mu(\mathbb{C})$. As above, using Fubini and permuting x, y, z, one shows that the triple integral in (25) equals $c^2_\varepsilon(\mu)/6$. □

Due to Proposition 12, the $T(1)$ theorem for the Cauchy transform can be rewritten in the following way:

Theorem 13. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. The Cauchy transform is bounded on $L^2(\mu)$ if and only if*

$$c^2(\mu_{|Q}) \leq C\mu(Q) \qquad \text{for all the squares } Q \subset \mathbb{C}.$$

Observe that this result is a restatement of the equivalence (a) $\Leftrightarrow$ (c) in Theorem 8, by an application of (24) to the measure $\mu_{|Q}$, for all the squares $Q \subset \mathbb{C}$.

5. Semiadditivity of γ_+ and its characterization in terms of curvature

We denote by $\Sigma(E)$ the set of Radon measures supported on E such that $\mu(B(x,r)) \leq r$ for all $x \in \mathbb{C}$, $r > 0$.

Theorem 14. *For any compact set $E \subset \mathbb{C}$ we have*

$$\begin{aligned}\gamma_+(E) &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1\,\forall\varepsilon > 0\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq \mu(E)\,\forall\varepsilon > 0\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \leq \mu(E)\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\big\}.\end{aligned}$$

In the statement above, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)}$ stands for the operator norm of $\mathcal{C}_\mu$ on $L^2(\mu)$. That is, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} = \sup_{\varepsilon>0} \|\mathcal{C}_{\mu,\varepsilon}\|_{L^2(\mu),L^2(\mu)}$.

Proof. We denote

$$\begin{aligned}S_1 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1\,\forall\varepsilon > 0\big\},\\ S_2 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq \mu(E)\,\forall\varepsilon > 0\big\},\\ S_3 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \leq \mu(E)\big\},\\ S_4 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\,\forall\varepsilon > 0\big\}.\end{aligned}$$

We will show that $\gamma_+(E) \lesssim S_1 \lesssim S_2 \lesssim S_3 \lesssim S_4 \lesssim \gamma_+(E)$. The inequality $S_3 \lesssim S_4$ requires more work than the others. We will give two proofs of it. One uses the $T(1)$ theorem and the other not (and so it is more elementary).

Proof of $\gamma_+(E) \lesssim S_1$. Let μ be supported on E such that $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$ with $\gamma_+(E) \leq 2\mu(E)$. It is enough to show that μ has linear growth and $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq C$ uniformly on $\varepsilon > 0$.

First we will prove the linear growth of μ. For any fixed $x \in \mathbb{C}$, by Fubini it turns out that for almost all $r > 0$,

$$\int_{|z-x|=r} \frac{1}{|z-x|}\, d\mu(z) < \infty.$$

For this r we have

$$\mu(B(x,r)) = -\int_{|z-x|=r} \mathcal{C}\mu(z)\, \frac{dz}{2\pi i} \leq r.$$

Now the linear growth of μ follows easily.

To deal with the $L^\infty(\mu)$ norm of $\mathcal{C}_\varepsilon$ we use a standard technique: we replace $\mathcal{C}_\varepsilon$ by the regularized operator $\widetilde{\mathcal{C}}_\varepsilon$, defined as

$$\widetilde{\mathcal{C}}_\varepsilon\mu(x) = \int r_\varepsilon(y-x)\, d\mu(y),$$

where r_ε is the kernel

$$r_\varepsilon(z) = \begin{cases} \dfrac{1}{z} & \text{if } |z| > \varepsilon, \\[2mm] \dfrac{\overline{z}}{\varepsilon^2} & \text{if } |z| \leq \varepsilon. \end{cases}$$

Then, $\widetilde{\mathcal{C}}_\varepsilon\mu$ is the convolution of the complex measure μ with the uniformly continuous kernel r_ε and so $\widetilde{\mathcal{C}}_\varepsilon\mu$ is a continuous function. Also, we have

$$r_\varepsilon(z) = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2},$$

where χ_ε is the characteristic function of $B(0,\varepsilon)$. Since μ is compactly supported, we have the following identity:

$$\widetilde{\mathcal{C}}_\varepsilon\mu = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \mu = \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \mathcal{C}\mu.$$

This equality must be understood in the sense of distributions, with $\mathcal{C}\mu$ being a function of $L^1_{loc}(\mathbb{C})$ with respect to Lebesgue planar measure. As a consequence, if $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$, we infer that $\|\widetilde{\mathcal{C}}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1$ for all $\varepsilon > 0$.

Since μ has linear growth, we have

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x) - \mathcal{C}_\varepsilon\mu(x)| = \frac{1}{\varepsilon^2}\left|\int_{|y-x|<\varepsilon} (\overline{y-x})d\mu(y)\right| \leq C, \tag{26}$$

and so $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq C$ uniformly on $\varepsilon > 0$.

Proof of $S_1 \lesssim S_2$. Trivial.

Proof of $S_2 \lesssim S_3$. This is a direct consequence of Proposition 12.

Proof of $S_3 \lesssim S_4$ *using the* $T(1)$ *theorem.* Let μ supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_3 \leq 2\mu(E)$. We set

$$A := \left\{x \in E : c^2_\mu(x) \leq 2\right\}.$$

By Tchebychev $\mu(A) \geq \mu(E)/2$. Moreover, for any set $B \subset \mathbb{C}$,

$$c^2(\mu_{|B\cap A}) \leq \iiint_{x\in B\cap A} c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z) = \int_{x\in B\cap A} c^2_\mu(x)\, d\mu(x) \leq 2\mu(B).$$

In particular, this estimate holds when B is any square in $\mathbb{C}$, and so $\mathcal{C}_{\mu_{|A}}$ is bounded on $L^2(\mu_{|A})$, by Theorem 13. Thus $S_4 \gtrsim \mu(A) \approx S_3$.

Proof of $S_3 \lesssim S_4$ without using the $T(1)$ theorem. Take μ supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_3 \leq 2\mu(E)$. To prove $S_3 \lesssim S_4$ we will show that there exists a measure ν supported on E with linear growth such that $\nu(E) \geq \mu(E)/4$ and $\|\mathcal{C}_\nu\|_{L^2(\nu),L^2(\nu)} \leq C$.

Given $C_4 > 0$, let

$$A_\varepsilon := \big\{x \in E : |\mathcal{C}_\varepsilon \mu(x)| \leq C_4 \text{ and } c^2_\mu(x) \leq C_4^2\big\}.$$

Since $\int c^2_\mu(x)\, d\mu(x) = c^2(\mu) \leq \mu(E)$ and, by Proposition 12, $\int |\mathcal{C}_\varepsilon \mu|^2\, d\mu \leq C\mu(E)$, we infer that $\mu(A_\varepsilon) \geq \mu(E)/2$ if C_4 is chosen big enough, by Tchebychev.

We want to show that the Cauchy integral operator $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{|A_\varepsilon})$. To this end we introduce an auxiliary "curvature operator": for $x, y \in A_\varepsilon$, consider the kernel $k(x,y) := \int c(x,y,z)^2\, d\mu(z)$, and let T be the operator

$$Tf(x) = \int k(x,y) f(y)\, d\mu(y).$$

By Schur's lemma, T is bounded on $L^p(\mu_{|A_\varepsilon})$ for all $p \in [1,\infty]$, because for all $x \in A_\varepsilon$,

$$\int k(x,y)\, d\mu_{|A_\varepsilon}(y) = \int k(y,x)\, d\mu_{|A_\varepsilon}(y) = \int_{y\in A_\varepsilon} c(x,y,z)^2\, d\mu(y) d\mu(z) \leq c^2_\mu(x) \leq C_4^2.$$

Given a non negative (real) function f supported on A_ε, by arguments similar to the ones in the proof of Proposition 12, we have

$$\begin{aligned} 4\int |\mathcal{C}_\varepsilon(f\, d\mu)|^2\, d\mu \;&=\; \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon\\ |y-z|>\varepsilon}} c(x,y,z)^2 f(x) f(y)\, d\mu(x) d\mu(y) d\mu(z) \\ &\quad - 2\mathrm{Re}\int (\mathcal{C}_\varepsilon \mu)\, \overline{\mathcal{C}_\varepsilon(f\, d\mu)}\, f\, d\mu + O(\|f\|^2_{L^2(\mu)}). \end{aligned}$$

See [Ve2, Lemma 1] for the details, for example. Thus,

$$\int |\mathcal{C}_\varepsilon(f\, d\mu)|^2\, d\mu \leq \frac{1}{4}\big|\langle Tf,\, f\rangle\big| + \frac{1}{2}\int \big|(\mathcal{C}_\varepsilon \mu)\, \mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu + C\|f\|^2_{L^2(\mu)}. \tag{27}$$

To estimate the first term on the right side we use the $L^2(\mu_{|A_\varepsilon})$ boundedness of T (recall that $\mathrm{supp}(f) \subset A_\varepsilon$):

$$\big|\langle Tf,\, f\rangle\big| \leq \|Tf\|_{L^2(\mu)}\, \|f\|_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)}.$$

To deal with the second integral on the right side of (27), notice that $|\mathcal{C}_\varepsilon \mu| \leq C_4$ on the support of f, and so

$$\int \big|(\mathcal{C}_\varepsilon \mu)\, \mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu \leq C_4 \int \big|\mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu \leq C_4 \|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \|f\|_{L^2(\mu)}.$$

By (27) we get

$$\|\mathcal{C}_\varepsilon(f\, d\mu)\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)} + \frac{C_4}{2}\, \|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \|f\|_{L^2(\mu)},$$

which implies that $\|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}$.

So far we have proved the $L^2(\mu_{|A_\varepsilon})$ boundedness of $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$. If A_ε were independent of ε, we would set $\nu := \mu_{|A_\varepsilon}$ and we would be done. Unfortunately this is not the case and we have to work a little more. We set

$$B_\varepsilon := \big\{x \in E : |\mathcal{C}_{\varepsilon,*}\mu(x)| \leq C_5 \text{ and } c^2_\mu(x) \leq C_5^2\big\},$$

where C_5 is some constant big enough (with $C_5 > C_4$) to be chosen below. By Theorem 7 and the discussion above, we know that $\mathcal{C}_{\varepsilon,*}$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu_{|A_\varepsilon})$ (with constants independent of ε). Thus,

$$\mu\big\{x \in A_\varepsilon : |\mathcal{C}_{\varepsilon,*}\mu(x)| > C_5\big\} \leq \frac{C\mu(E)}{C_5}.$$

If C_5 is big enough, the right hand side of the preceding inequality is $\leq \mu(E)/4 \leq \mu(A_\varepsilon)/2$. Thus, $\mu(B_\varepsilon) \geq \mu(E)/4$.

We set

$$B := \bigcap_{\varepsilon>0} B_\varepsilon.$$

Notice that, by definition, $B_\varepsilon \subset B_\delta$ if $\varepsilon > \delta$ and so we have

$$\mu(B) = \lim_{\varepsilon\to 0} \mu(B_\varepsilon) \geq \frac{1}{4}\,\mu(E).$$

By the same argument used for A_ε, it follows that $\mathcal{C}_{\mu_{|B_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{B_\varepsilon})$ (with constant independent of ε), and thus $\mathcal{C}_{\mu_{|B}}$ is bounded on $L^2(\mu_{|B})$. If we take $\nu := \mu_{|B}$, we are dome.

Proof of $S_4 \lesssim \gamma_+(E)$. This is a direct consequence of Lemma 10 and the fact that the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ implies its boundedness from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, as shown in Theorem 3. □

From the preceding theorem, since the term

$$\sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\big\}$$

is countably semiadditive, we deduce that γ_+ is also countably semiadditive.

Corollary 15. *The capacity γ_+ is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma_+\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma_+(E_i).$$

Another consequence of Theorem 14 is that the capacity γ_+ can be characterized in terms of the following potential, introduced by Verdera [Ve2]:

$$U_\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r} + c_\mu^2(x)^{1/2}. \tag{28}$$

The precise result is the following.

Corollary 16. *For any compact set $E \subset \mathbb{C}$ we have*

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, U_\mu(x) \leq 1\, \forall x \in \mathbb{C}\big\}.$$

The proof of this corollary follows easily from the fact that

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \leq \mu(E)\big\},$$

using Tchebychev. The details are left for the reader.

Let us remark that the preceding characterization of γ_+ in terms of U_μ is interesting because it suggests that some techniques of potential theory could be useful to study γ_+. See [To7] and [Ve2].

6. The comparability between γ and γ_+, and related results

6.1. Comparability between γ and γ_+. In [To8] the following result has been proved.

Theorem 17. *There exists an absolute constant C such that for any compact set $E \subset \mathbb{C}$ we have*

$$\gamma(E) \leq C\gamma_+(E).$$

As a consequence, $\gamma(E) \approx \gamma_+(E)$.

An obvious corollary of the preceding result and the characterization of γ_+ in terms of curvature obtained in Theorem 14 is the following.

Corollary 18. *Let $E \subset \mathbb{C}$ be compact. Then, $\gamma(E) > 0$ if and only if E supports a non zero Radon measure with linear growth and finite curvature.*

Since we know that γ_+ is countably semiadditive, the same happens with γ:

Corollary 19. *Analytic capacity is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma(E_i).$$

Notice that, by Theorem 14, to prove Theorem 17 it is enough to show that there exists some measure μ supported on E with linear growth, satisfying $\mu(E) \approx \gamma(E)$, and such that the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ with absolute constants. To implement this argument, the main tool used in [To8] is the $T(b)$ theorem of Nazarov, Treil and Volberg [NTV3]. To apply this theorem, one has to construct a suitable measure μ and a function $b \in L^\infty(\mu)$ fulfilling some suitable para-accretivity conditions. The construction of μ and b is the main difficulty which is overcome in [To8], by means of a bootstrapping argument which involves the potential U_μ of (28).

Let us remark that the comparability between γ and γ_+ had been previously proved by P. Jones for compact connected sets by geometric arguments, very different from the ones in [To8] (see [Pa, Chapter 3]). On the other hand, the case of Cantor sets was studied in [MTV1]. The proof of [To8] is inspired in part by the ideas in [MTV1].

Corollary 18 yields a characterization of removable sets for bounded analytic functions in terms of curvature of measures. Although this result has a definite geometric flavour, it is not clear if this is a really good geometric characterization. Nevertheless, in [To10] it has been shown that the characterization is invariant under bilipschitz mappings, using a corona type decomposition for non doubling measures. See also [GV] for an analogous result for some Cantor sets.

6.2. Other capacities. In [To9], some results analogous to Theorems 14 and 17 have been obtained for the continuous analytic capacity α. This capacity, introduced by Vitushkin, is defined like γ in (21), with the additional requirement that the functions f considered in the sup should extend continuously to the whole complex plane. In particular, in [To9] it is shown that α is semiadditive. This result has some nice consequences for the theory of uniform rational approximation on the complex plane. For example, it implies the so called *inner boundary conjecture*.

Volberg [Vo] has proved the natural generalization of Theorem 17 to higher dimensions. In this case, one should consider Lipschitz harmonic capacity instead of analytic capacity (see [MP] for the definition and properties of Lipschitz harmonic capacity). The main difficulty arises from the fact that in this case one does not have any good substitute of the notion of curvature of measures, and then one has to argue with a potential very different from the one defined in (28). See also [MT] for related results which avoid the use of any notion similar to curvature.

The techniques in Theorem 17 have also been used by Prat [Pr] and Mateu, Prat and Verdera [MPV] to study the capacities γ_α associated to α-dimensional signed Riesz kernels with α non integer:

$$k(x,y) = \frac{y-x}{|y-x|^{\alpha+1}}.$$

In [Pr] it is shown that sets with finite α-dimensional Hausdorff measure have vanishing capacity γ_α when $0 < \alpha < 1$. Moreover, for these α's it is proved in [MPV] that γ_α is comparable to one of the non linear Wolff's capacities. The case of non integer α with $\alpha > 1$ seems much more difficult to study, although in the AD regular situation some results have been obtained [Pr]. The results in [Pr] and [MPV] show that the behavior of γ_α with α non integer is very different from the one with α integer.

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Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, Spain

E-mail address: `xtolsa@mat.uab.es`