THE $T1$ THEOREM

V. CHOUSIONIS AND X. TOLSA

Introduction

These are the notes of a short course given by X. Tolsa at the Universitat Autònoma de Barcelona between November and December of 2012. The notes have been typed by V. Chousionis.

We present a "dyadic proof" of the classical $T1$ theorem of David and Journé. We have preferred to state the $T1$ theorem in terms of the boundedness over characteristic functions of cubes rather than in terms of the typical conditions $T1, T^*1 \in BMO$, which is the usual approach in the literature. Nevertheless, no attempt at originality is claimed.

The notation used below is standard. We use the shortcuts L^p or C^∞ for $L^p(\mathbb{R}^n)$ or $C^\infty(\mathbb{R}^n)$, as well as other analogous terminology. As usual, the letter C denotes some constant that may change its value at different occurrences, and typically depends on some absolute constants and other fixed parameters.

1. Statement and reductions

A kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\} \to \mathbb{R}$ is called Calderón-Zygmund standard, or simply standard, if there exist constants C, $\delta > 0$ such that for all distinct $x, y \in \mathbb{R}^n$ and for all x' such that $|x - x'| \leq \frac{|x-y|}{2}$,

(i) $|K(x,y)| \leq \dfrac{C}{|x-y|^n}$

(ii) $|K(x,y) - K(x',y)| + |K(y,x) + K(y,x')| \leq C\dfrac{|x-x'|^\delta}{|x-y|^{n+\delta}}$.

From now on K will always denote a standard kernel.

Given a function $f \in C_0^\infty$ we define

$$Tf(x) = \int K(x,y)f(y)dy \quad \text{for} \quad x \notin \operatorname{supp}(f).$$

We then say that T is Calderón-Zygmund operator (CZO) with kernel $K(\cdot,\cdot)$. If $f \in L^p$, for some $1 < p < \infty$, we define the truncated singular integrals T_ε associated with the standard kernel K by

$$T_\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)dy.$$

The operator T is said to be bounded in L^2 if there exists some constant C not depending on ε such that

$$\|T_\varepsilon(f)\|_2 \le C\|f\|_2$$

for all $f \in L^2$. The definition of the L^2 boundedness of T in terms of the truncated operators T_ε is convenient because it avoids the delicate question of the existence of principal values for Tf.

By T^* we denote the CZO associated with the kernel $\tilde{K}(x,y) = K(y,x)$, furthermore it holds that, for $f, g \in C_0^\infty$,

$$\langle T_\varepsilon^* f, g\rangle = \langle f, T_\varepsilon g\rangle.$$

The following theorem provides checkable criteria for the L^2-boundedness of CZO's.

Theorem 1 (David-Journé)**.** Let T be a CZO as above. Then T is bounded in L^2 if and only if there exists some constant $C > 0$ such that

(i) $\sup_{\varepsilon>0} \|T_\varepsilon \chi_Q\|_2 \le C m(Q)^{1/2}$
(ii) $\sup_{\varepsilon>0} \|T_\varepsilon^* \chi_Q\|_2 \le C m(Q)^{1/2}$

for all cubes Q.

The first step consists of regularizing the kernel. We define a function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \in C^\infty$, $0 \le \varphi \le 1$, such that $\varphi = 0$ in $[-1/2, 1/2]$ and $\varphi = 1$ in $\mathbb{R} \setminus [-1, 1]$. Given the function φ we regularize the kernel K at level ε by

$$\widetilde{K}_\varepsilon(x,y) = K(x,y)\varphi\left(\frac{|x-y|}{\varepsilon}\right).$$

Notice that $\widetilde{K}_\varepsilon(x,y) = K(x,y)$ when $|x-y| > \varepsilon$ and $\widetilde{K}_\varepsilon(x,y) = 0$ when $|x-y| \le \frac{\varepsilon}{2}$. It follows easily that $\widetilde{K}$ is also a standard kernel with constants not depending on ε. Next we consider the operators associated to $\widetilde{K}_\varepsilon$

$$\widetilde{T}_\varepsilon f(x) = \int \widetilde{K}_\varepsilon(x,y) f(y) dy.$$

Notice that, for $K_\varepsilon(x,y) = K(x,y)\chi_{\{r:|r|>\varepsilon\}}(|x-y|)$,

$$\begin{aligned}
|T_\varepsilon f(x) - \widetilde{T}_\varepsilon f(x)| &\le \int_{\frac{\varepsilon}{2}<|x-y|<\varepsilon} |K_\varepsilon(x,y) - \widetilde{K}_\varepsilon(x,y)||f(y)| dy \\
&\le \frac{C}{(\varepsilon/2)^n} \int_{|x-y|<\varepsilon} |f(y)| dy \le C M f(x)
\end{aligned}$$

where M denotes the usual Hardy-Littlewood maximal operator. Since M is bounded in L^p for all $1 < p \le \infty$ we deduce that $\widetilde{T}_\varepsilon$ is bounded in L^p, uniformly on ε, if and only if the operators T_ε are bounded in L^p uniformly on ε.

In the same manner, for $R > 2\varepsilon$, we can define the kernels

$$\widetilde{K}_{\varepsilon,R}(x,y) = \varphi\left(\frac{|x-y|}{\varepsilon}\right)\left(1 - \varphi\left(\frac{|x-y|}{R}\right)\right) K(x,y),$$

and the corresponding operators

$$\widetilde{T}_{\varepsilon,R}f(x) = \int \widetilde{K}_{\varepsilon,R}(x,y)f(y)dy.$$

Since $R > 2\varepsilon$ it follows that $\widetilde{T}_{\varepsilon,R} = \widetilde{T}_\varepsilon - \widetilde{T}_R$ therefore if $\widetilde{T}_\varepsilon : L^p \to L^p$ is bounded for all $\varepsilon > 0$ then $\widetilde{T}_{\varepsilon,R}$ is bounded as well for all positive R, ε such that $R > 2\varepsilon$. Furthermore, if $\widetilde{T}_{\varepsilon,R}$ is bounded in L^2 uniformly on ε and R, then $\widetilde{T}_\varepsilon$ is bounded uniformly on ε. To see this, let $f \in L^2$ and write

$$\widetilde{T}_\varepsilon f(x) = \widetilde{T}_{\varepsilon,R}f(x) + \widetilde{T}_R f(x).$$

Further, taking into account that

$$\int |\widetilde{K}_\varepsilon(x,y)||f(y)|dy \le \left(\int |\widetilde{K}_\varepsilon(x,y)|^2 dy\right)^{1/2} \left(\int |f(y)|^2\right)^{1/2} \le C(\varepsilon),$$

it follows that

$$|\widetilde{T}_R f(x)| \le \int_{|x-y|>R/2} |\widetilde{K}_\varepsilon(x,y)f(y)|dy \to 0 \quad \text{ as } R \to \infty.$$

Then, by the dominated convergence theorem we deduce that, for any $M > 1$,

$$\|\widetilde{T}_\varepsilon f\|_{L^2(B(0,M)} = \lim_{R\to\infty} \|\widetilde{T}_{\varepsilon,R}f\|_{L^2(B(0,M)} \le \sup_{R>2\varepsilon} \|\widetilde{T}_{\varepsilon,R}f\|_2.$$

Since this estimate is uniform on M, our claim is proven.

To summarize, if

$$\|T_\varepsilon \chi_Q\|_2 \le Cm(Q)^{1/2} \text{ and } \|T_\varepsilon^* \chi_Q\|_2 \le Cm(Q)^{1/2}$$

for all cubes Q uniformly on ε, we also have that

$$\|\widetilde{T}_{\varepsilon,R}\chi_Q\|_2 \le C'm(Q)^{1/2} \text{ and } \|\widetilde{T}_{\varepsilon,R}^*\chi_Q\|_2 \le C'm(Q)^{1/2}$$

for all cubes Q uniformly on ε and R. By the discussion above, the proof will be complete if we show that the above condition implies that $\widetilde{T}_{\varepsilon,R} : L^2 \to L^2$ is bounded uniformly on ε and R.

Hence it is enough to prove the following.

Theorem 2 (Reduced restatement of Theorem 1)**.** Let K be a standard Calderón-Zygmund kernel satisfying $\|K\|_\infty \le C_0$ and $K(x,y) = 0$ whenever $|x-y| \ge R$ for some $R > 0$. Let T be the associated integral operator:

$$Tf(x) = \int K(x,y)\,f(y)\,dy \quad \text{for } f \in L^p,\, x \in \mathbb{R}^n. \tag{1}$$

If furthermore

(i) $\|T\chi_Q\|_2 \le C_1 m(Q)^{1/2}$ and
(ii) $\|T^*\chi_Q\|_2 \le C_1 m(Q)^{1/2}$

for all cubes Q, then $\|T\|_{L^2\to L^2} \le C_2$ with C_2 depending on C_1 but independent of C_0 and R.

Remark that in (1) we assume the identity to hold for all $x \in \mathbb{R}^n$, not only for x away from supp$f$. Indeed, the above assumptions on the kernel ensure that the integral that defines $Tf(x)$ is absolutely convergent for all $x \in \mathbb{R}^n$ and $f \in L^p$, $1 \leq p \leq \infty$:

$$\int |K(x,y)\, f(y)|\, dy \leq \|f\|_p \left(\int_{|x-y|\leq R} |K(x,y)|^{p'} dy \right)^{1/p'} \leq C\, R^{n/p'} C_0\, \|f\|_p < \infty.$$

Further, the fact that

$$\int |K(x,y)|\, dy \leq C\, C_0\, R^n < \infty \qquad \text{for all } x \in \mathbb{R}^n$$

and the analogous estimate interchanging x and y guaranty that T is bounded in L^p. Indeed, for $f \in L^\infty$,

$$\begin{aligned}|T(f)(x)| &\leq \int |K(x,y)||f(y)|dy \leq \|f\|_\infty \int |K(x,y)|dy \\ &\leq C\, C_0\, R^n\, \|f\|_\infty,\end{aligned}$$

and analogously for the L^1 bound. By interpolation, we get the L^p boundedness of T for all $p \in [1,\infty]$, and in L^2 in particular. Clearly, the bound we obtain for $\|T\|_{L^2\to L^2}$ in this way depends on C_0 and R. To prove the theorem, our objective is to obtain some bound for $\|T\|_{L^2\to L^2}$ independent of C_0 and R.

2. The operators Δ_Q

Let $\mathcal{D}$ denote the family of dyadic cubes of $\mathbb{R}^n$ and let $\mathcal{D}_k \subset \mathcal{D}$ be the subfamily of dyadic cubes with side length $l(Q) = 2^{-k}$. For $Q \in \mathcal{D}$ and $f \in L^1_{loc}$ we define

$$\Delta_Q f(x) = \begin{cases} 0 & \text{if } x \notin Q, \\ m_P f - m_Q f & \text{if } x \in P \text{ and } P \text{ is a son of } Q, \end{cases}$$

where $m_Q f$ is the average of f on Q. The following proposition gathers several elementary but useful properties of the operators Δ_Q.

Proposition 3. For $Q, R \in \mathcal{D}$ and $f \in L^1_{loc}$,

(1) supp$\Delta_Q f \subset \overline{Q}$,
(2) Δ_Q is constant in every son of Q,
(3) $\int \Delta_Q f = 0$,
(4) $\langle \Delta_Q f, \Delta_R g\rangle = 0$ whenever $Q \neq R$,
(5) $\Delta_Q \circ \Delta_Q = \Delta_Q$.
(6) $\Delta_Q : L^2 \to L^2$ is bounded and $\Delta_Q^* = \Delta_Q$.

For simplicity from now on we will denote $\|\cdot\|_2 := \|\cdot\|$.

Proposition 4. If $f \in L^2$ then $f = \sum_{Q\in\mathcal{D}} \Delta_Q f$, the convergence is unconditional in L^2 and moreover $\|f\|^2 = \sum_{Q\in\mathcal{D}} \|\Delta_Q f\|^2$.

Proof. The essential step consists in showing that

$$\sum_{Q\in\mathcal{D}} \|\Delta_Q f\|^2 \leq \|f\|^2. \tag{2}$$

To see this, let $F \subset \mathcal{D}$ be finite and set $g = f - \sum_{Q\in F} \Delta_Q f$. Then $g \perp \sum_{Q\in F} \Delta_Q f$ because by (4),(5) and (6) of Proposition 3:

$$\langle f - \sum_{Q\in F} \Delta_Q f, \sum_{Q\in F} \Delta_Q f\rangle = \sum_{Q\in F} \langle f, \Delta_Q f\rangle - \sum_{Q\in F} \langle \Delta_Q f, \Delta_Q f\rangle = 0.$$

Hence $\|f\|^2 = \|g\|^2 + \sum_{Q\in F} \|\Delta_Q f\|^2$ and

$$\sum_{Q\in F} \|\Delta_Q f\|^2 \leq \|f\|^2.$$

Therefore since this holds for any finite subset $F \subset \mathcal{D}$, (2) follows.

Now we order $\mathcal{D} = \{Q_1, Q_2, \dots\}$ and we have to prove that $\lim_{m\to\infty} \sum_{i=1}^m \Delta_{Q_i} f = f$ in L^2. First we let $A = \cup_{k\in\mathbb{N}} A_k$, where A_k are the classes of functions with compact support such that they are constant on some dyadic cube in $\mathcal{D}_k$ and furthermore they have zero mean on each orthant. Then we notice that the result is true for the class of functions A and furthermore A is dense in L^2.

For any $\varepsilon > 0$ and all $f \in L^2$ there exists a function $g \in A$ such that $\|f - g\| < \varepsilon$. We write $\sum_{i=1}^m \Delta_{Q_i} f = \sum_{i=1}^m \Delta_{Q_i}(f-g) + \sum_{i=1}^m \Delta_{Q_i} g$ and we deduce using (2) that

$$\begin{aligned}
\left\| f - \sum_{i=1}^m \Delta_{Q_i} f \right\| &\leq \|f-g\| + \left\| g - \sum_{i=1}^m \Delta_{Q_i} g \right\| + \left\| \sum_{i=1}^m \Delta_{Q_i}(g-f) \right\| \\
&= \|f-g\| + \left\| g - \sum_{i=1}^m \Delta_{Q_i} g \right\| + \left(\sum_{i=1}^m \|\Delta_{Q_i}(g-f)\|^2 \right)^{1/2} \\
&\leq 2\|f-g\| + \left\| g - \sum_{i=1}^m \Delta_{Q_i} g \right\| \\
&\leq 2\varepsilon + \left\| g - \sum_{i=1}^m \Delta_{Q_i} g \right\|,
\end{aligned}$$

and the proof is complete after letting $m \to \infty$. □

3. Proof of the $T1$ theorem

We will prove that

$$|\langle Tf, g\rangle| \leq C\|f\|\|g\|, \tag{3}$$

for $f = \sum_{Q\in F_1} \Delta_Q f,\ g = \sum_{Q\in F_2} \Delta_Q g$ for finite $F_1, F_2 \subset \mathcal{D}$ and a constant C not depending on f or g. Then by Proposition 4, (3) holds for all $f, g \in L^2$ and Theorem 2 follows, taking into account that $\|T\|_{L^2\to L^2} < \infty$ by assumption.

We have

$$\langle Tf, g\rangle = \sum_{Q\in F_1, R\in F_2} \langle T\Delta_Q f, \Delta_R g\rangle.$$

We then observe the following two easy facts: first,

$$\|\Delta_Q f\|^2 \approx \|\Delta_Q f\|_\infty^2 m(Q) \tag{4}$$

and, second,

$$\|T(\Delta_Q f)\| \le C\|\Delta_Q f\|. \tag{5}$$

To see (4), suppose that $Q = \cup_{i=1}^{2^n} P_i$ where P_i are the sons of Q and let $c_i = m_{P_i} f - m_Q f$. Then,

$$\|\Delta_Q f\|^2 = \int |\Delta_Q f|^2 = \sum_{i=1}^{2^n} m(P_i) c_i^2 = \frac{m(Q)}{2^n} \sum_{i=1}^{2^n} c_i^2,$$

and because $\|\Delta_Q f\|_\infty = \max c_i$, we get

$$\frac{1}{2^n} m(Q)\|\Delta_Q f\|_\infty^2 \le \|\Delta_Q f\|^2 \le m(Q)\|\Delta_Q f\|_\infty^2.$$

Using assumption (i) of Theorem 2, (5) follows analogously.

In the following we prove an auxiliary lemma which deals with the case when two cubes are far each other.

Lemma 5. Let two functions φ_Q, ψ_R be such that $\mathrm{supp}\varphi_Q \subset \overline{Q}$, $\mathrm{supp}\psi_R \subset \overline{R}, \varphi_Q, \psi_R \in L^1, \int \varphi_Q = 0$ and $\mathrm{d}(Q, \mathrm{supp}\psi_R) \ge l(Q)$. Then,

$$|\langle T\varphi_Q, \psi_R\rangle| \le \begin{cases} C\dfrac{l(Q)^\delta}{\mathrm{d}(Q,\mathrm{supp}\psi_R)^{\delta+n}} \|\varphi_Q\|_1 \|\psi_R\|_1 \\[2ex] C\dfrac{l(Q)^\delta}{\mathrm{d}(Q,\mathrm{supp}\psi_R)^{\delta}} \|\varphi_Q\|_1 \|\psi_R\|_\infty. \end{cases} \tag{6}$$

Proof. Let x_Q be the center of Q. Using that K is a standard kernel and $\int \varphi_Q = 0$ we get

$$\begin{aligned} \big|\langle T\varphi_Q, \psi_R\rangle\big| &= \left|\int \left(\int K(x,y)\varphi_Q(x)\psi_R(y)dx\right) dy\right| \\ &= \left|\int \left(\int (K(x,y) - K(x,x_Q))\varphi_Q(y)\psi_R(y)dy\right) dx\right| \\ &\le C \iint_{\substack{y\in Q,\\ x\in \mathrm{supp}\psi_R}} \frac{|y-x_Q|^\delta}{|x-y|^{n+\delta}} |\varphi_Q(y)||\psi_R(x)| dxdy \\ &\le C\frac{l(Q)^\delta}{\mathrm{d}(Q,\mathrm{supp}\psi_R)^{\delta+n}} \|\varphi_Q\|_1 \|\psi_R\|_1. \end{aligned}$$

To complete the proof of the lemma notice that for $x \in Q$ and $y \in R$,

$$|x_Q - y| \le |x-y| + l(Q) \le |x-y| + d(Q, \mathrm{supp}\psi_R) \le 2|x-y|.$$

Therefore,

$$\begin{aligned}
&\iint_{\substack{x\in Q,\\ y\in \mathrm{supp}\psi_R}} \frac{|x-x_Q|^\delta}{|x-y|^{n+\delta}}|\varphi_Q(x)||\psi_R(y)|dxdy \\
&\qquad\qquad \leq Cl(Q)^\delta \|\varphi_Q\|_1\|\psi_R\|_\infty \int_{|x_Q-y|>\mathrm{d}(Q,\mathrm{supp}\psi_R)} \frac{1}{|x_Q-y|^{n+\delta}}dy \\
&\qquad\qquad \leq C\frac{l(Q)^\delta}{\mathrm{d}(Q,\mathrm{supp}\psi_R)^\delta}\|\varphi_Q\|_1\|\psi_R\|_\infty .
\end{aligned}$$

The last inequality follows because $\int_{|x_Q-y|>d}\frac{1}{|x_Q-y|^{n+\delta}}dy \lesssim d^{-\delta}$ for all $d>0$. This can be checked by integrating on annuli, for instance,

$$\begin{aligned}
\int_{|x_Q-y|>d}\frac{1}{|x_Q-y|^{n+\delta}}dy &\leq \sum_{i=0}^\infty \int_{B(x_Q,2^{i+1}d)\setminus B(x_Q,2^i d)} \frac{1}{|x_Q-y|^{n+\delta}}dy \\
&\leq \sum_{i=0}^\infty \frac{(d2^{i+1})^n}{(d2^i)^{n+\delta}} = \frac{2^n}{d^\delta}\sum_{i=0}^\infty\left(\frac{1}{2^\delta}\right)^i \\
&\leq Cd^{-\delta}.
\end{aligned}$$

□

Now we do the following splitting:

$$\begin{aligned}
\sum_{Q\in F_1, R\in F_2}\langle T\Delta_Q f,\Delta_R g\rangle &= \sum_{\substack{Q\in F_1,R\in F_2,\\ l(Q)\leq l(R)}}\langle T\Delta_Q f,\Delta_R g\rangle \\
&\quad + \sum_{\substack{Q\in F_1,R\in F_2,\\ l(Q)> l(R)}}\langle T\Delta_Q f,\Delta_R g\rangle \\
&= S_1+S_2.
\end{aligned}$$

We are going to bound S_1, the boundedness of S_2 follows analogously. We consider the following three cases for $Q,R\in\mathcal{D}$ such that $l(Q)\leq l(R)$:

(1) $d(\cup_i\partial P_i, Q)\leq l(Q)^\gamma l(R)^{1-\gamma}$ where the P_i's are the sons of R,
(2) $d(Q,R)>l(Q)^\gamma l(R)^{1-\gamma}$,
(3) $d(Q,\cup_i\partial P_i)>l(Q)l(R)^{1-\gamma}$ and $Q\subsetneq R$,

where $\gamma=\frac{\delta}{2(\delta+n)}$.

We will use the notation $(Q,R)\in(1)$ if Q,R satisfy (1) and so on. We have

$$S_1\leq \sum_{(Q,R)\in(1)}|\langle T\Delta_Q f,\Delta_R g\rangle| + \sum_{(Q,R)\in(2)}|\langle T\Delta_Q f,\Delta_R g\rangle| + \left|\sum_{(Q,R)\in(3)}\langle T\Delta_Q f,\Delta_R g\rangle\right|$$

and we will bound each of the three previous terms separately.

3.1. **Estimates for** $(Q, R) \in (1)$**.** First observe that

$$|\langle T\Delta_Q f, \Delta_R g\rangle| \leq C\|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2}. \tag{7}$$

To see this, let

$$\psi_R^1 = \Delta_R g \chi_{3Q^c} \text{ and } \psi_R^2 = \Delta_R g \chi_{3Q}.$$

Since $d(\text{supp}\psi_R^1, Q) \geq l(Q)$, by Lemma 5, Hölder's inequality and (4) we obtain

$$\begin{aligned}
|\langle T\Delta_Q f, \psi_R^1\rangle| &\leq C\|\Delta_Q f\|_1 \|\psi_R^1\|_\infty \\
&\leq C\|\Delta_Q f\|_1 \|\Delta_R g\|_\infty \\
&\leq C\|\Delta_Q f\| m(Q)^{1/2} \|\Delta_R g\| m(R)^{-1/2} \\
&= C\|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2}.
\end{aligned}$$

Furthermore, by (4), Hölder's inequality and (5)

$$\begin{aligned}
|\langle T\Delta_Q f, \psi_R^2\rangle| &\leq C\|\Delta_R g\|_\infty \int_{3Q} |T\Delta_Q f| \\
&\leq C\|\Delta_R g\| m(R)^{-1/2} m(3Q)^{1/2} \|T\Delta_Q f\| \\
&\leq C\|\Delta_R g\| m(R)^{-1/2} m(Q)^{1/2} \|\Delta_Q f\| \\
&= C\|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2}.
\end{aligned}$$

Therefore (7) is proved.

Lemma 6. We have

$$\sum_{(Q,R)\in(1)} \|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2} \leq C\|f\|\|g\|.$$

Proof. By Cauchy-Schwarz,

$$\begin{aligned}
\sum_{(Q,R)\in(1)} \|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2} &= \sum_{Q\in\mathcal{D}} \|\Delta_Q f\| \sum_{R:(Q,R)\in(1)} \|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2} \\
&\leq \left(\sum_{Q\in\mathcal{D}} \|\Delta_Q f\|^2\right)^{1/2} \left(\sum_{Q\in\mathcal{D}} \left(\sum_{R:(Q,R)\in(1)} \|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2}\right)^2\right)^{1/2} \\
&\leq \|f\| \left(\sum_{Q\in\mathcal{D}} \left(\sum_{R:(Q,R)\in(1)} \|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{n/2}\right)^2\right)^{1/2}.
\end{aligned}$$

Thus we only need to show that

$$\left(\sum_{Q\in\mathcal{D}}\left(\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|\left(\frac{l(Q)}{l(R)}\right)^{n/2}\right)^2\right)^{1/2}\le\|g\|. \tag{8}$$

We start with the following trick: we rewrite

$$\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|\left(\frac{l(Q)}{l(R)}\right)^{n/2}=\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|\left(\frac{l(Q)}{l(R)}\right)^{\frac{n-\varepsilon}{2}}\left(\frac{l(Q)}{l(R)}\right)^{\frac{\varepsilon}{2}}$$

where ε is some small number to be chosen later. Now we apply Cauchy-Schwarz and we get

$$\begin{aligned}&\left(\sum_{Q\in\mathcal{D}}\left(\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|\left(\frac{l(Q)}{l(R)}\right)^{n/2}\right)^2\right)^{1/2}\\ &\qquad\le\left(\sum_{Q\in\mathcal{D}}\left(\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|^2\left(\frac{l(Q)}{l(R)}\right)^{n-\varepsilon}\right)\left(\sum_{R:(Q,R)\in(1)}\left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}\right)\right)^{1/2}.\end{aligned}$$

For $(Q,R)\in(1)$ we have that $d(\cup_i\partial P_i,Q)\le l(R)$, where the P_i's are the sons of R. Now notice that for a fixed cube $Q\in\mathcal{D}$, and $k\in\mathbb{N}$,

$$\sharp\{R\in\mathcal{D}:(Q,R)\in(1)\text{ and }l(R)=2^k l(Q)\}\le C$$

where C only depends on n, the dimension of the space. If R_Q is the dyadic cube which contains Q and has length $2^k\ell(Q)$, then C is smaller than $(3^n-1)^2$ which is the cardinality of the set of the neighbors of all neighbors of R_Q. Thus,

$$\sum_{R:(Q,R)\in(1)}\left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}=\sum_{k\in\mathbb{N}}\sum_{\substack{R:(Q,R)\in(1),\\ l(R)=2^k l(Q)}}\left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}=C\sum_{k\in\mathbb{N}}2^{-k\varepsilon}\le C.$$

Furthermore, by Fubini,

$$\sum_{Q\in\mathcal{D}}\left(\sum_{R:(Q,R)\in(1)}\|\Delta_R g\|^2\left(\frac{l(Q)}{l(R)}\right)^{n-\varepsilon}\right)=\sum_{R\in\mathcal{D}}\|\Delta_R g\|^2\left(\sum_{Q:(Q,R)\in(1)}\left(\frac{l(Q)}{l(R)}\right)^{n-\varepsilon}\right).$$

Hence in order to prove (8) and thus settle the proof, we only need to show that

$$\left(\sum_{Q:(Q,R)\in(1)}\left(\frac{l(Q)}{l(R)}\right)^{n-\varepsilon}\right)\le C. \tag{9}$$

To prove this we split again:

$$\sum_{Q:(Q,R)\in(1)} \left(\frac{l(Q)}{l(R)}\right)^{n-\varepsilon} = \sum_{k\in\mathbb{N}} \sum_{\substack{Q:(Q,R)\in(1),\\ l(Q)=2^{-k}l(R)}} \left(\frac{l(Q)}{l(R)}\right)^{n} \left(\frac{l(R)}{l(Q)}\right)^{\varepsilon}$$
$$= \sum_{k\in\mathbb{N}} 2^{k\varepsilon} \sum_{\substack{Q:(Q,R)\in(1),\\ l(Q)=2^{-k}l(R)}} \left(\frac{l(Q)}{l(R)}\right)^{n}.$$

Let $d_k = 2\cdot 2^{-k\gamma} l(R)$, and for $A\subset\mathbb{R}^n$ denote

$$N_{d_k}(A) = \{x: \text{ there is } a\in A \text{ such that } d(a,y)\le d_k\}.$$

Then for $R\in\mathcal{D}$ fixed

$$\cup\{Q\in\mathcal{D} : (Q,R)\in(1),\, l(Q)=2^{-k}l(R)\}\subset N_{d_k}(\cup_i \partial P_i),$$

where P_i are the sons of R. Therefore,

$$\sum_{k\in\mathbb{N}} 2^{k\varepsilon} \sum_{\substack{Q:(Q,R)\in(1),\\ l(Q)=2^{-k}l(R)}} \left(\frac{l(Q)}{l(R)}\right)^{n} = \sum_{k\in\mathbb{N}} 2^{k\varepsilon} \sum_{\substack{Q:(Q,R)\in(1),\\ l(Q)=2^{-k}l(R)}} \frac{m(Q)}{m(R)}$$
$$\le \sum_{k\in\mathbb{N}} 2^{k\varepsilon} \frac{m(N_{d_k}(\cup_i\partial P_i))}{m(R)}$$
$$\le C\sum_{k\in\mathbb{N}} 2^{k\varepsilon} \frac{d_k\, l(R)^{n-1}}{l(R)^n}$$
$$\le C\sum_{k\in\mathbb{N}} 2^{k(\varepsilon-\gamma)}.$$

Hence if we choose $\varepsilon=\gamma/2$ the proof of (9) is complete and we are done. □

Combining (7) and Lemma 6 we derive the desired estimate

$$\sum_{(Q,R)\in(1)} |\langle T\Delta_Q f, \Delta_R g\rangle| \le C\|f\|\|g\|.$$

3.2. **Estimates for** $(Q,R)\in(2)$**.** In this subsection we are going to show that $\sum_{(Q,R)\in(2)} |\langle T\Delta_Q f,\Delta_R g\rangle| \le \|f\|\|g\|$. We start with an auxiliary lemma. As usual if I is an index set, $\ell^2(I)=\{(x_i)_{i\in I}: x_i\in\mathbb{R} \text{ and } \sum_{i\in I} x_i^2<\infty\}$.

Lemma 7 (Schur's lemma)**.** Let I be some set of indices, and for each $i\in I$ a number $w_i>0$. Suppose that, for some constant $a\ge 0$, the matrix $\{T_{i,j}\}_{i,j\in I}$ satisfies

$$\sum_j |T_{i,j}|\, w_j \le a\, w_i \quad \text{for each } i$$

and

$$\sum_i |T_{i,j}|\, w_i \le a\, w_j \quad \text{for each } j.$$

Then, the matrix $\{T_{i,j}\}_{i,j\in I}$ defines a bounded operator in $\ell^2(I)$ with norm $\le a$.

Proof. Let $\{x_i\}_{i\in I} \in \ell^2(I)$, and set $y_i = \sum_{j\in I} T_{i,j}x_j$. We intend to show that $\sum_i |y_i|^2 \le a^2 \sum_i |x_i|^2$. So we write

$$|T_{i,j}||x_j| = \left(|T_{i,j}|^{1/2}\, w_j^{1/2}\right)\left(|T_{i,j}|^{1/2}\, w_j^{-1/2}\, |x_j|\right),$$

and then by Cauchy-Schwarz,

$$|y_i|^2 \le \left(\sum_{j\in I} |T_{i,j}|\, w_j\right)\left(\sum_{j\in I} |T_{i,j}|\, w_j^{-1}\, |x_j|^2\right) \le a\, w_i \sum_{j\in I} |T_{i,j}|\, w_j^{-1}\, |x_j|^2.$$

Summing on i and interchanging the sums, we get

$$\sum_i |y_i|^2 \le a \sum_j \sum_i |T_{i,j}|\, w_i\, w_j^{-1}\, |x_j|^2 \le a^2 \sum_j |x_j|^2.$$

$\square$

For $(Q,R)\in(2)$ we have that $d(Q,\partial R)\ge l(Q)$ hence by Lemma 5 and Cauchy-Schwarz we have

$$\begin{aligned}|\langle T\Delta_Q f, \Delta_R g\rangle| &\le C\frac{l(Q)^\delta}{d(Q,R)^{\delta+n}}\|\Delta_Q f\|_1\|\Delta_R g\|_1\\ &\le C\frac{l(Q)^\delta}{d(Q,R)^{\delta+n}} m(Q)^{1/2}m(R)^{1/2}\|\Delta_Q f\|\|\Delta_R g\|.\end{aligned}$$

We denote $D(Q,R) = l(Q)+l(R)+d(Q,R)$. It then follows that

$$\frac{l(Q)^\delta}{d(Q,R)^{n+\delta}} \le C\frac{l(Q)^{\delta/2}l(R)^{\delta/2}}{D(Q,R)^{n+\delta}}. \tag{10}$$

To prove (10) we consider two cases. If $d(Q,R)\ge l(R)$ then $D(Q,R)\le 3d(Q,R)$ and furthermore $l(Q)^\delta \le l(Q)^{\delta/2}l(R)^{\delta/2}$ hence (10) follows. If $d(Q,R)\le l(R)$ then $D(Q,R)\le 3l(R)$ and recalling that for $(Q,R)\in(2),\ d(Q,R)\ge l(Q)^\gamma l(R)^{1-\gamma}$,

$$\begin{aligned}\frac{l(Q)^\delta}{d(Q,R)^{n+\delta}} &\le \frac{l(Q)^\delta}{l(Q)^{\gamma(n+\delta)}l(R)^{(1-\gamma)(n+\delta)}} = \frac{l(Q)^{\delta-\gamma(n+\delta)}l(R)^{\gamma(n+\delta)}}{l(R)^{(n+\delta)}}\\ &\le C\frac{l(Q)^{\delta/2}l(R)^{\delta/2}}{D(Q,R)^{n+\delta}},\end{aligned}$$

as $\gamma = \frac{\delta}{2(n+\delta)}$. Therefore,

$$|\langle T\Delta_Q f, \Delta_R g\rangle| \le C\frac{l(Q)^{\delta/2}l(R)^{\delta/2}}{D(Q,R)^{\delta+n}} m(Q)^{1/2}m(R)^{1/2}\|\Delta_Q f\|\|\Delta_R g\|. \tag{11}$$

Our goal now is to apply Schur's Lemma. To this end, consider the matrix $(T_{Q,R})_{(Q,R)\in(2)}$ defined by

$$T_{Q,R}=\frac{l(Q)^{\delta/2}l(R)^{\delta/2}}{D(Q,R)^{\delta+n}}m(Q)^{1/2}m(R)^{1/2}.$$

Applying Cauchy-Schwarz in $\ell^2(I)$, where I is the index set associated to the set $\{(Q,R)\in(2)\}$ we get,

$$\begin{aligned}\sum_{(Q,R)\in(2)}T_{Q,R}\|\Delta_Q f\|\|\Delta_R g\| &\leq \left(\sum_R\left(\sum_{Q:(Q,R)\in(2)}T_{Q,R}\|\Delta_Q f\|\right)^2\right)^{1/2}\left(\sum_R\|\Delta_R g\|^2\right)^{1/2}\\ &= \left(\sum_R\left(\sum_{Q:(Q,R)\in(2)}T_{Q,R}\|\Delta_Q f\|\right)^2\right)^{1/2}\|g\|.\end{aligned} \tag{12}$$

Hence we only need to show that

$$\sum_R\left(\sum_{Q:(Q,R)\in(2)}T_{Q,R}\|\Delta_Q f\|\right)^2\leq C\sum_{(Q,R)\in(2)}\|\Delta_Q f\|^2. \tag{13}$$

For $Q\in\mathcal{D}$ let $w_Q=m(Q)^{1/2}$. By Schur's lemma in order to prove (13) it is enough to show that

$$\sum_{R:(Q,R)\in(2)}T_{Q,R}w_R\leq Cw_Q\quad\text{and}\quad\sum_{Q:(Q,R)\in(2)}T_{Q,R}w_Q\leq Cw_R.$$

So it suffices to show that for all $Q\in\mathcal{D}$,

$$\sum_{R\in\mathcal{D}}T_{Q,R}w_R\leq Cw_Q. \tag{14}$$

Remark that in the sum above we do not assume $\ell(Q)\leq\ell(R)$. We write

$$\begin{aligned}\sum_R T_{Q,R}w_R &= \sum_{k\in\mathbb{Z}}\sum_{R:l(R)=2^k l(Q)}\frac{l(Q)^{\delta/2}2^{k\frac{\delta}{2}}l(Q)^{\delta/2}}{D(Q,R)^{n+\delta}}l(Q)^{n/2}l(R)^n\\ &= l(Q)^{n/2}\sum_{k\in\mathbb{Z}}2^{k\frac{\delta}{2}}l(Q)^{\delta}\sum_{R:l(R)=2^k l(Q)}\frac{l(R)^n}{D(Q,R)^{n+\delta}}.\end{aligned}$$

Let x_Q be the center of Q and denote $k_+ = \max(k,0)$. Then we have

$$\begin{aligned}\sum_{R:l(R)=2^k l(Q)} \frac{l(R)^n}{D(Q,R)^{n+\delta}} &= \sum_{R:l(R)=2^k l(Q)} \int_R \frac{1}{D(Q,R)^{n+\delta}}dx \\ &\leq C \sum_{R:l(R)=2^k l(Q)} \int_R \frac{1}{(|x_Q - x| + l(Q) + l(R))^{n+\delta}}dx \\ &\leq C \int \frac{1}{(|x_Q - x| + 2^{k_+} l(Q))^{n+\delta}}dx \\ &= C\bigg(\int_{|x-x_Q|\leq 2^{k_+}l(Q)} \frac{1}{(|x_Q - x| + 2^{k_+} l(Q))^{n+\delta}}dx \\ &\qquad + \int_{|x-x_Q|> 2^{k_+}l(Q)} \frac{1}{(|x_Q - x| + 2^{k_+} l(Q))^{n+\delta}}dx \bigg) \\ &= C(I_1 + I_2).\end{aligned}$$

For I_1,

$$I_1 \leq \int_{|x-x_Q|\leq 2^{k_+}l(Q)} \frac{1}{(2^{k_+} l(Q))^{n+\delta}}dx \leq \frac{(2^{k_+} l(Q))^n}{(2^{k_+} l(Q))^{n+\delta}} = (2^{k_+} l(Q))^{-\delta},$$

and for I_2 as usual after splitting the set $|x_Q - x| > 2^{k_+} l(Q)$ in annuli and integrating we get

$$I_2 \leq \int_{|x-x_Q|> 2^{k_+}l(Q)} \frac{1}{|x_Q - x|^{n+\delta}}dx \leq \frac{C}{(2^{k_+} l(Q))^{\delta}}.$$

Therefore,

$$\sum_{R:l(R)=2^k l(Q)} \frac{l(R)^n}{D(Q,R)^{n+\delta}} \leq \frac{C}{(2^{k_+} l(Q))^{\delta}},$$

and by (3.2),

$$\sum_{R:(Q,R)\in(2)} T_{Q,R} w_R \leq C l(Q)^{n/2} \sum_{k\in\mathbb{Z}} 2^{-|k|\frac{\delta}{2}} \leq C l(Q)^{n/2}.$$

3.3. **Estimates for** $(Q,R) \in (3)$**.** In this subsection we are going to show that $\left|\sum_{(Q,R)\in(3)} \langle T\Delta_Q f, \Delta_R g\rangle\right| \leq \|f\|\|g\|$. To this end we need the following discrete version of the famous embedding theorem of Carleson.

Theorem 8. [Carleson's Embedding Theorem] Let σ be a Radon measure on $\mathbb{R}^n$. Let $\mathcal{D}$ be the dyadic lattice from $\mathbb{R}^n$ and let $\{a_Q\}_{Q\in\mathcal{D}}$ be a family of non negative numbers. Suppose that for every cube $R \in \mathcal{D}$ we have

$$\sum_{Q\in\mathcal{D}:Q\subset R} a_Q \leq c_2\, \sigma(R). \tag{15}$$

Then, every family of non negative numbers $\{w_Q\}_{Q\in\mathcal{D}}$ satisfies

$$\sum_{Q\in\mathcal{D}} w_Q\, a_Q \leq c_2 \int \sup_{Q\ni x} w_Q\, d\sigma(x). \tag{16}$$

In particular, if $f\in L^2(\sigma)$,

$$\sum_{Q\in\mathcal{D}} |\langle f\rangle_{\sigma,Q}|^2\, a_Q \leq c\, c_2 \|f\|^2_{L^2(\sigma)}, \tag{17}$$

where $\langle f\rangle_{\sigma,Q} = \int_Q f\, d\sigma/\sigma(Q)$ and c is an absolute constant.

Proof. To prove (16), consider the characteristic function defined by $\chi(Q,t)=1$ if $0<t<w_Q$, and 0 otherwise. Then,

$$\sum_{Q\in\mathcal{D}} w_Q\, a_Q = \sum_{Q\in\mathcal{D}} \int_0^\infty \chi(Q,t)\, a_Q\, dt = \int_0^\infty \sum_{Q\in\mathcal{D}} \chi(Q,t)\, a_Q\, dt. \tag{18}$$

For each $t>0$, let

$$\Omega_t = \bigcup_{Q\in\mathcal{D}:\, w_Q>t} Q.$$

Notice that if $\chi(Q,t)=1$, then $Q\subset\Omega_t$, and thus

$$\sum_{Q\in\mathcal{D}} \chi(Q,t)\, a_Q \leq \sum_{Q\in\mathcal{D}:\, Q\subset\Omega_t} a_Q$$

For $m\geq 1$, let $I_m\subset\mathcal{D}$ be the subfamily of the cubes $Q\subset\Omega_t$ such that $\ell(Q)\leq 2^m$, and let $J_m\subset I_m$ be the subfamily of maximal cubes from I_m. Then we have

$$\sum_{Q\in\mathcal{D}:\, Q\subset\Omega_t} a_Q = \lim_{m\to\infty} \sum_{Q\in I_m} a_Q = \lim_{m\to\infty} \sum_{R\in J_m} \sum_{Q\subset R} a_Q.$$

By the assumption (15), for all $m\geq 1$,

$$\sum_{R\in J_m} \sum_{Q\subset R} a_Q \leq c_2 \sum_{R\in J_m} \sigma(R) \leq c_2\, \sigma(\Omega_t),$$

and so

$$\sum_{Q\in\mathcal{D}} \chi(Q,t)\, a_Q \leq c_2\, \sigma(\Omega_t).$$

Since Ω_t coincides with $\{x\in\mathbb{R}^d : w^*(x)>t\}$, where $w^*(x)=\sup_{Q\ni x} w_Q$, by (18) we obtain

$$\sum_{Q\in\mathcal{D}} w_Q\, a_Q \leq c_2 \int_0^\infty \sigma(\Omega_t) = c_2 \int w^*(x)\, d\sigma(x).$$

To prove the estimate (17), we take $w_Q = |\langle f\rangle_{\sigma,Q}|^2$, and then from (16) we deduce

$$\sum_{Q\in\mathcal{D}} |\langle f\rangle_{\sigma,Q}|^2\, a_Q \leq c_2 \|M_{\sigma,d} f\|^2_{L^2(\sigma)},$$

where $M_{\sigma,d}$ is the Hardy-Littlewood maximal dyadic operator with respect to σ. From the $L^2(\sigma)$ boundedness (with absolute constants) of this operator, we obtain (17). $\square$

We now observe that if R_Q is the son of R that contains Q, $\Delta_R g$ is constant in Q and

$$\Delta_R g|_{R_Q} = m_{R_Q} g - m_R g := c_{R,Q}(g).$$

Therefore we write,

$$\Delta_R g = \chi_{R\setminus R_Q}\Delta_R g + c_{R,Q}(g)\chi_{R_Q} = \chi_{R\setminus R_Q}\Delta_R g + c_{R,Q}(g)\cdot 1 - c_{R,Q}(g)\chi_{R_Q^c}.$$

Hence,

$$\begin{aligned}\left|\sum_{(Q,R)\in(3)}\langle T\Delta_Q f, \Delta_R g\rangle\right| &\le \sum_{(Q,R)\in(3)} |\langle T\Delta_Q f, \chi_{R\setminus R_Q}\Delta_R g\rangle| \\ &\quad + \sum_{(Q,R)\in(3)} |\langle T\Delta_Q f, c_{R,Q}(g)\chi_{R_Q^c}\rangle| + \left|\sum_{(Q,R)\in(3)}\langle T\Delta_Q f, c_{R,Q}(g)\rangle\right| \\ &= A_1 + A_2 + A_3.\end{aligned} \tag{19}$$

We will finish the proof of Theorem 2 by showing that $A_i \le C\|f\|\|g\|$ for $i = 1,2,3$. The following lemma deals with $A_1 + A_2$.

Lemma 9. We have

$$A_1 + A_2 = \sum_{(Q,R)\in(3)} |\langle T(\Delta_Q f), \chi_{R\setminus R_Q}\Delta_R g\rangle| + \sum_{(Q,R)\in(3)} |\langle T\Delta_Q f, c_{R,Q}(g)\chi_{R_Q^c}\rangle| \le C\|f\|\|g\|.$$

Proof. Regarding A_1, for $\psi_R := \chi_{R\setminus R_Q}\Delta_R g$, we have

$$d(Q, \mathrm{supp}\psi_R) \ge l(Q)^\gamma l(R)^{1-\gamma}.$$

Reasoning as in the proof of (11) and taking into account that $D(Q,R) \approx \ell(R)$,

$$\begin{aligned}|\langle T(\Delta_Q f), \chi_{R\setminus R_Q}\Delta_R g\rangle| &\le C\frac{l(Q)^{\delta/2} l(R)^{\delta/2}}{D(Q,R)^{\delta+n}} m(Q)^{1/2} m(R)^{1/2}\|\Delta_Q f\|_2\|\Delta_R g\|_2 \\ &\approx \frac{l(Q)^{n+\delta/2}}{\ell(R)^{n+\delta/2}}\|\Delta_Q f\|_2\|\Delta_R g\|_2.\end{aligned} \tag{20}$$

We know turn our attention to A_2. Since $d(R_Q^c, Q) \ge l(Q)^\gamma l(R)^{1-\gamma}$, by Lemma 5 applied to $\varphi_Q = \Delta_Q f$ and $\psi_R = \chi_{R_Q^c}$, we deduce

$$|\langle T\Delta_Q f, \chi_{R_Q^c}\rangle| \le C\|\Delta_Q f\|_1 \frac{l(Q)^\delta}{l(Q)^{\gamma\delta} l(R)^{(1-\gamma)\delta}} \le C\|\Delta_Q f\| m(Q)^{1/2}\left(\frac{l(Q)}{l(R)}\right)^{\delta(1-\gamma)},$$

Furthermore by (4),

$$|c_{R,Q}(g)| \leq \|\Delta_R g\|_\infty \approx \frac{\|\Delta_R g\|}{m(R)^{1/2}}.$$

Thus,

$$|c_{R,Q}(g)\langle T\Delta_Q f, \chi_{R_Q^c}\rangle| \leq C\|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{\delta(1-\gamma)} \left(\frac{m(Q)}{m(R)}\right)^{1/2}.$$

Together with (20), for $\varepsilon = \min(\delta/2, \delta(1-\gamma))$, this yields

$$\begin{aligned}
A_1 + A_2 &\leq C \sum_{(Q,R)\in(3)} \|\Delta_Q f\|\|\Delta_R g\| \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \left(\frac{m(Q)}{m(R)}\right)^{1/2} \\
&\leq C \sum_{R\in\mathcal{D}} \|\Delta_R g\| \sum_{Q:(Q,R)\in(3)} \|\Delta_Q f\| \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \left(\frac{m(Q)}{m(R)}\right)^{1/2} \\
&\leq C \left(\sum_{R\in\mathcal{D}} \|\Delta_R g\|^2\right)^{1/2} \left(\sum_{R\in\mathcal{D}} \left(\sum_{Q:Q\subset R} \|\Delta_Q f\| \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \left(\frac{m(Q)}{m(R)}\right)^{1/2}\right)^2\right)^{1/2} \\
&= C\|g\| \left(\sum_{R\in\mathcal{D}} \left(\sum_{Q:Q\subset R} \|\Delta_Q f\| \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon/2} \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon/2} \left(\frac{m(Q)}{m(R)}\right)^{1/2}\right)^2\right)^{1/2} \\
&\leq C\|g\| \left(\sum_{R\in\mathcal{D}} \left[\sum_{Q:Q\subset R} \|\Delta_Q f\|^2 \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}\right] \left[\sum_{Q:Q\subset R} \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \frac{m(Q)}{m(R)}\right]\right)^{1/2},
\end{aligned}$$

where we used twice Cauchy-Schwarz. For a cube $R \in \mathcal{D}$ let $\mathcal{D}_k(R) = \{Q \in \mathcal{D} : Q \subset R, l(Q) = 2^{-k} l(R)\}$. We then notice that,

$$\begin{aligned}
\sum_{Q:Q\subset R} \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \frac{m(Q)}{m(R)} &= \sum_{k\in\mathbb{N}} \sum_{Q\in\mathcal{D}_k(R)} \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon} \frac{m(Q)}{m(R)} \\
&= \sum_{k\in\mathbb{N}} 2^{-k\varepsilon} \sum_{Q\in\mathcal{D}_k(R)} \frac{m(Q)}{m(R)} = \sum_{k\in\mathbb{N}} 2^{-k\varepsilon} \leq C.
\end{aligned}$$

Therefore, by the previous two estimates and Fubini,

$$\begin{aligned}
A_1 + A_2 &\leq C\|g\| \left(\sum_{R\in\mathcal{D}} \sum_{Q:Q\subset R} \|\Delta_Q f\|^2 \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}\right)^{1/2} \\
&= C\|g\| \left(\sum_{Q\in\mathcal{D}} \|\Delta_Q f\|^2 \sum_{R:R\supset Q} \left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}\right)^{1/2} \\
&\leq C\|f\|\|g\|.
\end{aligned}$$

The last inequality follows, because as previously

$$\sum_{R\in\mathcal{D}:R\supset Q}\left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}=\sum_{k\in\mathbb{N}}\sum_{\substack{R:R\supset Q,\\ l(R)=2^k l(Q)}}\left(\frac{l(Q)}{l(R)}\right)^{\varepsilon}=\sum_{k\in\mathbb{N}}\left(\frac{1}{2^{\varepsilon}}\right)^{k}\leq C.$$

□

Finally we have to estimate A_3. Given a cube $Q\in\mathcal{D}$, let $(R_i)_{i\in\mathbb{N}}$ be the increasing sequence of cubes in $\mathcal{D}$ which strictly contain Q. Recall that R_{iQ} denotes the son of R_i which contains Q and notice that $R_{1Q}=Q$, $R_{i+1Q}=R_i$ and since $g=\sum_{R\in F_2}\Delta_R f$ and F_2 is a finite set of cubes, there exists some i_0 and some $F_{2'}\subset F_2$ such that

$$\text{supp} g\cap R_{i_0}=\cup_{R\in F_{2'}}R.$$

Hence, by Proposition 3,

$$\int_{R_{i_0}} g=\sum_{R\in F_{2'}}\int_R \Delta_R g=0$$

and $m_{R_{i_0}}g=0$. Furthermore,

$$\sum_{R\supsetneq Q}c_{R,Q}(g)=\sum_{i\in\mathbb{N}}c_{R_i,Q}g=\sum_{i\in\mathbb{N}}(m_{R_{iQ}}g-m_{R_i}g)$$

By the previous observations this sum is telescopic and since $m_{R_{i_0}}g=0$ we get

$$\sum_{R\supsetneq Q}c_{R,Q}(g)=m_Q g.$$

Now we can estimate:

$$\begin{aligned}\sum_{(Q,R):R\supsetneq Q}\langle T\Delta_Q f,c_{R,Q}(g)1\rangle &=\sum_{(Q,R):R\supsetneq Q}c_{R,Q}(g)\langle T\Delta_Q f,1\rangle\\ &=\sum_{Q\in\mathcal{D}}\left(\sum_{R\supsetneq Q}c_{R,Q}(g)\right)\langle T\Delta_Q f,1\rangle\\ &=\sum_{Q\in\mathcal{D}}m_Q(g)\langle \Delta_Q f,T^*1\rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{(Q,R)\in(3)}c_{R,Q}(g)\langle T\Delta_Q f,1\rangle &=\sum_{(Q,R):R\supsetneq Q}c_{R,Q}(g)\langle T\Delta_Q f,1\rangle-\sum_{\substack{(Q,R):R\supsetneq Q,\\(Q,R)\notin(3)}}c_{R,Q}(g)\langle T\Delta_Q f,1\rangle\\ &=\sum_{(Q,R):R\supsetneq Q}m_Q(g)\langle \Delta_Q f,T^*1\rangle-\sum_{\substack{(Q,R):R\supsetneq Q,\\(Q,R)\notin(3)}}c_{R,Q}(g)\langle T\Delta_Q f,1\rangle\\ &=S_3-S_4.\end{aligned}\tag{21}$$

Therefore $|A_3| \leq |S_3| + |S_4|$, and in order to finish the proof we only need to bound the terms S_3 and S_4.

Lemma 10. For S_4 as in (21) we have

$$|S_4| \leq C\|f\|\|g\|.$$

Proof. We have,

$$\begin{aligned}
|S_4| &\leq \sum_{\substack{(Q,R):R\supsetneq Q,\\ d(Q,R)\leq l(Q)^{\gamma} l(R)^{1-\gamma}}} |c_{R,Q}(g)||\langle T\Delta_Q f, 1\rangle| \\
&\leq \sum_{\substack{(Q,R):R\supsetneq Q,\\ d(Q,R)\leq l(Q)^{\gamma} l(R)^{1-\gamma}}} |c_{R,Q}(g)|\|T\Delta_Q f\|_1.
\end{aligned}$$

By the second estimate in (6) and duality, it is clear that

$$\|(T\Delta_Q f)\chi_{(3Q)^c}\|_1 \leq C\|\Delta_Q f\|_1 \leq C\|\Delta_Q f\| m(Q)^{1/2}.$$

Using Again by Hölder's inequality and (5),

$$\|(T\Delta_Q f)\chi_{3Q}\|_1 \leq C\|T\Delta_Q f\| m(Q)^{1/2} \leq C\|\Delta_Q f\| m(Q)^{1/2}.$$

On the other hand, by (4),

$$|c_{R,Q}(g)| \leq \|\Delta_R g\|_\infty \approx \frac{\|\Delta_R g\|}{m(R)^{1/2}}.$$

So by all the previous estimates

$$|S_4| \leq C \sum_{\substack{(Q,R):R\supsetneq Q,\\ (Q,R)\notin(3)}} \|\Delta_R g\|\|\Delta_Q f\| \left(\frac{m(Q)}{m(R)}\right)^{1/2}.$$

Now notice that the pairs $(Q,R) \notin (3)$ such that $R \supsetneq Q$ belong to the case (1), and thus by Lemma 6,

$$|S_4| \leq C\|f\|\|g\|.$$

□

Finally we need to bound $|S_3|$. By (5) and (6) of Proposition 3,

$$\begin{aligned}
S_3 &= \sum_{Q\in\mathcal{D}} m_Q g\langle \Delta_Q f, T^*1\rangle = \sum_{Q\in\mathcal{D}} m_Q g\langle f, \Delta_Q(T^*1)\rangle \\
&= \sum_{Q\in F_1} m_Q g\langle f, \Delta_Q(T^*1)\rangle = \langle f, \sum_{Q\in F_1} m_Q g\Delta_Q(T^*1)\rangle.
\end{aligned}$$

Notice that the functions $g_Q = m_Q g\Delta_Q(T^*1)$ are orthogonal, and thus

$$\begin{aligned}|S_3| &\le \left|\langle f, \sum_{Q\in F_1} m_Q g\Delta_Q(T^*1)\rangle\right| \le \|f\| \left\|\sum_{Q\in F_1} m_Q g\Delta_Q(T^*1)\right\| \\ &= \|f\| \left(\sum_{Q\in F_1} |m_Q g|^2 \|\Delta_Q(T^*1)\|^2\right)^{1/2}.\end{aligned}$$

The following lemma settles the case for S_3.

Lemma 11. We have

$$\sum_{Q\in\mathcal{D}} |m_Q g|^2 \|\Delta_Q T^*1\|^2 \le C\|g\|^2. \tag{22}$$

Proof. By Theorem 8 it suffices to prove that

$$\sum_{Q\subset P, Q\in\mathcal{D}} \|\Delta_Q T^*1\|^2 \le Cm(P) \tag{23}$$

for all $P \in \mathcal{D}$.

By Proposition 4, since $(T^*1 - m_P(T^*1))\chi_P \in L^2$, we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}, Q\subset P} \|\Delta_Q T^*1\|^2 &= \sum_{Q\in\mathcal{D}, Q\subset P} \|\Delta_Q((T^*1 - m_P(T^*1))\chi_P)\|^2 \\ &= \sum_{Q\in\mathcal{D}} \|\Delta_Q((T^*1 - m_P(T^*1))\chi_P)\|^2 \\ &= \|(T^*1 - m_P(T^*1))\chi_P\|^2.\end{aligned}$$

Thus in order to complete the proof it is enough to show that for all $P \in \mathcal{D}$,

$$\frac{1}{m(P)} \int_P |T^*1 - m_P(T^*1)|^2 \le C. \tag{24}$$

This is equivalent to saying that $T^*1 \in BMO_d^2$, the dyadic BMO space.

Lemma 12. If $\|T^*\chi_{2P}\|^2 \le Cm(P)$,

$$\frac{1}{m(P)} \int_P |T^*1 - m_P(T^*1)|^2 \le C.$$

Proof. We have

$$\begin{aligned}T^*1 - m_P(T^*1) &= T^*\chi_{2P} - m_P(T^*\chi_{2P}) + T^*\chi_{(2P)^c} - m_P(T^*\chi_{(2P)^c}) \\ &= A + B.\end{aligned}$$

We first deal with A:

$$\begin{aligned}\int_P |A|^2 &= \int_P |T^*\chi_{2P} - m_P(T^*\chi_{2P})|^2 \\ &\lesssim \|T^*\chi_{2P}\|^2 + m(P)|m_P(T^*\chi_{2P})|^2 \\ &\leq Cm(P) + m(P)|m_P(T^*\chi_{2P})|^2.\end{aligned}$$

But by Hölder's inequality,

$$\begin{aligned}|m_P(T^*\chi_{2P})|^2 &\leq \left(\frac{1}{m(P)}\int_P |T^*\chi_{2P}|\right)^2 \\ &\leq C\frac{1}{m(P)}\|T^*\chi_{2P}\|^2 \\ &\leq C\frac{1}{m(P)}m(P) = C.\end{aligned}$$

Hence,

$$\int_P |A|^2 \leq Cm(P).$$

Now we will estimate $\int_P |B|^2$. The first step is to show that for all $x_1, x_2 \in P$,

$$|T^*\chi_{(2P)^c}(x_1) - T^*\chi_{(2P)^c}(x_2)| \leq C. \tag{25}$$

To see this,

$$\begin{aligned}|T^*\chi_{(2P)^c}(x_1) - T^*\chi_{(2P)^c}(x_2)| &\leq \int_{(2P)^c} |K(y,x_1) - K(y,x_2)|dy \\ &\leq C\int_{(2P)^c} \frac{|x_1-x_2|^\delta}{|x_1-y|^{n+\delta}}dy \\ &\leq Cl(P)^\delta \int_{(2P)^c} \frac{1}{|x_1-y|^{n+\delta}}dy \\ &\leq C\frac{l(P)^\delta}{l(P)^\delta} = C,\end{aligned}$$

where in the last inequality we used integration in annuli as usual.

Now,

$$\begin{aligned}
\int_P |B|^2 &= \int_P \left| T^*\chi_{(2P)^c}(x_1) - \frac{1}{m(P)}\int_P T^*\chi_{(2P)^c}(x_2)dx_2 \right|^2 dx_1 \\
&= \int_P \left| \frac{1}{m(P)}\int_P T^*\chi_{(2P)^c}(x_1)dx_2 - \frac{1}{m(P)}\int_P T^*\chi_{(2P)^c}(x_2)dx_2 \right|^2 dx_1 \\
&= \int_P \frac{1}{m(P)^2} \left| \int_P (T^*\chi_{(2P)^c}(x_1) - T^*\chi_{(2P)^c}(x_2))dx_2 \right|^2 dx_1 \\
&\leq \int_P \frac{1}{m(P)^2} \left(\int_P |T^*\chi_{(2P)^c}(x_1) - T^*\chi_{(2P)^c}(x_2)|^2 \right) m(P) dx_1 \\
&\leq Cm(P).
\end{aligned}$$

For the last two inequalities we used Hölder's inequality and (25). This finishes the proof of Lemma 12 and the proof of Lemma 11 as well. Hence the proof of Theorem 2 is completed. □

Remark 13. Expressions of the form

$$\Pi_h g = \sum_{Q\in\mathcal{D}} m_Q g \Delta_Q h$$

are called paraproducts of g associated to h. In our case $\Pi g = \sum_{Q\in\mathcal{D}} m_Q g \Delta_Q(T^*1)$ is the paraproduct associated to $T^*(1)$. Using Lemma 11 and orthogonality it follows that $\|\Pi g\| \leq C\|g\|$.

□